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New Comparison Theorems for the Even-Order Neutral Delay Differential Equation

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Received: 12 March 202; Accepted: 9 April 2020; Published: 6 May 2020

Abstract: The aim of this study was to examine the asymptotic properties and oscillation of the even-order neutral differential equations. The results obtained are based on the Riccati transformation and the theory of comparison with first- and second-order delay equations. Our results improve and complement some well-known results. We obtain Hille and Nehari type oscillation criteria to ensure the oscillation of the solutions of the equation. One example is provided to illustrate these results.

Keywords: even-order differential equations; neutral delay; oscillation

1. Introduction

During the past years, research activity has focused on the oscillatory behavior of solutions to different classes of neutral differential equations. In a related field, the asymptotic behavior of the solutions to delay and neutral delay differential equations was discussed in many works, awith fruitful achievements [1–28]. One of the main reasons for this lies in delay differential equations arising in many applied problems in natural sciences, technology, and automatic control [25].

This paper is concerned with oscillation of the even-order nonlinear neutral differential equation of the form

$$\left(r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + p(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma} + q(\varsigma)\, y^{\gamma}(\delta(\varsigma)) = 0, \tag{1}$$

where $\varsigma \geq \varsigma_0$, $n \geq 4$ is an even natural number and

$$u(\varsigma) := y(\varsigma) + c(\varsigma)\, y(g(\varsigma)). \tag{2}$$

Throughout this paper, we assume that the following conditions are satisfied:

(C_1) γ is a quotient of odd natural numbers;
(C_2) $r \in C^1([\varsigma_0,\infty))$, $r(\varsigma) > 0$, $r'(\varsigma) \geq 0$;
(C_3) $c, p, q \in C([\varsigma_0,\infty))$, $p(\varsigma) > 0$, $q(\varsigma) > 0$, $0 \leq c(\varsigma) < c_0 < 1$, q is not identically zero for large ς; and
(C_4) $g \in C^1([\varsigma_0,\infty))$, $\delta \in C([\varsigma_0,\infty))$, $g'(\varsigma) > 0$, $g(\varsigma) \leq \varsigma$ and $\lim_{\varsigma\to\infty} g(\varsigma) = \lim_{\varsigma\to\infty} \delta(\varsigma) = \infty$.

Definition 1. *A function $y \in C^{n-1}[\varsigma_y,\infty)$, $\varsigma_y \geq \varsigma_0$, is called a solution of Equation (1), if $r\left(y^{(n-1)}\right)^{\gamma} \in C^1([\varsigma_y,\infty))$, and y satisfies (1) on $[\varsigma_y,\infty)$.*

If a solution of Equation (1) has arbitrarily large zeros on $[\varsigma_y, \infty)$, then it is called oscillatory, and otherwise is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the following, we present some background details that motivated our research.

Theorem 1. *(See [17]) If there exists function $\rho \in C^1([\varsigma_0, \infty), (0, \infty))$ and $M > 1$, $\theta \in (0,1)$, such that*

$$\limsup_{\varsigma \to \infty} \frac{1}{H(\varsigma, \varsigma_1)} \int_{\varsigma_1}^{\varsigma} \left(H(\varsigma, s) \rho(s) q(s) - \left(\frac{h^{\gamma+1}(\varsigma, s)}{p} \right)^p \frac{\rho(s) r(s)}{(H(\varsigma, s) G(s))^{p-1}} \right) \mathrm{d}s = \infty,$$

where $G(s) = \theta M g^{n-2}(s) g'(s)$, then the equation

$$L_y' + p(\varsigma) \left| \left(y^{(n-1)}(\varsigma) \right) \right|^{p-2} y^{(n-1)}(\varsigma) + q(\varsigma) \left| (y(g(\varsigma))) \right|^{p-2} y(g(\varsigma)) = 0, \tag{3}$$

where $L_y = r(\varsigma) \left| \left(y^{(n-1)}(\varsigma) \right) \right|^{p-2} y^{(n-1)}(\varsigma)$, p is a real number satisfying $p > 1$.

As a special case of Equation (1), when $p(\varsigma) = 0$. Zafer [26] and Zhang and Yan [27] studied the equation

$$u^{(n)}(\varsigma) + q(\varsigma) y(\delta(\varsigma)) = 0 \tag{4}$$

and established some new sufficient conditions for oscillation.

Theorem 2. *(See [26]) Let $n \geq 2$ such that*

$$\limsup_{\varsigma \to \infty} \int_{\delta(\varsigma)}^{\varsigma} \pi(s) \mathrm{d}s > (n-1) 2^{(n-1)(n-2)}, \quad \delta'(\varsigma) \geq 0$$

or

$$\liminf_{\varsigma \to \infty} \int_{\delta(\varsigma)}^{\varsigma} \pi(s) \mathrm{d}s > \frac{(n-1) 2^{(n-1)(n-2)}}{\mathrm{e}} \tag{5}$$

where $\pi(\varsigma) := \delta^{n-1}(\varsigma)(1 - p(\delta(\varsigma))) q(\varsigma)$, then every solution of Equation (4) is oscillatory.

Theorem 3. *(See [27]) Let $0 \leq p(\varsigma) < p_0 < 1$ and $n \geq 2$ such that*

$$\liminf_{\varsigma \to \infty} \int_{\delta(\varsigma)}^{\varsigma} \pi(s) \mathrm{d}s > \frac{(n-1)!}{\mathrm{e}} \tag{6}$$

or

$$\limsup_{\varsigma \to \infty} \int_{\delta(\varsigma)}^{\varsigma} \pi(s) \mathrm{d}s > (n-1)!, \quad \delta(\varsigma) \geq 0$$

where $\pi(\varsigma) := \delta^{n-1}(\varsigma)(1 - p(\delta(\varsigma))) q(\varsigma)$, then every solution of Equation (4) is oscillatory.

To prove this, we apply the previous results to the equation

$$\left(y(\varsigma) + \frac{1}{2} y\left(\frac{1}{2} \varsigma \right) \right)^{(4)} + \frac{q_0}{\varsigma^4} y\left(\frac{9}{10} \varsigma \right) = 0, \ \varsigma \geq 1, \tag{7}$$

then we find that Equation (7) is oscillatory if

The condition	Equation (5)	Equation (6)
The criterion	$q_0 > 1839.2$	$q_0 > 59.5$

Hence, [27] improved the results in [26].

In this paper, using the theory of comparison with first- and second-order delay equations, new oscillatory criteria are established of Equation (1). We improve the results in [26,27]. An example is provided to illustrate the criteria.

Here, we define the next notation:

$$\begin{aligned} \eta_{\varsigma_0}(\varsigma) &:= \exp\left(\int_{\varsigma_0}^{\varsigma}\frac{p(u)}{r(u)}\mathrm{d}u\right), \\ \tilde{\eta}_0(\varsigma) &:= \left(\frac{1}{\eta_{\varsigma_1}(\varsigma)\,r(\varsigma)}\int_{\varsigma}^{\infty} q(s)\,\eta_{\varsigma_1}(s)\,c_2^{\gamma}(\delta(s))\,\mathrm{d}s\right)^{1/\gamma} \\ \tilde{\eta}_k(\varsigma) &:= \int_{\varsigma}^{\infty}\tilde{\eta}_{k-1}(s)\,\mathrm{d}s,\ \ k=1,2,...,n-2 \end{aligned}$$

and

$$c_m(\varsigma) := \frac{1}{c(g^{-1}(\varsigma))}\left(1-\frac{\left(g^{-1}(g^{-1}(\varsigma))\right)^{m-1}}{\left(g^{-1}(\varsigma)\right)^{m-1}c\left(g^{-1}(g^{-1}(\varsigma))\right)}\right),\ m=2,n.$$

We establish the oscillatory behavior of Equation (1) under the conditions

$$\delta(\varsigma) < g(\varsigma),\ \delta'(\varsigma)\ge 0 \text{ and } \left(g^{-1}(\varsigma)\right)' > 0 \tag{8}$$

and

$$\int_{\varsigma_0}^{\infty}\left(\frac{1}{r(s)}\exp\left(-\int_{\varsigma_0}^{s}\frac{p(u)}{r(u)}du\right)\right)^{1/\gamma} ds = \infty. \tag{9}$$

2. Some Auxiliary Lemmas

We employ the following lemmas:

Lemma 1. *[15] If the function y satisfies $y^{(i)}(\varsigma) > 0$, $i = 0,1,...,n$, and $y^{(n+1)}(\varsigma) < 0$, then*

$$\frac{y(\varsigma)}{\varsigma^n/n!} \ge \frac{y'(\varsigma)}{\varsigma^{n-1}/(n-1)!}.$$

Lemma 2. *([1] (Lemma 2.2.3)) Let $y \in C^n([\varsigma_0,\infty),(0,\infty))$. Assume that $y^{(n)}(\varsigma)$ is of fixed sign and not identically zero on $[\varsigma_0,\infty)$ and that there exists a $\varsigma_1 \ge \varsigma_0$, such that $y^{(n-1)}(\varsigma)\,y^{(n)}(\varsigma) \le 0$ for all $\varsigma \ge \varsigma_1$. If $\lim_{\varsigma\to\infty} y(\varsigma) \ne 0$, then for every $\mu \in (0,1)$, there exists $\varsigma_\mu \ge \varsigma_1$ such that*

$$y(\varsigma) \ge \frac{\mu}{(n-1)!}\varsigma^{n-1}\left|y^{(n-1)}(\varsigma)\right| \text{ for } \varsigma \ge \varsigma_\mu.$$

Lemma 3. *([21] (Lemma 1.2)) Assume that Equation (9) holds and y is an eventually positive solution of Equation (1). Then, there exist two possible cases:*

$$\begin{aligned} (\mathbf{I}_1): &\quad u(\varsigma) > 0,\ u'(\varsigma) > 0,\ u''(\varsigma) > 0,\ u^{(n-1)}(\varsigma) > 0 \text{ and } u^{(n)}(\varsigma) < 0, \\ (\mathbf{I}_2): &\quad u(\varsigma) > 0,\ u^{(j)}(\varsigma) > 0,\ u^{(j+1)}(\varsigma) < 0 \text{ for all odd integer} \\ &\quad j \in \{1,2,...,n-3\},\ u^{(n-1)}(\varsigma) > 0 \text{ and } u^{(n)}(\varsigma) < 0, \end{aligned}$$

for $\varsigma \ge \varsigma_1$, where $\varsigma_1 \ge \varsigma_0$ is sufficiently large.

Lemma 4. *Assume that Equation (9) holds and y is an eventually positive solution of (1). Then*

$$\left(\eta_{\varsigma_0}(\varsigma)\,r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \eta_{\varsigma_0}(\varsigma)\,q(\varsigma)\,(1-c(\delta(\varsigma)))^{\gamma}\,u^{\gamma}(\delta(\varsigma)) \le 0, \text{ for } c_0 < 1 \tag{10}$$

and

$$\left(\eta_{\varsigma_0}(\varsigma)\, r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \frac{\eta_{\varsigma_0} q(\varsigma)}{c^{\gamma}\left(g^{-1}(\delta(\varsigma))\right)}\left(u\left(g^{-1}(\delta(\varsigma))\right) - \frac{u\left(g^{-1}\left(g^{-1}(\delta(\varsigma))\right)\right)}{c\left(g^{-1}\left(g^{-1}(\delta(\varsigma))\right)\right)}\right)^{\gamma} \leq 0, \tag{11}$$

for $\varsigma \geq \varsigma_1$, *where* $\varsigma_1 \geq \varsigma_0$ *is sufficiently large.*

Proof. Let y be an eventually positive solution of Equation (1). It is not difficult to see that

$$\begin{aligned}
\frac{1}{\eta_{\varsigma_0}(\varsigma)}\frac{\mathrm{d}}{\mathrm{d}\varsigma}&\left(\eta_{\varsigma_0}(\varsigma)\, r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)\\
&= \frac{1}{\eta_{\varsigma_0}(\varsigma)}\left(\eta_{\varsigma_0}(\varsigma)\left(r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \eta'_{\varsigma_0}(\varsigma)\, r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)\\
&= \left(r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \frac{\eta'_{\varsigma_0}(\varsigma)}{\eta_{\varsigma_0}(\varsigma)} r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\\
&= \left(r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + p(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}.
\end{aligned} \tag{12}$$

Considering Equation (2) and $u'(\varsigma) > 0$, we determine that $y(\varsigma) \geq (1 - c(\varsigma))\, u(\varsigma)$. Thus, from Equations (1) and (12), we have that Equation (10) holds.

From Equation (2), we obtain

$$\begin{aligned}
c\left(g^{-1}(\varsigma)\right) y(\varsigma) &= u\left(g^{-1}(\varsigma)\right) - y\left(g^{-1}(\varsigma)\right)\\
&= u\left(g^{-1}(\varsigma)\right) - \left(\frac{u\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)}{c\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)} - \frac{y\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)}{c\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)}\right)\\
&\geq u\left(g^{-1}(\varsigma)\right) - \frac{1}{c\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)} u\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right),
\end{aligned} \tag{13}$$

which, with Equations (1), (12), and (13), gives Equation (11). The proof is complete. □

3. Comparison Theorems with First-Order Equations

In this section, we compare the oscillatory behavior of Equation (1) with the first-order differential equations.

Theorem 4. *Assume that* $c_0 < 1$ *and Equation (9) hold. If the differential equation*

$$v'(\varsigma) + (1 - c(\delta(\varsigma)))^{\gamma} q(\varsigma)\frac{\eta_{\varsigma_0}(\varsigma)}{\eta_{\varsigma_0}(\delta(\varsigma))}\left(\frac{\mu\delta^{n-1}(\varsigma)}{(n-1)!\, r^{1/\gamma}(\delta(\varsigma))}\right)^{\gamma} v(\delta(\varsigma)) = 0 \tag{14}$$

is oscillatory, then every solution of Equation (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of Equation (1). Then, we suppose that $y(\varsigma)$, $y(g(\varsigma))$, and $y(\delta(\varsigma))$ are positive for all $\varsigma \geq \varsigma_1$ that are sufficiently large. From Lemma 4, we obtain that Equation (10) holds. Using Lemma 2, we obtain that

$$u(\varsigma) \geq \frac{\mu}{(n-1)!}\varsigma^{n-1} u^{(n-1)}(\varsigma), \tag{15}$$

for some $\mu \in (0,1)$. From Equations (1), (10), and (15), we see that

$$\left(\eta_{\varsigma_0}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \eta_{\varsigma_0}(\varsigma) q(\varsigma)(1-c(\delta(\varsigma)))^{\gamma}\left(\frac{\mu\delta^{n-1}(\varsigma)}{(n-1)!}\right)^{\gamma}\left(u^{(n-1)}(\delta(\varsigma))\right)^{\gamma} \le 0.$$

Then, if we set $v(\varsigma) = \eta_{\varsigma_0}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}$, then we have that $v > 0$ is a solution of the delay inequality:

$$v'(\varsigma) + (1-c(\delta(\varsigma)))^{\gamma} q(\varsigma)\frac{\eta_{\varsigma_0}(\varsigma)}{\eta_{\varsigma_0}(\delta(\varsigma))}\left(\frac{\mu\delta^{n-1}(\varsigma)}{(n-1)!r^{1/\gamma}(\delta(\varsigma))}\right)^{\gamma} v(\delta(\varsigma)) \le 0.$$

It is well known ([24] (Theorem 1)) that the corresponding Equation (14) also has a positive solution, which is a contradiction. The proof is complete. □

Theorem 5. *Assume that Equations (8) and (9) hold. If the differential equations*

$$w'(\varsigma) + q(\varsigma)\frac{\eta_{\varsigma_0}(\varsigma)}{\eta_{\varsigma_0}(g^{-1}(\delta(\varsigma)))}\left(\frac{\mu\left(g^{-1}(\delta(\varsigma))\right)^{n-1} c_n(\delta(\varsigma))}{(n-1)!r^{1/\gamma}(g^{-1}(\delta(\varsigma)))}\right)^{\gamma} w\left(g^{-1}(\delta(\varsigma))\right) = 0 \tag{16}$$

and

$$\omega'(\varsigma) + g^{-1}(\delta(\varsigma))\,\widetilde{\eta}_{n-3}(\varsigma)\,\omega\left(g^{-1}(\delta(\varsigma))\right) = 0 \tag{17}$$

are oscillatory, then every solution of Equation (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of (1). Then, we suppose that $y(\varsigma), y(g(\varsigma))$, and $y(\delta(\varsigma))$ are positive for all $\varsigma \ge \varsigma_1$ that are sufficiently large. From Lemma 3, we have two possible cases $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$.

In the case where $(\mathbf{I}_1)$ holds, from Lemma 1, we obtain $u(\varsigma) \ge \frac{1}{(n-1)}\varsigma u'(\varsigma)$ and then $\left(\varsigma^{1-n}u(\varsigma)\right)' \le 0$. Thus, we obtain

$$u\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right) \le \frac{\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right)^{n-1}}{\left(g^{-1}(\varsigma)\right)^{n-1}} u\left(g^{-1}(\varsigma)\right). \tag{18}$$

Using Lemma 4, we have that Equation (11), given by Equation (18):

$$\left(\eta_{\varsigma_1}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \eta_{\varsigma_1}(\varsigma) q(\varsigma) c_n^{\gamma}(\delta(\varsigma)) u^{\gamma}\left(g^{-1}(\delta(\varsigma))\right) \le 0. \tag{19}$$

From Lemma 2, we obtain Equation (15). Therefore, from Equation (19), we obtain:

$$\left(\eta_{\varsigma_1}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + \eta_{\varsigma_1}(\varsigma) q(\varsigma)\left(\frac{\mu c_n(\delta(\varsigma))}{(n-1)!}\left(g^{-1}(\delta(\varsigma))\right)^{n-1}\right)^{\gamma}\left(u^{(n-1)}\left(g^{-1}(\delta(\varsigma))\right)\right)^{\gamma} \le 0. \tag{20}$$

Then, if we set $w(\varsigma) = \eta_{\varsigma_0}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}$, then we have that $w > 0$ is a solution of the delay inequality:

$$w'(\varsigma) + q(\varsigma)\frac{\eta_{\varsigma_1}(\varsigma)}{\eta_{\varsigma_1}(g^{-1}(\delta(\varsigma)))}\left(\frac{\mu\left(g^{-1}(\delta(\varsigma))\right)^{n-1} c_n(\delta(\varsigma))}{(n-1)!r^{1/\gamma}(g^{-1}(\delta(\varsigma)))}\right)^{\gamma} w\left(g^{-1}(\delta(\varsigma))\right) \le 0.$$

It is well known ([24] (Theorem 1)) that the corresponding Equation (16) also has a positive solution, which is a contradiction.

In the case where $(\mathbf{I}_2)$ holds, from Lemma 1, we obtain:

$$u(\varsigma) \geq \varsigma u'(\varsigma) \tag{21}$$

and then $\left(\varsigma^{-1} u(\varsigma)\right)' \leq 0$. Hence, since $g^{-1}(\varsigma) \leq g^{-1}\left(g^{-1}(\varsigma)\right)$, we get:

$$u\left(g^{-1}\left(g^{-1}(\varsigma)\right)\right) \leq \frac{g^{-1}\left(g^{-1}(\varsigma)\right)}{g^{-1}(\varsigma)} u\left(g^{-1}(\varsigma)\right), \tag{22}$$

which, with Equation (11), yields:

$$\left(\eta_{\varsigma_1}(\varsigma) r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}\right)' + q(\varsigma) \eta_{\varsigma_1}(\varsigma) c_2^{\gamma}(\delta(\varsigma)) u^{\gamma}\left(g^{-1}(\delta(\varsigma))\right) \leq 0. \tag{23}$$

Integrating Equation (23) from ς to ∞, we obtain:

$$\begin{aligned} -u^{(n-1)}(\varsigma) &\leq -\left(\frac{1}{\eta_{\varsigma_1}(\varsigma) r(\varsigma)} \int_{\varsigma}^{\infty} q(s) \eta_{\varsigma_1}(s) c_2^{\gamma}(\delta(s)) u^{\gamma}\left(g^{-1}(\delta(s))\right) \mathrm{d}s\right)^{1/\gamma} \\ &\leq -\widetilde{\eta}_0(\varsigma) u\left(g^{-1}(\delta(\varsigma))\right). \end{aligned}$$

Integrating this inequality $n-3$ times from ς to ∞, we obtain:

$$u''(\varsigma) + \widetilde{\eta}_{n-3}(\varsigma) u\left(g^{-1}(\delta(\varsigma))\right) \leq 0, \tag{24}$$

which, with Equation (21), gives:

$$u''(\varsigma) + g^{-1}(\delta(\varsigma)) \widetilde{\eta}_{n-3}(\varsigma) u'\left(g^{-1}(\delta(\varsigma))\right) \leq 0.$$

Thus, if we set $\omega(\varsigma) := u'(\varsigma)$, then we conclude that $\omega > 0$ is a solution of:

$$\omega'(\varsigma) + g^{-1}(\delta(\varsigma)) \widetilde{\eta}_{n-3}(\varsigma) \omega\left(g^{-1}(\delta(\varsigma))\right) \leq 0. \tag{25}$$

It is well known ([24] (Theorem 1)) that the corresponding Equation (17) also has a positive solution, which is a contradiction. The proof is complete. $\square$

4. Comparison Theorems with Second-Order Equations

In this section, we compare the oscillatory behavior of Equation (1) with the second-order differential equations.

It is well known [2] that the differential equation

$$\left[a(\varsigma)\left(y'(\varsigma)\right)^{\gamma}\right]' + q(\varsigma) y^{\gamma}(g(\varsigma)) = 0, \quad \varsigma \geq \varsigma_0, \tag{26}$$

where $\gamma > 0$ is the ratio of odd positive integers, $a, q \in C[\varsigma_0, \infty)$, is nonoscillatory if and only if there exists a number $\varsigma \geq \varsigma_0$, and a function $v \in C^1[\varsigma, \infty)$, satisfying the inequality:

$$v'(\varsigma) + \gamma a^{\frac{-1}{\gamma}}(\varsigma)(v(\varsigma))^{(1+\gamma)/\gamma} + q(\varsigma) \leq 0, \quad \text{on } [\varsigma, \infty).$$

In what follows, we compare the oscillatory behavior of Equation (1) with the second-order half-linear equations of the type in Equation (26).

Theorem 6. *Assume that Equations (8) and (9) hold. If the differential equations*

$$\left(\frac{((n-2)!)^{\gamma}\gamma\eta_{\varsigma_1}(\delta(\varsigma))r(\delta(\varsigma))}{(\mu_1\delta'(\varsigma)\delta^{n-2}(\varsigma))^{\gamma}}(y'(\varsigma))^{\gamma}\right)' + \eta_{\varsigma_1}(\varsigma)c_n^{\gamma}(\delta(\varsigma))q(\varsigma)y^{\gamma}(\varsigma) = 0 \tag{27}$$

and

$$\frac{1}{\delta'(\varsigma)}y''(\varsigma) + \widetilde{\eta}_{n-3}(\varsigma)y(\varsigma) = 0 \tag{28}$$

are oscillatory for some constant $\mu_1 \in (0,1)$, then every solution of Equation (1) is oscillatory.

Proof. Assume the contrary that y is a positive solution of Equation (1). Then, we can suppose that $y(\varsigma)$, $y(g(\varsigma))$ and $y(\delta(\varsigma))$ are positive for all $\varsigma \geq \varsigma_1$ that are sufficiently large. From Lemma 3, we have two possible cases: $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$.

In the case where $(\mathbf{I}_1)$ holds, as in the proof of Theorem 5, we arrive at Equation (19). Now, we define a function ϕ by

$$\phi(\varsigma) = \eta_{\varsigma_1}(\varsigma)r(\varsigma)\frac{\left(u^{(n-1)}(\varsigma)\right)^{\gamma}}{u^{\gamma}(\delta(\varsigma))}.$$

Then, $\phi(\varsigma) > 0$, for all $\varsigma \geq \varsigma_1$. Differentiating ϕ and using Equation (19), we get:

$$\phi'(\varsigma) \leq -\eta_{\varsigma_1}(\varsigma)q(\varsigma)c_n^{\gamma}(\delta(\varsigma))\frac{u^{\gamma}(g^{-1}(\delta(\varsigma)))}{u^{\gamma}(\delta(\varsigma))} - \frac{\eta_{\varsigma_1}(\varsigma)r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}}{u^{2\gamma}(\delta(\varsigma))}\gamma u^{\gamma-1}(\delta(\varsigma))u'(\delta(\varsigma))\delta'(\varsigma). \tag{29}$$

From Lemma 2, we have:

$$u'(\delta(\varsigma)) \geq \frac{\mu}{(n-2)!}\delta^{n-2}(\varsigma)u^{(n-1)}(\delta(\varsigma)). \tag{30}$$

Since $\eta_{\varsigma_1}(\varsigma)r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma}$ is decreasing, we have:

$$\eta_{\varsigma_1}(\varsigma)r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma} \leq \eta_{\varsigma_1}(\delta(\varsigma))r(\delta(\varsigma))\left(u^{(n-1)}(\delta(\varsigma))\right)^{\gamma}, \text{ for all } \varsigma \geq \delta(\varsigma), \tag{31}$$

that is,

$$\frac{1}{\eta_{\varsigma_1}^{1/\gamma}(\delta(\varsigma))r^{1/\gamma}(\delta(\varsigma))}(\eta_{\varsigma_1}(\varsigma)r(\varsigma))^{1/\gamma}u^{(n-1)}(\varsigma) \leq u^{(n-1)}(\delta(\varsigma)), \tag{32}$$

from Equations (30) and (32), we have:

$$u'(\delta(\varsigma)) \geq \frac{\mu}{(n-2)!}\frac{\delta^{n-2}(\varsigma)}{\eta_{\varsigma_1}^{1/\gamma}(\delta(\varsigma))r^{1/\gamma}(\delta(\varsigma))}(\eta_{\varsigma_1}(\varsigma)r(\varsigma))^{1/\gamma}u^{(n-1)}(\varsigma). \tag{33}$$

Since $g^{-1}(\varsigma) > \varsigma$ and $u'(\varsigma) > 0$, we have $u\left(g^{-1}(\varsigma)\right) > u(\varsigma)$ and so

$$\frac{u\left(g^{-1}(\delta(\varsigma))\right)}{u(\delta(\varsigma))} > 1. \tag{34}$$

By using Equations (34) and (33) in Equation (29), we have:

$$\phi'(\varsigma) \leq -\eta_{\varsigma_1}(\varsigma)q(\varsigma)c_n^{\gamma}(\delta(\varsigma)) - \frac{\eta_{\varsigma_1}(\varsigma)r(\varsigma)\left(u^{(n-1)}(\varsigma)\right)^{\gamma+1}}{u^{\gamma+1}(\delta(\varsigma))}\gamma\frac{\mu\delta'(\varsigma)\delta^{n-2}(\varsigma)}{(n-2)!}\left(\frac{\eta_{\varsigma_1}(\varsigma)r(\varsigma)}{\eta_{\varsigma_1}(\delta(\varsigma))r(\delta(\varsigma))}\right)^{1/\gamma}, \tag{35}$$

From the definition of ϕ, we have:

$$\phi'(\varsigma) \leq -\eta_{\varsigma_1}(\varsigma)q(\varsigma)c_n^{\gamma}(\delta(\varsigma)) - \frac{\gamma\mu\delta'(\varsigma)\delta^{n-2}(\varsigma)}{(n-2)!(\eta_{\varsigma_1}(\delta(\varsigma))r(\delta(\varsigma)))^{1/\gamma}}\phi^{(\gamma+1)/\gamma}(\varsigma),$$

that is,

$$\phi'(\varsigma) + \frac{\gamma\mu\delta'(\varsigma)\,\delta^{n-2}(\varsigma)}{(n-2)!\,(\eta_{\varsigma_1}(\delta(\varsigma))\,r(\delta(\varsigma)))^{1/\gamma}}\phi^{(\gamma+1)/\gamma}(\varsigma) + \eta_{\varsigma_1}(\varsigma)\,c_n^{\gamma}(\delta(\varsigma))\,q(\varsigma) \le 0. \tag{36}$$

Thus, we conclude that Equation (36) is nonoscillatory for every constant $\mu \in (0,1)$. From [2], we see that Equation (27) is nonoscillatory for every constant $\mu_1 \in (0,1)$, which is a contradiction.

In the case where $(\mathbf{I}_2)$ holds, as in the proof of Theorem 5, we arrive at Equation (24). Now, we define a function φ by:

$$\varphi(\varsigma) = \frac{u'(\varsigma)}{u(\delta(\varsigma))}.$$

Then $\phi(\varsigma) > 0$, for all $\varsigma \ge \varsigma_1$. Differentiating ϕ, we obtain:

$$\varphi'(\varsigma) = \frac{u''(\varsigma)}{u(\delta(\varsigma))} - \frac{u'(\varsigma)}{u^2(\delta(\varsigma))}u'(\delta(\varsigma))\,\delta'(\varsigma),$$

since $u''(\varsigma) < 0$, we have $u'(\delta(\varsigma)) > u'(\varsigma)$ for all $\varsigma \ge \delta(\varsigma)$. Thus

$$\varphi'(\varsigma) \le \frac{u''(\varsigma)}{u(\delta(\varsigma))} - \left(\frac{u'(\varsigma)}{u(\delta(\varsigma))}\right)^2 \delta'(\varsigma). \tag{37}$$

From Equation (24), we obtain:

$$\varphi'(\varsigma) \le -\frac{\widetilde{\eta}_{n-3}(\varsigma)\,u\left(g^{-1}(\delta(\varsigma))\right)}{u(\delta(\varsigma))} - \left(\frac{u'(\varsigma)}{u(\delta(\varsigma))}\right)^2 \delta'(\varsigma).$$

Since $g^{-1}(\varsigma) > \varsigma$ and $u'(\varsigma) > 0$, we have $u\left(g^{-1}(\varsigma)\right) > u(\varsigma)$, and so:

$$\varphi'(\varsigma) \le -\widetilde{\eta}_{n-3}(\varsigma) - \left(\frac{u'(\varsigma)}{u(\delta(\varsigma))}\right)^2 \delta'(\varsigma), \tag{38}$$

From the definition of φ, we have

$$\varphi'(\varsigma) \le -\widetilde{\eta}_{n-3}(\varsigma) - \delta'(\varsigma)\,\varphi^2(\varsigma),$$

that is,

$$\varphi'(\varsigma) + \delta'(\varsigma)\,\varphi^2(\varsigma) + \widetilde{\eta}_{n-3}(\varsigma) \le 0. \tag{39}$$

Thus, we conclude that Equation (39) is nonoscillatory. From [2] we see that Equation (28) is nonoscillatory, which is a contradiction. Thus, the proof is complete. □

Corollary 1. *Assume that $c_0 < 1$ and Equation (9) hold. If*

$$\liminf_{\varsigma\to\infty} \int_{\delta(\varsigma)}^{\varsigma} (1 - c(\delta(s)))^{\gamma}\, q(s)\, \frac{\eta_{\varsigma_0}(s)}{\eta_{\varsigma_0}(\delta(s))} \left(\frac{\mu\delta^{n-1}(s)}{r^{1/\gamma}(\delta(s))}\right)^{\gamma} \mathrm{d}s > \frac{((n-1)!)^{\gamma}}{\mathrm{e}} \tag{40}$$

is oscillatory, then every solution of Equation (1) is oscillatory.

Corollary 2. *Assume that Equations (8) and (9) hold. If*

$$\liminf_{\varsigma\to\infty} \int_{g^{-1}(\delta(\varsigma))}^{\varsigma} q(s)\, \frac{\eta_{\varsigma_0}(s)}{\eta_{\varsigma_0}\left(g^{-1}(\delta(s))\right)} \left(\frac{\mu\left(g^{-1}(\delta(s))\right)^{n-1} c_n(\delta(s))}{r^{1/\gamma}\left(g^{-1}(\delta(s))\right)}\right)^{\gamma} \mathrm{d}s > \frac{((n-1)!)^{\gamma}}{\mathrm{e}} \tag{41}$$

and

$$\liminf_{\varsigma\to\infty}\int_{g^{-1}(\delta(\varsigma))}^{\varsigma} g^{-1}\left(\delta\left(s\right)\right)\widetilde{\eta}_{n-3}\left(s\right)\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{42}$$

are oscillatory, then every solution of Equation (1) is oscillatory.

It is well known [23] that if

$$\int_{\varsigma_0}^{\infty}\frac{1}{a\left(\varsigma\right)}\mathrm{d}\varsigma=\infty,\text{ and }\liminf_{\varsigma\to\infty}\left(\int_{\varsigma_0}^{\varsigma}\frac{1}{a\left(s\right)}\mathrm{d}s\right)\int_{\varsigma}^{\infty}q\left(s\right)\mathrm{d}s>\frac{1}{4},$$

then Equation (26) with $\gamma=1$ is oscillatory.

Based on the above results and Corollary (3), we easily obtain the following Hille and Nehari type oscillation criteria for Equation (1) with $\gamma=1$.

Corollary 3. *Let $\gamma=1$. Assume that Equations (8) and (9) hold. If*

$$\int_{\varsigma_0}^{\infty}\frac{\mu_1\delta'\left(\varsigma\right)\delta^{n-2}\left(\varsigma\right)}{(n-2)!\eta_{\varsigma_1}\left(\delta\left(\varsigma\right)\right)r\left(\delta\left(\varsigma\right)\right)}\mathrm{d}\varsigma=\infty$$

and

$$\liminf_{\varsigma\to\infty}\left(\int_{\varsigma_0}^{\varsigma}\frac{\mu_1\delta'\left(s\right)\delta^{n-2}\left(s\right)}{(n-2)!\eta_{\varsigma_1}\left(\delta\left(s\right)\right)r\left(\delta\left(s\right)\right)}\mathrm{d}s\right)\int_{\varsigma}^{\infty}q\left(s\right)\eta_{\varsigma_1}\left(s\right)c_n^{\gamma}\left(\delta\left(s\right)\right)\mathrm{d}s>\frac{1}{4},$$

also, if

$$\int_{\varsigma_0}^{\infty}\delta'\left(\varsigma\right)\mathrm{d}\varsigma=\infty$$

and

$$\liminf_{\varsigma\to\infty}\left(\int_{\varsigma_0}^{\varsigma}\int_{\varsigma_0}^{\infty}\delta'\left(s\right)\mathrm{d}s\right)\int_{\varsigma}^{\infty}\widetilde{\eta}_{n-3}\left(s\right)\mathrm{d}s>\frac{1}{4},$$

are oscillatory for some constant $\mu_1\in(0,1)$, then every solution of Equation (1) is oscillatory.

Example 1. *For $\varsigma\geq 1$, consider the equation*

$$u^{(4)}\left(\varsigma\right)+\frac{1}{\varsigma}u^{(3)}\left(\varsigma\right)+\frac{q_0}{\varsigma^4}y\left(\frac{\varsigma}{2}\right)=0, \tag{43}$$

where $u\left(\varsigma\right)=y\left(\varsigma\right)+\frac{1}{2}y\left(\frac{\varsigma}{3}\right)$ and $q_0>0$ is a constant. Note that $\gamma=1$, $n=4$, $r\left(\varsigma\right)=1$, $p\left(\varsigma\right)=1/\varsigma$, $q\left(\varsigma\right)=q_0/\varsigma^4$, $\delta\left(\varsigma\right)=\varsigma/2$, $g^{-1}\left(\varsigma\right)=(3/2)\,\varsigma$ and $g\left(\varsigma\right)=\varsigma/3$.So, we obtain:

$$\eta_{\varsigma_0}\left(\varsigma\right)=\varsigma,\ \eta_{\varsigma_0}\left(\delta\left(\varsigma\right)\right)=\varsigma/2.$$

Thus, we find:

$$\begin{aligned}&\liminf_{\varsigma\to\infty}\int_{\delta(\varsigma)}^{\varsigma}\left(1-c\left(\delta\left(s\right)\right)\right)^{\gamma}q\left(s\right)\frac{\eta_{\varsigma_0}\left(s\right)}{\eta_{\varsigma_0}\left(\delta\left(s\right)\right)}\left(\frac{\mu\delta^{n-1}\left(s\right)}{r^{1/\gamma}\left(\delta\left(s\right)\right)}\right)^{\gamma}\mathrm{d}s\\ =\ &\liminf_{\varsigma\to\infty}\int_{\varsigma/2}^{\varsigma}\frac{q_0}{\varsigma^4}\left(\frac{\varsigma^3}{8}\right)\mathrm{d}s=\frac{q_0}{8}\ln 2.\end{aligned}$$

Hence, the condition becomes:

$$q_0>\frac{48}{\mathrm{e}\ln 2}. \tag{44}$$

Therefore, by Corollary 1, all solutions of Equation (43) are oscillatory if $q_0>25.5$.

Remark 1. *Consider Equation (7) by Corollary 1; all solutions of Equation (7) are oscillatory if* $q_0 > 57.5$*, whereas the criterion obtained from the results of [26,27] are* $q_0 > 1839.2$ *and* $q_0 > 59.5$*, respectively. Hence, our results improve the results in [26].*

Remark 2. *The results obtained in [26,27] are a special case of the results obtained in this study.*

Remark 3. *The results in this paper can be extended to the more general equation of the form*

$$\left(r\left(\varsigma\right)\left(u^{(n-1)}\left(\varsigma\right)\right)^{\gamma}\right)' + p\left(\varsigma\right)\left(u^{(n-1)}\left(\varsigma\right)\right)^{\gamma} + q\left(\varsigma\right)f\left(y\left(\delta\left(\varsigma\right)\right)\right) = 0,$$

where $f\left(y\right) \geq ky^{\beta} > 0$*. The statement and the formulation of the results are left to the interested reader.*

5. Conclusions

This paper is concerned with the oscillatory behavior of solutions of Equation (1). Using comparison with first- and second-order delay equations, a new asymptotic criterion for Equation (1) is presented. We obtained Hille and Nehari type oscillation criteria to ensure oscillation of the solutions of Equation (1). In future work, we obtain some Philos type oscillation criteria of Equation (1).

Author Contributions: The authors have contributed equally and significantly in this paper. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The authors thank the reviewers for for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: There are no competing interests between the authors.

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