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A Note on Parametric Kinds of the Degenerate Poly-Bernoulli and Poly-Genocchi Polynomials

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Abstract: Recently, the parametric kind of some well known polynomials have been presented by many authors. In a sequel of such type of works, in this paper, we introduce the two parametric kinds of degenerate poly-Bernoulli and poly-Genocchi polynomials. Some analytical properties of these parametric polynomials are also derived in a systematic manner. We will be able to find some identities of symmetry for those polynomials and numbers.

Keywords: degenerate poly-Bernoulli polynomials; degenerate poly-Genocchi polynomials; stirling numbers

1. Introduction

Special functions, polynomials and numbers play a prominent role in the study of many areas of mathematics, physics and engineering. In particular, the Appell polynomials and numbers are frequently used in the development of pure and applied mathematics related to functional equations in differential equations, approximation theories, interpolation problems, summation methods, quadrature rules and their multidimensional extensions (see [1]).The sequence of Appell polynomials $A_j(z)$ can be signified as follows:

$$\frac{d}{dz}A_j(z) = jA_{j-1}(z), \quad A_0(z) \neq 0, z = \eta + i\xi \in \mathbb{C}, \quad j \in \mathbb{N}, \tag{1}$$

or equivalently

$$A(z)e^{\eta z} = \sum_{j=0}^{\infty} A_j(\eta)\frac{z^j}{j!}, \tag{2}$$

where

$$A(z) = A_0 + A_1\frac{z}{1!} + A_2\frac{z^2}{2!} + \cdots + A_j\frac{z^j}{j!} + \cdots, \ A_0 \neq 0,$$

is a formal power series with coefficients A_j known as Appell numbers.

The well known degenerate exponential function is defined by (see [2])

$$e_\mu^\eta(z) = (1 + \mu z)^{\frac{\eta}{\mu}}, \qquad e_\mu(z) = e_\mu^1(z), (\mu \in \mathbb{R}). \tag{3}$$

In 1956 and 1979, Carlitz [3,4] introduced and investigated the following degenerate Bernoulli and Euler polynomials:

$$\frac{z}{e_\mu(z)-1}e_\mu^\eta(z)=\frac{z}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}}=\sum_{s=0}^{\infty}\beta_s(\eta;\mu)\frac{z^s}{s!}, \tag{4}$$

and

$$\frac{2}{e_\mu(z)+1}e_\mu^\eta(z)=\frac{2}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}}=\sum_{s=0}^{\infty}\mathfrak{E}_s(\eta;\mu)\frac{z^s}{s!}. \tag{5}$$

Note that

$$\lim_{\mu\longrightarrow 0}\beta_s(\eta;\mu)=B_s(\eta),\quad \lim_{\mu\longrightarrow 0}\mathfrak{E}_s(\eta;\mu)=E_s(\eta),$$

where $B_s(\eta)$ and $E_s(\eta)$ are the classical Bernoulli and Euler polynomials (see [5,6]).

Lim [7] introduced the degenerate Genocchi polynomials $G_j^{(p)}(\eta;\mu)$ of order p by means of the undermentioned generating function:

$$\left(\frac{2z}{e_\mu(z)+1}\right)^p e_\mu^\eta(z)=\left(\frac{2z}{(1+\mu z)^{\frac{1}{\mu}}-1}\right)^p(1+\mu z)^{\frac{\eta}{\mu}}=\sum_{j=0}^{\infty}G_j{}^{(p)}(\eta;\mu)\frac{z^j}{j!}, \tag{6}$$

so that

$$G_j^{(p)}(\eta;\mu)=\sum_{s=0}^{j}\binom{j}{s}G_s^{(p)}(\mu)\left(\frac{\eta}{\mu}\right)_{j-s}. \tag{7}$$

From Equation (6), we note that

$$\lim_{\mu\longrightarrow 0}\sum_{s=0}^{\infty}G_j^{(p)}(\eta;\mu)\frac{z^j}{j!}=\lim_{\mu\longrightarrow 0}\left(\frac{2z}{(1+\mu z)^{\frac{1}{\mu}}-1}\right)^p(1+\mu z)^{\frac{\eta}{\mu}}$$

$$=\left(\frac{2z}{e^z+1}\right)^p e^{\eta z}=\sum_{j=0}^{\infty}G_j^{(p)}(\eta)\frac{z^j}{j!},$$

where $G_j^{(p)}(\eta)$ are the generalized Genocchi polynomials of order p (see [8–11]).

The degenerate poly-Bernoulli and poly-Genocchi polynomials are defined by (see [12–14])

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^\eta(z)=\frac{\mathrm{Li}_k(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}}=\sum_{s=0}^{\infty}B_s^{(k)}(\eta;\mu)\frac{z^s}{s!},(k\in\mathbb{Z}), \tag{8}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^\eta(z)=\frac{2\mathrm{Li}_k(1-e^{-z})}{(1+\mu z)^{\frac{1}{\lambda}}+1}(1+\mu z)^{\frac{\eta}{\mu}}=\sum_{s=0}^{\infty}G_s^{(k)}(\eta;\mu)\frac{z^s}{s!},(k\in\mathbb{Z}). \tag{9}$$

Here, we note that (see [5,15]).

$$\lim_{\mu\longrightarrow 0}B_s^{(k)}(\eta;\mu)=B_s^{(k)}(\eta),\quad \lim_{\mu\longrightarrow 0}G_s^{(k)}(\eta;\mu)=G_s^{(k)}(\eta),$$

The Stirling numbers of the first kind are given by (see, [16–18])

$$(a)_s = a(a-1)\cdots(a-s+1) = \sum_{k=0}^{s} S^{(1)}(s,k)a^k, (k \geq 0), \tag{10}$$

and the Stirling numbers of the second kind are defined by (see [19,20])

$$a^s = \sum_{k=0}^{s} S^{(2)}(k,s)(a)_k. \tag{11}$$

The degenerate Stirling numbers of the of the second kind are defined by (see [10,21,22])

$$\frac{1}{k!}(e_\mu(t)-1)^k = \sum_{k=s}^{\infty} S_\mu^{(2)}(k,s)\frac{t^k}{k!}, (k \geq 0). \tag{12}$$

Note that $\lim_{\mu \longrightarrow 0} S_\mu^{(2)}(k,s) = S^{(2)}(k,s), (s,k \geq 0)$.

In the year (2017, 2018), Jamei et al. [23,24] introduced the two parametric kinds of exponential functions as follows (see also [6,23–25]):

$$e^{\eta z}\cos \xi z = \sum_{k=0}^{\infty} C_k(\eta,\xi)\frac{z^k}{k!}, \tag{13}$$

and

$$e^{\eta z}\sin \xi z = \sum_{k=0}^{\infty} S_k(\eta,\xi)\frac{z^k}{k!}, \tag{14}$$

where

$$C_k(\eta,\xi) = \sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} (-1)^j \eta^{k-2j}\xi^{2j}, \tag{15}$$

and

$$S_k(\eta,\xi) = \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k}{2j+1} (-1)^j \eta^{k-2j-1}\xi^{2j+1}. \tag{16}$$

Recently, Kim et al. [2] introduced the following degenerate type parametric exponential functions:

$$e_\mu^\eta(z)\cos_\mu^\xi(z) = \sum_{k=0}^{\infty} C_{k,\mu}(\eta,\xi)\frac{z^k}{k!}, \tag{17}$$

and

$$e_\mu^\eta(z)\sin_\mu^\xi(z) = \sum_{k=0}^{\infty} S_{k,\mu}(\eta,\xi)\frac{z^k}{k!}, \tag{18}$$

where

$$C_{r,\mu}(\eta,\xi) = \sum_{k=0}^{[\frac{r}{2}]}\sum_{q=2k}^{r} \binom{r}{q} (-1)^k \mu^{q-2k}\xi^{2k}S^1(q,2k)(\eta)_{r-q,\mu}, \tag{19}$$

and

$$S_{r,\mu}(\eta,\xi) = \sum_{k=0}^{[\frac{r-1}{2}]}\sum_{q=2k+1}^{r} \binom{r}{q} (-1)^k \mu^{q-2k-1}\xi^{2k+1}S^1(q,2k+1)(\eta)_{r-q,\mu}. \tag{20}$$

Motivated by the importance and potential applications in certain problems in number theory, combinatorics, classical and numerical analysis and physics, several families of degenerate Bernoulli and Euler polynomials and degenerate versions of special polynomials have been recently studied by many authors, (see [3–5,11–13,16]). Recently, Kim and Kim [2] have introduced the degenerate Bernoulli and degenerate Euler polynomials of a complex variable. By separating the real and imaginary parts, they introduced the parametric kinds of these degenerate polynomials.

The main object of this article is to present the parametric kinds of degenerate poly-Bernoulli and poly-Genocchi polynomials in terms of the degenerate type parametric exponential functions. We also investigate some fundamental properties of our introduced parametric polynomials.

2. Parametric Kinds of the Degenerate Poly-Bernoulli Polynomials

In this section, we define the two parametric kinds of degenerate poly-Bernoulli polynomials by means of the two special generating functions involving the degenerate exponential as well as trigonometric functions.

It is well known that (see [2])

$$e^{(\eta+i\xi)z}=e^{\eta z}e^{i\xi z}=e^{\eta z}(\cos\xi z+i\sin\xi z), \tag{21}$$

The degenerate trigonometric functions are defined by (see [19])

$$\cos_\mu z=\frac{e_\mu^{i}(z)+e_\mu^{-i}(z)}{2},\quad \sin_\mu z=\frac{e_\mu^{i}(z)-e_\mu^{-i}(z)}{2i}. \tag{22}$$

Note that, we have

$$\lim_{\mu\to0}\cos_\mu z=\cos z,\quad \lim_{\mu\to0}\sin_\mu z=\sin z.$$

In view of Equation (8), we have

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta+i\xi}(z)=\sum_{j=0}^{\infty}B_{j,\mu}^{(k)}(\eta+i\xi)\frac{z^j}{j!}, \tag{23}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta-i\xi}(z)=\sum_{j=0}^{\infty}B_{j,\mu}^{(k)}(\eta-i\xi)\frac{z^j}{j!}. \tag{24}$$

From Equations (23) and (24), we note that

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)=\sum_{j=0}^{\infty}\left(\frac{B_{j,\mu}^{(k)}(\eta+i\xi)+B_{j,\mu}^{(k)}(\eta-i\xi)}{2}\right)\frac{z^j}{j!}, \tag{25}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta}(z)\sin_\mu^{\xi}(z)=\sum_{j=0}^{\infty}\left(\frac{B_{j,\mu}^{(k)}(\eta+i\xi)-B_{j,\mu}^{(k)}(\eta-i\xi)}{2i}\right)\frac{z^j}{j!}. \tag{26}$$

Definition 1. *The degenerate cosine-poly-Bernoulli polynomials $B_{p,\mu}^{(k,c)}(\eta,\xi)$ and degenerate sine-poly-Bernoulli polynomials $B_{p,\mu}^{(k,s)}(\eta,\xi)$ for nonnegative integer p are defined, respectively, by*

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)=\sum_{p=0}^{\infty}B_{p,\mu}^{(k,c)}(\eta,\xi)\frac{z^p}{p!}, \tag{27}$$

and

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta}(z)\sin_\mu^{\xi}(z)=\sum_{p=0}^{\infty}B_{p,\mu}^{(k,s)}(\eta,\xi)\frac{z^p}{p!}. \tag{28}$$

For $\eta=\xi=0$ in Equations (27) and (28), we get

$$B^{(k,c)}_{p,\mu}(0,0) = B^{(k)}_{p,\mu}, B^{(k,s)}_{p,\mu}(0,0) = 0, (p \geq 0).$$

Note that $\lim_{\mu \longrightarrow 0} B^{(k,c)}_{p,\mu}(\eta,\xi) = B^{(k,c)}_{p}(\eta,\xi)$, $\lim_{\mu \longrightarrow 0} B^{(k,s)}_{p,\mu}(\eta,\xi) = B^{(k,s)}_{p}(\eta,\xi)$, $(p \geq 0)$, *where* $B^{(k,c)}_{p}(\eta,\xi)$ *and* $B^{(k,s)}_{p}(\eta,\xi)$ *are the new type of poly-Bernoulli polynomials.*

Based on Equations (25)–(28), we determine

$$B^{(k,c)}_{p,\mu}(\eta,\xi) = \frac{B^{(k)}_{p,\mu}(\eta+i\xi) + B^{(k)}_{p,\mu}(\eta-i\xi)}{2}, \tag{29}$$

and

$$B^{(k,s)}_{p,\mu}(\eta,\xi) = \frac{B^{(k)}_{p,\mu}(\eta+i\xi) - B^{(k)}_{p,\mu}(\eta-i\xi)}{2i}. \tag{30}$$

Theorem 1. *Let $k \in \mathbb{Z}$ and $j \geq 0$. Then*

$$\begin{aligned} B^{(k)}_{j,\mu}(\eta+i\xi) &= \sum_{q=0}^{j} \binom{j}{q} B^{(k)}_{j-q,\mu}(\eta)(i\xi)_{q,\mu} \\ &= \sum_{q=0}^{j} \binom{j}{q} B^{(k)}_{j-q,\mu}\ (\eta+i\xi)_{q,\mu}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} B^{(k)}_{j,\mu}(\eta-i\xi) &= \sum_{q=0}^{j} \binom{j}{q} B^{(k)}_{j-q,\mu}(\eta)(-1)^q(i\xi)_{q,\mu} \\ &= \sum_{q=0}^{j} \binom{j}{q} B^{(k)}_{j-q,\mu}(\eta-i\xi)_{q,\mu}. \end{aligned} \tag{32}$$

Proof. From Equation (23), we have

$$\begin{aligned} \sum_{j=0}^{\infty} B^{(k)}_{j,\mu}(\eta+i\xi)\frac{z^j}{j!} &= \frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1} e^{\eta}_{\mu}(z) e^{i\xi}_{\mu}(z) \\ &= \left(\sum_{j=0}^{\infty} B^{(k)}_{j,\mu}(\eta)\frac{z^j}{j!}\right)\left(\sum_{q=0}^{\infty}(i\xi)_{q,\mu}\frac{z^q}{q!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\binom{j}{q} B^{(k)}_{j-q,\mu}(\eta)(i\xi)_{q,\mu}\right)\frac{z^j}{j!}. \end{aligned} \tag{33}$$

Similarly, we find

$$\begin{aligned} \frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1} e^{\eta}_{\mu}(z) e^{i\xi}_{\mu}(z) &= \left(\sum_{j=0}^{\infty} B^{(k)}_{j,\mu}\frac{z^j}{j!}\right)\left(\sum_{q=0}^{\infty}(\eta+i\xi)_{q,\mu}\frac{z^q}{q!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\binom{j}{q} B^{(k)}_{j-q,\mu}(\eta+i\xi)_{q,\mu}\right)\frac{z^j}{j!}. \end{aligned} \tag{34}$$

In view of Equations (33) and (34), we obtain our first claimed result shown in Equation (31). Similarly, we can establish our second result shown in Equation (32). □

Theorem 2. *The following results hold true:*

$$B_{j,\mu}^{(k,c)}(\eta,\xi)=\sum_{r=0}^{j}\binom{j}{r}B_{r,\mu}^{(k)}\,C_{j-r,\mu}(\eta,\xi)$$

$$=\sum_{r=0}^{[\frac{q}{2}]}\sum_{q=2r}^{j}\binom{j}{q}\mu^{q-2r}(-1)^{r}\xi^{2r}S^{(1)}(q,2r)B_{j-q,\mu}^{(k)}(\eta), \tag{35}$$

and

$$B_{j,\mu}^{(k,s)}(\eta,\xi)=\sum_{r=0}^{j}\binom{j}{r}B_{r,\mu}^{(k)}S_{j-r,\mu}(\eta,\xi)$$

$$=\sum_{r=0}^{[\frac{q-1}{2}]}\sum_{q=2r+1}^{j}\binom{j}{q}\mu^{q-2r-1}(-1)^{r}\xi^{2r+1}S^{(1)}(q,2r+1)B_{j-q,\mu}^{(k)}(\eta). \tag{36}$$

Proof. From Equations (27) and (17), we see

$$\sum_{j=0}^{\infty}B_{j,\mu}^{(k,c)}(\eta,\xi)\frac{z^{j}}{j!}=\frac{\mathrm{Li}_{k}(1-e^{-z})}{e_{\mu}(z)-1}e_{\mu}^{\eta}(z)\cos_{\mu}^{\xi}(z)$$

$$=\left(\sum_{r=0}^{\infty}B_{r,\mu}^{(k)}\frac{z^{r}}{r!}\right)\left(\sum_{j=0}^{\infty}C_{j,\mu}(\eta,\xi)\frac{z^{j}}{j!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}B_{r,\mu}^{(k)}C_{j-r,\mu}(\eta,\xi)\right)\frac{z^{j}}{j!}. \tag{37}$$

Now, by using Equations (27) and (10), we find

$$\frac{\mathrm{Li}_{k}(1-e^{-z})}{e_{\mu}(z)-1}e_{\mu}^{\eta}(z)\cos_{\mu}^{\xi}(z)=\sum_{j=0}^{\infty}B_{j,\mu}^{(k)}(\eta)\frac{z^{j}}{j!}\sum_{p=0}^{\infty}\sum_{r=0}^{[\frac{q}{2}]}\mu^{l-2r}(-1)^{r}y^{2r}S^{(1)}(q,2r)\frac{z^{r}}{r!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\sum_{r=0}^{[\frac{q}{2}]}\binom{j}{q}\mu^{q-2r}(-1)^{r}\xi^{2r}S^{(1)}(q,2r)B_{j-r,\mu}^{(k)}(\eta)\right)\frac{z^{j}}{j!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{[\frac{q}{2}]}\sum_{q=2r}^{j}\binom{j}{q}\mu^{q-2r}(-1)^{r}\xi^{2r}S^{(1)}(q,2r)B_{j-q,\mu}^{(k)}(\eta)\right)\frac{z^{j}}{j!}. \tag{38}$$

Therefore, from Equations (37) and (38), we attain our needed result, Equation (35). Similarly, we can obtain Equation (36). □

Theorem 3. *Each of the following identities holds true:*

$$B_{r,\mu}^{(2,c)}(\eta,\xi)=\sum_{q=0}^{r}\binom{r}{q}\frac{q!B_{q}}{q+1}B_{r-q,\mu}^{(c)}(\eta,\xi), \tag{39}$$

and

$$B_{r,\mu}^{(2,s)}(\eta,\xi)=\sum_{q=0}^{r}\left(\begin{array}{c}r\\q\end{array}\right)\frac{q!B_q}{q+1}B_{r-q,\mu}^{(s)}(\eta,\xi). \tag{40}$$

Proof. In view of Equation (27), we have

$$\sum_{r=0}^{\infty}B_{r,\mu}^{(k,c)}(\eta,\xi)\frac{z^r}{r!}=\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)$$

$$=\frac{e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)}{e_\mu(z)-1}\int_0^z\underbrace{\frac{1}{e^u-1}\int_0^u\frac{1}{e^u-1}\cdots\frac{1}{e^u-1}\int_0^u\frac{u}{e^u-1}}_{(k-1)-\text{times}}du\cdots du. \tag{41}$$

Upon setting $k=2$, we obtain

$$\sum_{r=0}^{\infty}B_{r,\mu}^{(2,c)}(\eta,\xi)\frac{z^r}{r!}=\frac{e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)}{e_\mu(z)-1}\int_0^z\frac{u}{e^u-1}du$$

$$=\left(\sum_{q=0}^{\infty}\frac{B_qz^q}{(q+1)}\right)\frac{e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)}{e_\mu(z)-1}$$

$$=\left(\sum_{q=0}^{\infty}\frac{q!B_qz^q}{(q+1)q!}\right)\left(\sum_{r=0}^{\infty}B_{r,\mu}^{(c)}(\eta,\xi)\frac{z^r}{r!}\right).$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{r}\left(\begin{array}{c}r\\q\end{array}\right)\frac{q!B_q}{q+1}B_{r-q,\mu}^{(c)}(\eta,\xi)\frac{z^r}{r!},$$

which gives our required result, Equation (39). The proof of Equation (40) is similar; therefore, we omit the proof. □

Theorem 4. *Let $k\in\mathbb{Z}$, then*

$$B_{j,\mu}^{(k,c)}(\eta,\xi)=\sum_{r=0}^{j}\left(\begin{array}{c}j\\r\end{array}\right)\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}l!S_2(r+1,q)}{q^k(r+1)}\right)B_{j-r,\mu}^{(c)}(\eta,\xi), \tag{42}$$

and

$$B_{j,\mu}^{(k,s)}(\eta,\xi)=\sum_{r=0}^{j}\left(\begin{array}{c}j\\r\end{array}\right)\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}q!S_2(r+1,q)}{q^k(r+1)}\right)B_{j-r,\mu}^{(s)}(\eta,\xi). \tag{43}$$

Proof. From Equations (27) and (11), we see

$$\sum_{j=0}^{\infty}B_{j,\mu}^{(k,c)}(\eta,\xi)\frac{z^j}{j!}=\left(\frac{\mathrm{Li}_k(1-e^{-z})}{z}\right)\left(\frac{ze_\mu^{\eta}(z)\cos_\mu^{\xi}(z)}{e_\mu(z)-1}\right). \tag{44}$$

Now

$$\frac{1}{z}\mathrm{Li}_k(1-e^{-z})=\frac{1}{z}\sum_{q=1}^{\infty}\frac{(1-e^{-z})^q}{q^k}$$

$$=\frac{1}{z}\sum_{q=1}^{\infty}\frac{(-1)^q}{q^k}q!\sum_{r=l}^{\infty}(-1)^rS_2(r,q)\frac{z^r}{r!}$$

$$=\frac{1}{z}\sum_{r=q}^{\infty}\sum_{q=1}^{r}\frac{(-1)^{q+r}}{q^k}q!S_2(r,q)\frac{z^r}{r!}$$

$$= \sum_{r=0}^{\infty} \left(\sum_{q=1}^{q+1} \frac{(-1)^{q+r+1}}{q^k} l! \frac{S_2(r+1,q)}{r+1} \right) \frac{z^r}{r!}. \tag{45}$$

On using Equation (45) in (44), we find

$$\sum_{j=0}^{\infty} B_{j,\mu}^{(k,c)}(\eta,\xi) \frac{z^j}{j!} = \sum_{r=0}^{\infty} \left(\sum_{q=1}^{r+1} \frac{(-1)^{q+r+1}}{q^k} l! \frac{S_2(r+1,q)}{r+1} \right) \frac{z^r}{r!} \left(\sum_{j=0}^{\infty} B_{j,\mu}^{(c)}(\eta,\xi) \frac{z^j}{j!} \right).$$

Replacing j by $j-r$ in the right side of above expression and after equating the coefficients of z^j, we obtain our needed result, Equation (42). Similarly, we can derive our second result, Equation (43). □

Theorem 5. *The following recurrence relation holds true:*

$$B_{j,\mu}^{(k,c)}(\eta+1,\xi) - B_{j,\mu}^{(k,c)}(\eta,\xi)$$

$$= \sum_{r=1}^{j} \binom{j}{r} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r,q+1) \right) C_{j-r,\mu}(\eta,\xi), \tag{46}$$

and

$$B_{j,\mu}^{(k,s)}(\eta+1,\xi) - B_{j,\mu}^{(k,s)}(\eta,\xi)$$

$$= \sum_{r=1}^{j} \binom{j}{r} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r,q+1) \right) S_{j-r,\mu}(\eta,\xi). \tag{47}$$

Proof. In view of Equation (27), we have

$$\sum_{j=0}^{\infty} B_{j,\mu}^{(k,c)}(\eta+1,\xi) \frac{z^j}{j!} - \sum_{j=0}^{\infty} B_{j,\mu}^{(k,c)}(\eta,\xi) \frac{z^j}{j!}$$

$$= \frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1} e_\mu^{(\eta+1)}(z) \cos_\mu^{\xi}(z) - \frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1} e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= \mathrm{Li}_k(1-e^{-z}) e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= \sum_{q=0}^{\infty} \frac{(1-e^{-z})^{q+1}}{(q+1)^k} e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= \sum_{r=1}^{\infty} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r,q+1) \right) \frac{z^r}{r!} e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= \left(\sum_{r=1}^{\infty} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r,q+1) \right) \frac{z^r}{r!} \right) \left(\sum_{j=0}^{\infty} C_{j,\mu}(\eta,\xi) \frac{z^j}{j!} \right),$$

which upon replacing j by $j-r$ in the right side of above expression and after equating the coefficients of z^j, yields our first claimed result, Equation (46). Similarly, we can establish our second result, Equation (47). □

Theorem 6. *Let $k \in \mathbb{Z}$ and $j \geq 0$, then we have*

$$B_{j,\mu}^{(k,c)}(\eta+\gamma,\xi) = \sum_{r=0}^{j} \binom{j}{r} B_{j-r,\mu}^{(k,c)}(\eta,\xi)(\gamma)_{r,\mu}, \tag{48}$$

and

$$B^{(k,s)}_{j,\mu}(\eta+\gamma,\xi)=\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right) B^{(k,s)}_{j-r,\mu}(\eta,\xi)(\gamma)_{r,\mu}. \tag{49}$$

Proof. On using Equation (27), we find

$$\sum_{j=0}^{\infty} B^{(k,c)}_{j,\mu}(\eta+\gamma,\xi)\frac{z^j}{j!}=\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}e_\mu^{(\eta+\gamma)}(z)\cos^{\xi}_{\mu}(z)$$

$$=\left(\sum_{j=0}^{\infty} B^{(k,c)}_{j,\mu}(\eta,\xi)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}(\gamma)_{r,\mu}\frac{z^r}{r!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right) B^{(k,c)}_{j-r,\mu}(\eta,\xi)(\gamma)_{r,\mu}\right)\frac{z^j}{j!}.$$

By comparing the coefficients of z^j on both sides, we obtain the result, Equation (48). The proof of Equation (49) is similar to Equation (48).
□

Theorem 7. *If $k\in\mathbb{Z}$ and $j\geq 0$, then*

$$B^{(k,c)}_{j,\mu}(\eta,\xi)=\sum_{r=0}^{j}\sum_{q=0}^{r}\left(\begin{array}{c} j \\ r \end{array}\right)(\eta)_q\, S^{(2)}_{\mu}(r,q)B^{(k,c)}_{j-r,\mu}(0,\xi), \tag{50}$$

and

$$B^{(k,s)}_{j,\mu}(\eta,\xi)=\sum_{r=0}^{j}\sum_{q=0}^{r}\left(\begin{array}{c} j \\ r \end{array}\right)(\eta)_q\, S^{(2)}_{\mu}(r,q)B^{(k,s)}_{j-r,\mu}(0,\xi). \tag{51}$$

Proof. From Equations (27) and (12), we find

$$\sum_{j=0}^{\infty} B^{(k,c)}_{j,\mu}(\eta,\xi)\frac{z^j}{j!}=\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}(e_\mu(z)-1+1)^{\eta}\cos^{\xi}_{\mu}(z)$$

$$=\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}\sum_{q=0}^{\infty}\left(\begin{array}{c} \eta \\ q \end{array}\right)(e_\mu(z)-1)^{q}\cos^{\xi}_{\mu}(z)$$

$$=\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1}\cos^{\xi}_{\mu}(z)\sum_{q=0}^{\infty}(\eta)_q\sum_{r=q}^{\infty}S^{(2)}_{\mu}(r,q)\frac{z^r}{r!}$$

$$=\sum_{j=0}^{\infty} B^{(k,c)}_{j,\mu}(0,\xi)\frac{z^j}{j!}\sum_{r=0}^{\infty}\left(\sum_{q=0}^{r}(\eta)_q S^{(2)}_{\mu}(r,q)\right)\frac{z^r}{r!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{q=0}^{r}\left(\begin{array}{c} j \\ r \end{array}\right)(\eta)_q S^{(2)}_{\mu}(r,qB^{(k,c)}_{j-r,\mu}(0,\xi)\right)\frac{z^j}{j!}.$$

On comparing the coefficients of z^j on both sides, we obtain our required result, Equation (50). The proof of Equation (51) is similar to Equation (50).
□

3. Parametric Kinds of Degenerate Poly-Genocchi Polynomials

In this section, we introduce the two parametric kinds of degenerate poly-Genocchi polynomials by defining the two special generating functions involving the degenerate exponential as well as trigonometric functions.

In view of Equation (9), we have

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta+i\xi}(z)=\sum_{j=0}^{\infty}G_{j,\mu}^{(k)}(\eta+i\xi)\frac{z^j}{j!},\tag{52}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta-i\xi}(z)=\sum_{j=0}^{\infty}G_{j,\mu}^{(k)}(\eta-i\xi)\frac{z^j}{j!}.\tag{53}$$

From Equations (52) and (53), we can easily get

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)=\sum_{j=0}^{\infty}\left(\frac{G_{j,\mu}^{(k)}(\eta+i\xi)+G_{j,\mu}^{(k)}(\eta-i\xi)}{2}\right)\frac{z^j}{j!},\tag{54}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta}(z)\sin_\mu^{\xi}(z)=\sum_{j=0}^{\infty}\left(\frac{G_{j,\mu}^{(k)}(\eta+i\xi)-G_{j,\mu}^{(k)}(\eta-i\xi)}{2i}\right)\frac{z^j}{j!}.\tag{55}$$

Definition 2. *The degenerate cosine-poly-Genocchi polynomials $G_{j,\mu}^{(k,c)}(\eta,\xi)$ and degenerate sine-poly-Genocchi polynomials $G_{j,\mu}^{(k,s)}(\eta,\xi)$ for nonnegative integer j are defined, respectively, by*

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta}(z)\cos_\mu^{\xi}(z)=\sum_{j=0}^{\infty}G_{j,\mu}^{(k,c)}(\eta,\xi)\frac{z^j}{j!},\tag{56}$$

and

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}e_\mu^{\eta}(z)\sin_\mu^{\xi}(z)=\sum_{j=0}^{\infty}G_{j,\mu}^{(k,s)}(\eta,\xi)\frac{z^j}{j!}.\tag{57}$$

On setting $\eta=\xi=0$ in Equations (56) and (57), we get

$$G_{j,\mu}^{(k,c)}(0,0)=G_{j,\mu}^{(k)},\quad G_{j,\mu}^{(k,s)}(0,0)=0,(j\geq 0).$$

Note that $\lim_{\mu\longrightarrow 0}G_{j,\mu}^{(k,c)}(\eta,\xi)=G_j^{(k,c)}(\eta,\xi)$, $\lim_{\mu\longrightarrow 0}G_{j,\mu}^{(k,s)}(\eta,\xi)=G_j^{(k,s)}(\eta,\xi)$, $(j\geq 0)$, where $G_n^{(k,c)}(\eta,\xi)$ and $G_j^{(k,s)}(\eta,\xi)$ are the new type of poly-Genocchi polynomials.

From Equations (54)–(57), we determine

$$G_{j,\mu}^{(k,c)}(\eta,\xi)=\frac{G_{j,\mu}^{(k)}(\eta+i\xi)+G_{j,\mu}^{(k)}(\eta-i\xi)}{2}\tag{58}$$

and

$$G_{j,\mu}^{(k,s)}(\eta,\xi)=\frac{G_{j,\mu}^{(k)}(\eta+i\xi)-G_{j,\mu}^{(k)}(\eta-i\xi)}{2i}.\tag{59}$$

Theorem 8. *For $k\in\mathbb{Z}$ and $j\geq 0$, we have*

$$G_{j,\mu}^{(k)}(\eta+i\xi)=\sum_{q=0}^{j}\binom{j}{q}G_{j-q,\mu}^{(k)}(\eta)(i\xi)_{q,\mu}$$

$$= \sum_{q=0}^{j} \binom{j}{q} G_{j-q,\mu}^{(k)} (\eta + i\xi)_{q,\mu}, \tag{60}$$

and

$$G_{j,\mu}^{(k)}(\eta - i\xi) = \sum_{q=0}^{j} \binom{j}{q} G_{j-q,\mu}^{(k)}(\eta)(-1)^q (i\xi)_{q,\mu}$$

$$= \sum_{q=0}^{j} \binom{j}{q} G_{j-q,\mu}^{(k)}(\eta - i\xi)_{q,\mu}. \tag{61}$$

Proof. On using Equation (52), we see

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k)}(\eta + i\xi)\frac{z^j}{j!} = \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{\eta}(z) e_\mu^{i\xi}(z)$$

$$= \left(\sum_{j=0}^{\infty} G_{j,\mu}^{(k)}(\eta)\frac{z^j}{j!}\right)\left(\sum_{q=0}^{\infty} (i\xi)_{q,\mu}\frac{z^q}{q!}\right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{q=0}^{j} \binom{j}{q} G_{j-q,\mu}^{(k)}(\eta)(i\xi)_{q,\mu}\right)\frac{z^j}{j!}. \tag{62}$$

Similarly, we find

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{\eta}(z) e_\mu^{i\xi}(z) = \left(\sum_{j=0}^{\infty} G_{j,\mu}^{(k)}\frac{z^j}{j!}\right)\left(\sum_{q=0}^{\infty} (\eta + i\xi)_{q,\mu}\frac{z^q}{q!}\right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{q=0}^{j} \binom{j}{q} G_{j-q,\mu}^{(k)}(\eta + i\xi)_{q,\mu}\right)\frac{z^j}{j!}. \tag{63}$$

By comparing the coefficients of z^j on both sides in Equations (62) and (63), we obtain our desired result, Equation (60). The proof of Equation (61) is similar to Equation (60). □

Theorem 9. *If $k \in \mathbb{Z}$ and $j \geq 0$, then*

$$G_{j,\mu}^{(k,c)}(\eta, \xi) = \sum_{r=0}^{j} \binom{j}{r} G_{r,\mu}^{(k)} C_{j-r,\mu}(\eta, \xi)$$

$$= \sum_{r=0}^{[\frac{q}{2}]} \sum_{q=2r}^{j} \binom{j}{q} \mu^{q-2r}(-1)^r \xi^{2r} S^{(1)}(q, 2r) G_{j-q,\mu}^{(k)}(\xi), \tag{64}$$

and

$$G_{j,\mu}^{(k,s)}(\eta, \xi) = \sum_{r=0}^{j} \binom{j}{r} B_{r,\mu}^{(k)} S_{j-r,\mu}(\eta, \xi)$$

$$= \sum_{r=0}^{[\frac{q-1}{2}]} \sum_{q=2r+1}^{j} \binom{j}{q} \mu^{q-2r-1}(-1)^r \xi^{2r+1} S^{(1)}(q, 2r+1) G_{j-q,\mu}^{(k)}(\eta). \tag{65}$$

Proof. From Equations (56) and (10), we see

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta, \xi)\frac{z^j}{j!} = \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{\eta}(t) \cos_\mu^{\xi}(z)$$

$$= \left(\sum_{r=0}^{\infty} G_{r,\mu}^{(k)} \frac{z^r}{r!} \right) \left(\sum_{j=0}^{\infty} C_{j,\mu}(\eta,\xi) \frac{z^j}{j!} \right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \binom{j}{r} G_{r,\mu}^{(k)} C_{j-r,\mu}(\eta,\xi) \right) \frac{z^j}{j!}. \tag{66}$$

Similarly, we find

$$\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^\eta(z) \cos_\mu^\xi(z) = \sum_{j=0}^{\infty} G_{j,\mu}^{(k)}(\eta) \frac{z^j}{j!} \sum_{q=0}^{\infty} \sum_{r=0}^{[\frac{q}{2}]} \mu^{q-2r} (-1)^r \xi^{2r} S^{(1)}(q,2r) \frac{z^r}{r!}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} \sum_{m=0}^{[\frac{l}{2}]} \binom{j}{l} \mu^{l-2m} (-1)^m \xi^{2m} S^{(1)}(q,2r) G_{j-q,\mu}^{(k)}(\eta) \right) \frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{[\frac{q}{2}]} \sum_{q=2r}^{j} \binom{j}{q} \mu^{q-2r} (-1)^r \xi^{2r} S^{(1)}(q,2r) G_{j-q,\mu}^{(k)}(\eta) \right) \frac{z^j}{j!}. \tag{67}$$

By comparing the coefficients of z^j on both sides of Equations (66) and (67), we easily get our first claimed result, Equation (64). Similarly, we can establish our second needed result, Equation (65). □

Theorem 10. *Let $j \geq 0$. Then, we have*

$$G_{j,\mu}^{(2,c)}(\eta,\xi) = \sum_{r=0}^{j} \binom{j}{r} \frac{r!B_r}{r+1} G_{j-r,\mu}^{(c)}(\eta,\xi), \tag{68}$$

and

$$G_{j,\mu}^{(2,s)}(\eta,\xi) = \sum_{r=0}^{j} \binom{j}{r} \frac{r!B_r}{r+1} G_{j-r,\mu}^{(s)}(\eta,\xi). \tag{69}$$

Proof. By using Equation (56), we determine

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta,\xi) \frac{z^j}{j!} = \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^\eta(z) \cos_\mu^\xi(z)$$

$$= \frac{2e_\mu^\eta(z)\cos_\mu^\xi(z)}{e_\mu(z)+1} \int_0^z \underbrace{\frac{1}{e^u-1} \int_0^u \frac{1}{e^u-1} \cdots \frac{1}{e^u-1} \int_0^u \frac{u}{e^u-1}}_{(k-1)-\text{times}} du \cdots du. \tag{70}$$

On setting $k=2$ in Equation (70), we find

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(2,c)}(\eta,\xi) \frac{z^j}{j!} = \frac{2e_\mu^\eta(z)\cos_\mu^\xi(z)}{e_\mu(z)+1} \int_0^z \frac{u}{e^u-1} dz$$

$$= \left(\sum_{r=0}^{\infty} \frac{r!B_r z^r}{(r+1)r!} \right) \frac{2ze_\mu^\eta(z)\cos_\mu^\xi(z)}{e_\mu(z)+1}$$

$$= \left(\sum_{r=0}^{\infty} \frac{r!B_r z^r}{(r+1)r!} \right) \left(\sum_{j=0}^{\infty} G_{j,\mu}^{(c)}(\eta,\xi) \frac{z^j}{j!} \right).$$

On replacing j by $j-r$ in the above equation, we obtain

$$=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\frac{r!B_r}{r+1}G^{(c)}_{j-r,\mu}(\eta,\xi)\frac{z^j}{j!}.$$

Finally, by equating the coefficients of the like powers of z in the last expression, we get the result, Equation (68). The proof of Equation (69) is similar to Equation (68). □

Theorem 11. *For $k\in\mathbb{Z}$ and $j\geq 0$, we have*

$$G^{(k,c)}_{j,\mu}(\eta,\xi)=\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}q!S_2(r+1,q)}{q^k(r+1)}\right)G^{(c)}_{j-r,\mu}(\eta,\xi), \tag{71}$$

and

$$G^{(k,s)}_{j,\mu}(\eta,\xi)=\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}q!S_2(r+1,q)}{q^k(r+1)}\right)G^{(s)}_{j-r,\mu}(\eta,\xi). \tag{72}$$

Proof. In view of Equations (56) and (11), we see

$$\sum_{j=0}^{\infty}G^{(k,c)}_{j,\mu}(\eta,\xi)\frac{z^j}{j!}=\left(\frac{2\mathrm{Li}_k(1-e^{-z})}{z}\right)\left(\frac{ze^{\eta}_{\mu}(z)\cos^{\xi}_{\mu}(z)}{e_{\mu}(z)+1}\right). \tag{73}$$

Now

$$\begin{aligned}\frac{1}{z}\mathrm{Li}_k(1-e^{-z})&=\frac{1}{z}\sum_{q=1}^{\infty}\frac{(1-e^{-z})^q}{q^k}\\&=\frac{1}{z}\sum_{q=1}^{\infty}\frac{(-1)^q}{q^k}q!\sum_{r=l}^{\infty}(-1)^rS_2(r,q)\frac{z^r}{r!}\\&=\frac{1}{z}\sum_{r=1}^{\infty}\sum_{q=1}^{r}\frac{(-1)^{q+r}}{q^k}q!S_2(r,q)\frac{t^r}{r!}\\&=\sum_{r=0}^{\infty}\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}}{q^k}q!\frac{S_2(r+1,q)}{q+1}\right)\frac{z^r}{r!}.\end{aligned} \tag{74}$$

Using Equation (74) in (73), we find

$$\sum_{j=0}^{\infty}G^{(k,c)}_{j,\mu}(\eta,\xi)\frac{z^j}{j!}=\sum_{r=0}^{\infty}\left(\sum_{q=1}^{r+1}\frac{(-1)^{q+r+1}}{q^k}q!\frac{S_2(r+1,q)}{r+1}\right)\frac{z^r}{r!}\left(\sum_{j=0}^{\infty}G^{(c)}_{j,\mu}(\eta,\xi)\frac{z^j}{j!}\right),$$

which on comparing the coefficients of z^j on both sides, yields our desired result, Equation (71). Similarly, we can derive our second result, Equation (72). □

Theorem 12. *Let $k\in\mathbb{Z}$ and $j\geq 0$, then we have*

$$\begin{aligned}&\frac{1}{2}\left[G^{(k,c)}_{j,\mu}(\eta+1,\xi)+G^{(k,c)}_{j,\mu}(\eta,\xi)\right]\\&=\sum_{r=1}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\left(\sum_{q=0}^{r-1}\frac{(-1)^{q+r+1}}{(q+1)^k}(q+1)!S_2(r,q+1)\right)C_{j-r,\mu}(\eta,\xi),\end{aligned} \tag{75}$$

and

$$\frac{1}{2}\left[G^{(k,s)}_{j,\mu}(\eta+1,\xi)+G^{(k,s)}_{j,\mu}(\eta,\xi)\right]$$

$$= \sum_{r=1}^{j} \left(\begin{array}{c} j \\ r \end{array} \right) \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r, q+1) \right) S_{j-r,\mu}(\eta, \xi). \tag{76}$$

Proof. Taking

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta+1, \xi) \frac{z^j}{j!} + \sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta, \xi) \frac{z^j}{j!}$$

$$= \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{(\eta+1)}(z) \cos_\mu^{\xi}(z) + \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= 2\mathrm{Li}_k(1-e^{-z}) e_\mu^{(\eta)}(z) \cos_\mu^{\xi}(z)$$

$$= \sum_{q=0}^{\infty} \frac{(1-e^{-z})^{q+1}}{(q+1)^k} 2 e_\mu^{\eta}(z) \cos_\mu^{(\xi)}(z)$$

$$= \sum_{r=1}^{\infty} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r, q+1) \right) \frac{z^r}{r!} 2 e_\mu^{x}(z) \cos_\mu^{(\xi)}(z)$$

$$= 2 \left(\sum_{r=1}^{\infty} \left(\sum_{q=0}^{r-1} \frac{(-1)^{q+r+1}}{(q+1)^k} (q+1)! S_2(r, q+1) \right) \frac{z^r}{r!} \right) \left(\sum_{j=0}^{\infty} C_{j,\mu}(\eta, \xi) \frac{z^j}{j!} \right).$$

On replacing j by $j-r$ in the right side of the above equation, and after comparing the coefficients of z^j on both sides, we acquire the desired result, Equation (75). Similarly, we can obtain the result, Equation (76). □

Theorem 13. *For $k \in \mathbb{Z}$ and $j \geq 0$, we have*

$$G_{j,\mu}^{(k,c)}(\eta+\alpha, \xi) = \sum_{m=0}^{j} \left(\begin{array}{c} j \\ m \end{array} \right) G_{j-m,\mu}^{(k,c)}(\eta, \xi) (\alpha)_{m,\mu}, \tag{77}$$

and

$$G_{j,\mu}^{(k,s)}(\eta+\alpha, \xi) = \sum_{m=0}^{j} \left(\begin{array}{c} j \\ m \end{array} \right) G_{j-m,\mu}^{(k,s)}(\eta, \xi) (\alpha)_{m,\mu}. \tag{78}$$

Proof. By using Equation (56), we have

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta+\alpha, \xi) \frac{z^j}{j!} = \frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1} e_\mu^{(\eta+\alpha)}(z) \cos_\mu^{(\xi)}(z)$$

$$= \left(\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta, \xi) \frac{z^j}{j!} \right) \left(\sum_{m=0}^{\infty} (\alpha)_{m,\mu} \frac{z^m}{m!} \right)$$

$$= \sum_{j=0}^{\infty} \left(\sum_{m=0}^{j} \left(\begin{array}{c} j \\ m \end{array} \right) G_{j-m,\mu}^{(k,c)}(\eta, \xi) (\alpha)_{m,\mu} \right) \frac{z^j}{j!}.$$

By comparing the coefficients of z^j on both sides in the last expression, we acquire our desired result, Equation (77). Similarly, we can derive our second result, Equation (78).

□

Theorem 14. *If $k \in \mathbb{Z}$ and $j \geq 0$, then*

$$G_{j,\mu}^{(k,c)}(\eta, \xi) = \sum_{r=0}^{j} \sum_{q=0}^{r} \left(\begin{array}{c} j \\ r \end{array} \right) (\eta)_l S_\mu^{(2)}(r, q) G_{j-r,\mu}^{(k,c)}(0, \xi), \tag{79}$$

and

$$G_{j,\mu}^{(k,s)}(\eta,\xi)=\sum_{r=0}^{j}\sum_{q=0}^{r}\left(\begin{array}{c} j \\ r \end{array}\right)(\eta)_l S_{\mu}^{(2)}(r,q) G_{j-r,\mu}^{(k,s)}(0,\xi). \tag{80}$$

Proof. From Equations (56) and (12), we have

$$\sum_{j=0}^{\infty} G_{j,\mu}^{(k,c)}(\eta,\xi)\frac{z^j}{j!}=\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}(e_\mu(z)-1+1)^{\eta}\cos_{\mu}^{\xi}(z)$$

$$=\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}\sum_{q=0}^{\infty}\left(\begin{array}{c} \eta \\ q \end{array}\right)(e_\mu(z)-1)^{q}\cos_{\mu}^{\xi}(z)$$

$$=\frac{2\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)+1}\cos_{\mu}^{\xi}(z)\sum_{q=0}^{\infty}(\eta)_q\sum_{r=q}^{\infty}S_{\mu}^{(2)}(r,q)\frac{z^r}{r!}$$

$$=\sum_{j=0}^{\infty}G_{j,\mu}^{(k,c)}(0,\xi)\frac{z^j}{j!}\sum_{r=0}^{\infty}\left(\sum_{q=0}^{r}(\eta)_q\, S_{\mu}^{(2)}(r,q)\right)\frac{z^r}{r!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{q=0}^{r}\left(\begin{array}{c} j \\ r \end{array}\right)(\eta)_q S_{\mu}^{(2)}(r,q) G_{j-r,\mu}^{(k,c)}(0,\xi)\right)\frac{z^j}{j!}.$$

Finally, by comparing the coefficients of z^j on both sides in the last expression, we arrive at our claimed result, Equation (79). Similarly, we can establish our second result, Equation (80). □

4. Conclusions

In the present article, we have considered the parametric kinds of degenerate poly-Bernoulli and poly-Genocchi polynomials by making use of the degenerate type exponential as well as trigonometric functions. We have also derived some analytical properties of our newly introduced parametric polynomials by using the series manipulation technique. Furthermore, it is noticed that, if we consider any Appell polynomials of a complex variable (as discussed in the present article), then we can easily define its parametric kinds by separating the complex variable into real and imaginary parts.

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Abbreviations

The following abbreviations are used in this manuscript:

MKdV modified Korteweg–de Vries equation

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