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New Generalized Hermite-Hadamard Inequality and Related Integral Inequalities Involving Katugampola Type Fractional Integrals

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Abstract: In this paper, a new identity for the generalized fractional integral is defined. Using this identity we studied a new integral inequality for functions whose first derivatives in absolute value are convex. The new generalized Hermite-Hadamard inequality for generalized convex function on fractal sets involving Katugampola type fractional integral is established. This fractional integral generalizes Riemann-Liouville and Hadamard's integral, which possess a symmetric property. We derive trapezoid and mid-point type inequalities connected to this generalized Hermite-Hadamard inequality.

Keywords: convex function; generalized convex function; Hermite–Hadamard inequality; Katugampola fractional integral

1. Introduction

The emergence of convexity theory, in the field of mathematical analysis, has been considered as the remarkable development. Due to the wide applications of convexity, variety of new convex functions have being reported and widely studied in the literature. The definition of a classical convex function is given below.

Definition 1. *A function $\mathcal{G} : V \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\mathcal{G}(\vartheta m + (1 - \vartheta n)) \leq \vartheta \mathcal{G}(m) + (1 - \vartheta)\mathcal{G}(n),$$

holds for all $\mathrm{m}, \mathrm{n} \in \mathbb{R}$ *and* $\vartheta \in [0, 1]$.

This notion has inspired many to formulate new inequalities. Many new classes of inequalities that are related to the convex functions have been derived and applied to other field of studies, see [1,2]. Among the interesting classes of such inequalities are those of Hermite-Hadamard's type, which have been applied to many problems in finance, engineering and science. Similar to the convexity, convexity inequality, for a function $\mathcal{G} : V \subseteq \mathbb{R} \to \mathbb{R}$, the Hermite-Hadamard inequality can also be defined as

$$\mathcal{G}\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_m^n \mathcal{G}(x)dx \leq \frac{\mathcal{G}(m)+\mathcal{G}(n)}{2}. \tag{1}$$

In the literature, many generalizations of Hermite-Hadamard type inequalities are established by applying the generalizations of convexity. For example, very recently, a new type of integral inequality for regular convex function was studied by [3]. Furthermore, many researchers have been studying the generalization of inequality in (1) motivated by various modifications of the notion of convexity, such as s-convexity and generalized s-convexity, for example see the details in ([4–7]), where Hermite-Hadamard inequality were extended in order to include the problems that related to fractional calculus, a branch of calculus dealing with derivatives and integrals of non-integer order (see [8–13]). Nowadays, the real-life applications of fractional calculus exist in most areas of studies [14,15]. Based on the application of fractional calculus, the mathematicians defined its derivatives and integrals differently. Thus there are many type of fractional derivatives. One of the most widely used approaches is the Riemann-Liouville operator method. The detail of this method can be found in the following references [16,17]. The work of Sarikaya et al. [18] on the formulation of Hermite-Hadamard inequality, via Riemann-Liouville fractional integral, has fascinated many researchers to contribute to this field. Next, we recall the Sarikaya's inequality as follows.

Theorem 1. *Let $\mathcal{G} : [m,n] \to \mathbb{R}$ be a positive function with $0 \leq m < n$ and $\mathcal{G} \in L[m,n]$. If $\mathcal{G}$ is a convex function on $[m,n]$, then the following inequalities hold*

$$\mathcal{G}\left(\frac{m+n}{2}\right) \leq \frac{\Gamma(\lambda+1)}{2(n-m)^{\lambda}}\left[J^{\lambda}_{m^+}\mathcal{G}(n) + J^{\lambda}_{n^-}\mathcal{G}(m)\right] \leq \frac{\mathcal{G}(m)+\mathcal{G}(n)}{2} \tag{2}$$

with $\lambda > 0$. Where the Riemann-Liouville integrals $J^{\lambda}_{m^+}\mathcal{G}$ and $J^{\lambda}_{n^-}\mathcal{G}$ of order $\lambda \in \mathbb{R}_+$ are defined by

$$J^{\lambda}_{m^+}\mathcal{G}(x) = \frac{1}{\Gamma(\lambda)}\int_m^x (x-\vartheta)^{\lambda-1}\mathcal{G}(\vartheta)d\vartheta, \quad x > m,$$

and

$$J^{\lambda}_{n^-}\mathcal{G}(x) = \frac{1}{\Gamma(\lambda)}\int_x^n (\vartheta-x)^{\lambda-1}\mathcal{G}(\vartheta)d\vartheta, \quad x < n,$$

respectively.

Using the above approach, many new inequalities have been obtained and reported in the literature. For example, an important theorem was established through the Riemann-Liouville fractional calculus and reported in [19] as follows.

Theorem 2. *Suppe that $\mathcal{G} : [m,n] \to \mathbb{R}$ is a differentiable function on (m,n), where $m < n$. If $|\mathcal{G}'|$ is convex on $[m,n]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\lambda+1)}{2(n-m)^{\lambda}}\left[J^{\lambda}_{m^+}\mathcal{G}(n) + J^{\lambda}_{n^-}\mathcal{G}(n)\right] - \mathcal{G}\left(\frac{m+n}{2}\right)\right| \\ \leq &\frac{n-m}{4(\lambda+1)}\left(\lambda+3-\frac{1}{2^{\lambda-1}}\right)\left[|\mathcal{G}'(m)| + |\mathcal{G}'(n)|\right].\end{aligned} \tag{3}$$

Other similar improvements on Hermite-Hadamard type inequalities, including an introduction to generalized convex function on fractal sets, can be seen in [20]. For example, a very new study was carried out on the improvement of Hermite-Hadamard type inequalities via generalized convex functions on fractal set, see [21], and we provide the definition of this concept as

Definition 2. *Let $\mathcal{G} : V \subset \mathbb{R} \to \mathbb{R}^{\alpha}(0 < \alpha \leq 1)$. If the following inequality*

$$\mathcal{G}(\vartheta m + (1-\vartheta)n) \leq \vartheta^{\alpha}\mathcal{G}(m) + (1-\vartheta)^{\alpha}\mathcal{G}(n) \tag{4}$$

holds for any $m, n \in V$ and $\vartheta \in [0,1]$, then $\mathcal{G}$ is called a generalized convex on V.

The Riemann-Liouville fractional integral, along the Hadamard's fractional integral that possesses a symmetric property given in [22], is a generalized through the recent work of Katugampola. These two integrals were combined and given in a single form (see [23,24]).

Definition 3. *Let $[m,n] \subset \mathbb{R}$ be a finite interval. Then, the left-and right-sided Katugampola fractional integrals of order $\lambda > 0$ for $\mathcal{G} \in X_c^p(m,n)$ are defined by*

$$ {}^{\rho}I_{m+}^{\lambda}\mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)}\int_{\mathrm{m}}^{x}\frac{\vartheta^{\rho-1}}{(x^{\rho}-\vartheta^{\rho})^{1-\lambda}}\mathcal{G}(\vartheta)d\vartheta \quad \text{and} \quad {}^{\rho}I_{n-}^{\lambda}\mathcal{G}(x) = \frac{\rho^{1-\lambda}}{\Gamma(\lambda)}\int_{x}^{n}\frac{\vartheta^{\rho-1}}{(\vartheta^{\rho}-x^{\rho})^{1-\lambda}}\mathcal{G}(\vartheta)d\vartheta, $$

with $m < x < n$, $\rho > 0$. Given the space of complex-valued Lebesgue measurable function ω as $X_c^p(m,n)(c \in \mathbb{R}, 1 \leq p \leq \infty)$, we define the norm of the function on $[m,n]$ as follows

$$ \|\mathcal{G}\|_{X_c^p} = \left(\int_m^n |\vartheta^c\mathcal{G}(\vartheta)|^p\frac{d\vartheta}{\vartheta}\right)^{1/p} < \infty, $$

whereby $1 \leq p < \infty, c \in \mathbb{R}$. If $p = \infty$, we obtain

$$ \|\mathcal{G}\|_{X_c^\infty} = \operatorname*{ess\ sup}_{m \leq \vartheta \leq n}\left[\vartheta^c|\mathcal{G}(\vartheta)|\right]. $$

Other related works including the generalization of Hermite-Hadamard inequality for Katugampola fractional integrals [25], given in the following lemma, as well as the theorem that follows immediately.

Lemma 1. *Let $\mathcal{G} : [m^{\rho}, n^{\rho}] \to \mathbb{R}$ be a differentiable mapping on (m^{ρ}, n^{ρ}), with $0 \leq m < n$. If the fractional integrals exist, we obtain the following equality,*

$$ \begin{aligned} \frac{\mathcal{G}(m^{\rho})+\mathcal{G}(n^{\rho})}{2} - \frac{\lambda\rho^{\lambda}\Gamma(\lambda+1)}{2(n^{\rho}-m^{\rho})^{\lambda}}\left[{}^{\rho}I_{m+}^{\lambda}\mathcal{G}(n^{\rho}) +{}^{\rho} I_{n-}^{\lambda}\mathcal{G}(m^{\rho})\right] =& \frac{n^{\rho}-m^{\rho}}{2}\int_0^1\left[(1-\vartheta^{\rho})^{\lambda}-\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1} \\ &\times \mathcal{G}'\left(\vartheta^{\rho}m^{\rho}+(1-\vartheta^{\rho})n^{\rho}\right)d\vartheta. \end{aligned} \tag{5} $$

Theorem 3. *Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^{\rho}, n^{\rho}] \to \mathbb{R}$ be a non-negative function with $0 \leq m < n$ and $\mathcal{G} \in X_c^p(m^{\rho}, n^{\rho})$. If $\mathcal{G}$ is also a convex function on $[m,n]$, then we have*

$$ \mathcal{G}\left(\frac{m^{\rho}+n^{\rho}}{2}\right) \leq \frac{\rho^{\lambda}\Gamma(\lambda+1)}{2(n^{\rho}-m^{\rho})^{\lambda}}\left[{}^{\rho}I_{m+}^{\lambda}\mathcal{G}(n^{\rho}) +{}^{\rho} I_{n-}^{\lambda}\mathcal{G}(m^{\rho})\right] \leq \frac{\mathcal{G}(m^{\rho})+\mathcal{G}(n^{\rho})}{2}, \tag{6} $$

whereby the fractional integrals are given for the function $\mathcal{G}(x^{\rho})$ and evaluated at m and n, respectively.

Katugampola fractional integrals have many applications in the fields of science and technology, some of which can be found in the following references [26,27]. Therefore, many generalizations of different inequalities are studied via these fractional integrals. For example, Kermausuor [28] and Mumcu et al. [29] generalized Ostrowski-type and Hermite-Hadamard type inequalities for harmonically convex functions, respectively. Tekin et al. [30] proposed Hermite-Hadamard inequality for p-convex functions for Katugampola fractional integrals. Other inequalities generalized via Katugampola fractional integrals include Grüss inequality, [31,32] and Lyapunov inequality [33].

Therefore, the aim of this paper is to generalize the Hermite-Hadamard inequality for generalized convex functions on fractal sets via Katugampola fractional integrals. This can be the generalization of the work of Chen and Katugampola [25], who proposed the inequality stated in Theorem 3. Another objective of this study is to define a new identity for generalized fractional integrals, through which generalized Hermite-Hadamard type inequalities for convex function are derived. The trapezoid and mid-point type inequalities are also proposed for the generalized convex function involving

Katugampola fractional integrals, which would generalize the Riemann-Liouville and the Hadamard integrals into a single form.

2. New Generalized Fractional Integrals Identity and New Integral Inequality for Katugampola Fractional Integrals

In order to improve the identity established in [19] for generalized fractional integrals, the following lemma can be used to prove our results.

Lemma 2. *Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a differentiable mapping on (m^ρ, n^ρ), where $m < n$. The following equality holds if the fractional integrals exist,*

$$\begin{aligned}&\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}[{}^\rho J^\lambda_{m^+}\mathcal{G}(n^\rho)+{}^\rho J^\lambda_{n^-}\mathcal{G}(m^\rho)]-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\\ =&\frac{n^\rho-m^\rho}{2}\left[\int_0^1 M\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta-\int_0^1\left[(1-\vartheta^\rho)^\lambda-\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right],\end{aligned} \tag{7}$$

where

$$M=\begin{cases}\vartheta^{\rho-1}, & 0\le\vartheta<\frac{1}{\rho\sqrt{2}}\\ -\vartheta^{\rho-1}, & \frac{1}{\rho\sqrt{2}}\le\vartheta<1.\end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}I=&\left[\int_0^{\frac{1}{\rho\sqrt{2}}}\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]+\left[-\int_{\frac{1}{\rho\sqrt{2}}}^1\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]\\ &+\left[-\int_0^1\left[(1-\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]+\left[\int_0^1\left[\vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\right]\\ =&I_1+I_2+I_3+I_4.\end{aligned} \tag{8}$$

Integrating by parts, we get I_1 and I_2 as follows,

$$\begin{aligned}I_1=&\int_0^{\frac{1}{\rho\sqrt{2}}}\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)dt=\frac{1}{m^\rho-n^\rho}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)\Big|_0^{\frac{1}{\rho\sqrt{2}}}\\ =&\frac{1}{\rho(n^\rho-m^\rho)}\left[-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)+\mathcal{G}(n^\rho)\right],\end{aligned} \tag{9}$$

$$\begin{aligned}I_2=&-\int_{\frac{1}{\rho\sqrt{2}}}^1\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta=\frac{-1}{m^\rho-n^\rho}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)\Big|_{\frac{1}{\rho\sqrt{2}}}^1\\ =&\frac{1}{n^\rho-m^\rho}\left[\mathcal{G}(m^\rho)-\mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right].\end{aligned} \tag{10}$$

Set $x^\rho=\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho$ for calculating I_3 and I_4,

$$\begin{aligned}I_3=&-\int_0^1(1-\vartheta^\rho)^\lambda\mathcal{G}'(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n^\rho)d\vartheta\\ =&-\frac{(1-\vartheta^\rho)^\lambda}{m^\rho-n^\rho}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n)\Big|_0^1-\frac{\lambda}{m^\rho-n^\rho}\int_0^1(1-\vartheta^\rho)^{\lambda-1}\mathcal{G}(\vartheta^\rho m^\rho+(1-\vartheta^\rho)n)d\vartheta\\ =&-\frac{\mathcal{G}(n^\rho)}{n^\rho-m^\rho}+\frac{\lambda}{n^\rho-m^\rho}\int_{n^\rho}^{m^\rho}\left(\frac{x^\rho-m^\rho}{n^\rho-m^\rho}\right)^{\lambda-1}\mathcal{G}(x^\rho)\frac{x^\rho}{m^\rho-n^\rho}dx\\ =&-\frac{\mathcal{G}(n^\rho)}{\rho(n^\rho-m^\rho)}+\frac{\lambda\rho^{\lambda-1}\Gamma(\lambda+1)}{(n^\rho-m^\rho)^{\lambda+1}}{}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho),\end{aligned} \tag{11}$$

$$I_4 = -\int_0^1 \vartheta^{\rho\lambda} \cdot \vartheta^{\rho-1} \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right) d\vartheta = -\frac{\mathcal{G}\left(m^\rho\right)}{\rho\left(n^\rho - m^\rho\right)} + \frac{\lambda\rho^{\lambda-1}\Gamma(\lambda+1)^\rho}{\left(n^\rho - m^\rho\right)^{\lambda+1}}\, I^\lambda_{m+}\mathcal{G}\left(n^\rho\right). \quad (12)$$

Now substituting inequalities (9), (10), (11) and (12) into (8) completes the proof. □

Remark 1. *If $\rho = 1$, then the identity (7) in Lemma 2 reduces to identity (3) in Lemma 2.1 [19].*

Using Lemma 2, the following result for differentiable function is obtained.

Theorem 4. *Let $\mathcal{G} : [m^\rho, n^\rho] \to \mathbb{R}$ be a differentiable mapping on (m^ρ, n^ρ) with $0 \le m < n$. If $|\mathcal{G}'|$ is convex on $[m^\rho, n^\rho]$, then the following inequality holds:*

$$\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}\left(n^\rho\right) + {}^\rho I^\alpha_{n-}\mathcal{G}\left(m^\rho\right)\right] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \le \frac{n^\rho - m^\rho}{4\rho(\lambda+1)}\left[3+\lambda-\frac{1}{2^{\lambda-1}}\right]\left[\left|\mathcal{G}'\left(m^\rho\right)\right| + \left|\mathcal{G}'\left(n^\rho\right)\right|\right]. \quad (13)$$

Proof. Usining Lemma 2 and the convexity of $|\mathcal{G}'|$, we get

$$\begin{aligned}
&\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}\left(n^\rho\right) + {}^\rho I^\lambda_{n-}\mathcal{G}\left(m^\rho\right)\right] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \\
\le & \frac{n^\rho - m^\rho}{2}\left[\int_0^{\frac{1}{\rho\sqrt{2}}} \vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta + \int_{\frac{1}{\rho\sqrt{2}}}^1 t^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta\right. \\
&\left. + \int_0^1 |(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}|\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta\right] \qquad (14)\\
\le & \frac{n^\rho - m^\rho}{2}\left[\int_0^{\frac{1}{\rho\sqrt{2}}} \vartheta^{\rho-1}[\vartheta^\rho|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)|\mathcal{G}'(n^\rho)|]d\vartheta + \int_{\frac{1}{\rho\sqrt{2}}}^1 \vartheta^{\rho-1}[\vartheta^\rho|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)|\mathcal{G}'(n^\rho)|]d\vartheta\right. \\
&\left. + \int_0^1 |(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}|\vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta\right].
\end{aligned}$$

Thus,

$$\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}\left(n^\rho\right) + {}^\rho I^\lambda_{n-}\mathcal{G}\left(m^\rho\right)\right] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| = \frac{n^\rho - m^\rho}{2}[I_1 + I_2 + I_3], \quad (15)$$

whereby I_1, I_2 and I_3 are the first, second and third integrals in inequality (14).

When calculating I_1 and I_2, we get the following

$$I_1 = \int_0^{\frac{1}{\rho\sqrt{2}}} \vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)|d\vartheta = \frac{1}{\rho 8}\left|\mathcal{G}'(m^\rho)\right| + \frac{3}{\rho 8}\left|\mathcal{G}'(n^\rho)\right|, \quad (16)$$

$$I_2 = \int_{\frac{1}{\rho\sqrt{2}}}^1 \vartheta^{\rho-1}|\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)| = \frac{3}{\rho 8}\left|\mathcal{G}'(m^\rho)\right| + \frac{1}{\rho 8}\left|\mathcal{G}'(n^\rho)\right|. \quad (17)$$

A similar line of argument for the proof of Theorem 2.5 in [25] can be used to calculate I_3,

$$I_3 = \int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta)^\rho b^\rho)dt = \frac{1}{\rho(\lambda+1)}\left(1-\frac{1}{2^\lambda}\right)\left[\left|\mathcal{G}'\left(m^\rho\right)\right| + \left|\mathcal{G}'\left(n^\rho\right)\right]\right|. \quad (18)$$

Submitting inequalities (16), (17) and (18) in (15), we get (13). This completes the proof. □

Remark 2. *i.* *Choosing $\rho = 1$ in Theorem 4 reduces inequality (13) to inequality (3) of Theorem 2.*

ii. *Choosing $\rho = 1$ and $\lambda = 1$ reduces inequality (13) to inequality (16) in [19], which is given as follows*

$$\left|\frac{1}{n-m}\int_m^n \mathcal{G}(x)dx - \mathcal{G}\left(\frac{m+n}{2}\right)\right| \le \frac{3(n-m)}{8}\left(\left|\mathcal{G}'(m)\right| + \left|\mathcal{G}'(n)\right|\right).$$

3. Generalized Hermite-Hadamard Inequality and Related Integral Inequalities for Katugampola Fractional Integral on Fractal Sets

The following theorem generalizes the result obtained by [25] of the Hermite-Hadamard inequality involving the Katugampola fractional integrals for generalized convex function on fractal sets.

Theorem 5. *Suppose that $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ $(0 < \alpha \le 1)$ is a positive function with $0 \le m < n$ and $\mathcal{G} \in X_c^p(m^\rho, n^\rho)$ for $\lambda > 0$ and $\rho > 0$. If $\mathcal{G}$ is a generalized convex function on $[m^\rho, n^\rho]$, then we obtain*

$$\begin{aligned}\mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right) &\le \frac{\rho^\lambda \Gamma(\lambda+1)}{2^\alpha (n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho)\right] \\ &\le \frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2^\alpha}.\end{aligned} \tag{19}$$

Proof. Suppose that $x, y \in [m, n], \lambda > 0$, defined by $x^\rho = \vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho$ and $y^\rho = \vartheta^\rho n^\rho + (1-\vartheta^\rho) m^\rho$, where $\vartheta \in [0, 1]$. Since $\mathcal{G}$ is generalized convex function, we have

$$2^\alpha \mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right) \le \mathcal{G}\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) + \mathcal{G}\left(\vartheta^\rho n^\rho + (1-\vartheta^\rho) m^\rho\right). \tag{20}$$

Multiplying both sides of the inequality (20) by $\vartheta^{\lambda\rho-1}$, for $\lambda > 0$ and then integrating over $[0, 1]$ with respect to ϑ, we obtain the following

$$\begin{aligned}\frac{2^\alpha}{\lambda\rho}\mathcal{G}\left(\frac{m^\rho + n^\rho}{2}\right) &\le \int_0^1 \vartheta^{\lambda\rho-1}\mathcal{G}\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta + \int_0^1 \vartheta^{\lambda\rho-1}\mathcal{G}\left(\vartheta^\rho n^\rho + (1-\vartheta^\rho) m^\rho\right) d\vartheta \\ &= \int_n^m \left(\frac{n^\rho - x^\rho}{n^\rho - m^\rho}\right)^{\lambda-1} \mathcal{G}(x^\rho)\frac{x^{\rho-1}}{m^\rho - n^\rho}dx \\ &\quad + \int_m^n \left(\frac{y^\rho - m^\rho}{n^\rho - m^\rho}\right)^{\lambda-1} \mathcal{G}(y^\rho)\frac{y^{\rho-1}}{n^\rho - m^\rho}dy \\ &= \frac{\rho^{\lambda-1}\Gamma(\lambda)}{(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho)\right].\end{aligned} \tag{21}$$

This establishes the first inequality. When proving the second inequality (19), we first observe generalized convex functions $\mathcal{G}$, which is given as

$$\mathcal{G}\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) \le (\vartheta^\rho)^\alpha \mathcal{G}(m^\rho) + (1-\vartheta^\rho)^\alpha \mathcal{G}(n^\rho), \tag{22}$$

and

$$\mathcal{G}\left(\vartheta^\rho n^\rho + (1-\vartheta^\rho) m^\rho\right) \le (\vartheta^\rho)^\alpha \mathcal{G}(n^\rho) + (1-\vartheta^\rho)^\alpha \mathcal{G}(m^\rho). \tag{23}$$

Summing the above inequalities, we have

$$\mathcal{G}\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) + \mathcal{G}\left(\vartheta^\rho n^\rho + (1-\vartheta^\rho) m^\rho\right) \le \mathcal{G}(m^\rho) + \mathcal{G}(n^\rho). \tag{24}$$

Multiplying both sides of inequality (24) by $\vartheta^{\lambda\rho-1}$, for $\lambda > 0$ and integrating the result over $[0, 1]$ with respect to ϑ, we obtain

$$\frac{\rho^{\lambda-1}\Gamma(\lambda)}{(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho)\right] \le \frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{\lambda\rho}. \tag{25}$$

This completes the proof. □

Remark 3. *Taking $\alpha = 1$ in inequality (19) of Theorem 5 reduces the result to inequality (6) of Theorem 3.*

Now, we derive the mid-point type inequalities via generalized convex functions on the fractal set for the Katugampola fractional integral. Therefore, the definition of generalized beta function is given as follows

$$\beta_\rho(m,n) = \int_0^1 \rho\,(1-x^\rho)^{n-1}\,(x^\rho)^{m-1}\,x^{\rho-1}dx.$$

Note that, as $\rho \to 1, \beta_\rho(m,n) \to \beta(m,n)$.

Theorem 6. *Suppose that $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ $(0 < \alpha \le 1)$ be a differentiable function on (m^ρ, n^ρ), and $\mathcal{G}' \in L^1[m,n]$ with $0 \le m < n$. If $|\mathcal{G}'|^q$ is generalized convex on $[m^\rho, n^\rho]$, we obtain*

$$\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}[{}^\rho J^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho J^\lambda_{n^-}\mathcal{G}(m^\rho)] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \le \frac{n^\rho-m^\rho}{2}\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho}\right][|\mathcal{G}'(m^\rho)| + |\mathcal{G}'(n^\rho)|]. \tag{26}$$

Proof. From Lemma 2, we have

$$\begin{aligned}
&\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}[{}^\rho J^\lambda_{m^+}\mathcal{G}(n^\rho) + {}^\rho J^\lambda_{n^-}\mathcal{G}(m^\rho)] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \\
\le\ & \frac{n^\rho-m^\rho}{2}\left|\int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta + \int_{\frac{1}{\sqrt[\rho]{2}}}^1 \vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta\right. \\
&\left. - \int_0^1\left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)d\vartheta\right| \\
\le\ & \frac{n^\rho-m^\rho}{2}\left|\int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta + \int_{\frac{1}{\sqrt[\rho]{2}}}^1 t^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta\right| \\
+\ & \left|\int_0^1[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)d\vartheta\right|.
\end{aligned} \tag{27}$$

Using the fact that the function $|\mathcal{G}'|$ is generalized convex on $[m^\rho, n^\rho]$, we obtain the following

$$\begin{aligned}
\left|\int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta\right| &\le \int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho-1}[\vartheta^{\rho\alpha}|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)^\alpha|\mathcal{G}'(n^\rho)|]dt \\
&\le |\mathcal{G}'(m^\rho)|\int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho\alpha+\rho-1}d\vartheta + |\mathcal{G}'(n^\rho)|\int_0^{\frac{1}{\sqrt[\rho]{2}}} \vartheta^{\rho-1}(1-\vartheta^\rho)^\alpha d\vartheta \\
&= |\mathcal{G}'(m^\rho)|\left[\frac{1}{2^{\alpha+1}\rho(\alpha+1)}\right] + |\mathcal{G}'(n^\rho)|\left[\frac{2^{\alpha+1}-1}{2^{\alpha+1}\rho(\alpha+1)}\right].
\end{aligned} \tag{28}$$

In the same way, we have

$$\left|\int_{\frac{1}{\sqrt[\rho]{2}}}^1 \vartheta^{\rho-1}[\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)]d\vartheta\right| \le |\mathcal{G}'(m^\rho)|\left[\frac{2^{\alpha+1}-1}{2^{\alpha+1}\rho(\alpha+1)}\right] + |\mathcal{G}'(n^\rho)|\left[\frac{1}{2^{\alpha+1}\rho(\alpha+1)}\right] \tag{29}$$

and

$$\left|\int_0^1 [(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}]\vartheta^{\rho-1}\mathcal{G}'(\vartheta^\rho m^\rho + (1-\vartheta^\rho)n^\rho)d\vartheta\right| \leq \int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right]\vartheta^{\rho-1}\left|\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right)\right| d\vartheta$$
$$\leq |\mathcal{G}'(m^\rho)| \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\alpha(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\alpha(\vartheta^\rho)^\lambda]d\vartheta$$
$$+|\mathcal{G}'(n^\rho)| \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\alpha + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\alpha]d\vartheta \quad (30)$$
$$\leq |\mathcal{G}'(m^\rho)|\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]$$
$$+|\mathcal{G}'(n^\rho)|\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right].$$

Substituting inequalities (28), (29) and (30) in (27), we deduce the inequality (26). □

The following corollary is derived to show the estimates of the difference between mid-point-type and the integral of $\mathcal{G}$ on $[m^\rho, n^\rho]$ when $\lambda = \alpha = \frac{2}{3}$.

Corollary 1. *In Theorem 6, if we take $\lambda = \alpha = \frac{2}{3}$ in inequality (26), we have*

$$\left|\frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}[{}^\rho J^\lambda_{m^+}\mathcal{G}(n^\rho) +{}^\rho J^\lambda_{n^-}\mathcal{G}(m^\rho)] - \mathcal{G}\left(\frac{m^\rho+n^\rho}{2}\right)\right| \leq \frac{(n^\rho-m^\rho)\beta_\rho(\frac{5}{3},\frac{5}{3})}{\rho}\left[\frac{|\mathcal{G}'(m^\rho)|+|\mathcal{G}'(n^\rho)|}{2}\right].$$

The trapezoid-type inequalities via generalized convex function on fractal sets for Katugampola fractional integrals can be derived using Lemma 1.

Theorem 7. *Suppose that $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ be a differentiable function on (m^ρ, n^ρ), and $\mathcal{G}' \in L^1[m, n]$ with $0 \leq m < n$. If $|\mathcal{G}'|^q$ is generalized convex on $[m^\rho, n^\rho]$ for $q \geq 1$, we obtain*

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(b^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m^+}\mathcal{G}(n^\rho) +{}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho)\right]\right| \leq \frac{n^\rho-m^\rho}{2}\left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]^{\frac{1}{q}}$$
$$\times\left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}. \quad (31)$$

Proof. From Lemma 1, we have

$$\left|\frac{\mathcal{G}(m^\rho)+\mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I^\alpha_{m^+}\mathcal{G}(n^\rho) +{}^\rho I^\lambda_{n^-}\mathcal{G}(m^\rho)\right]\right| \leq \frac{n^\rho-m^\rho}{2}\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\right.$$
$$\left.\times\,\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right)d\vartheta\right|. \quad (32)$$

In the first case, suppose that $q = 1$. Since the function $|\mathcal{G}'|$ is generalized convex on $[m^\rho, n^\rho]$, we have

$$\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right) \leq (\vartheta^\rho)^\alpha|\mathcal{G}'(m^\rho)| + (1-\vartheta^\rho)^\alpha|\mathcal{G}'(n^\rho)|.$$

Therefore,

$$\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right)d\vartheta\right| \leq \int_0^1 \left[(1-\vartheta^\rho)^\lambda + \vartheta^{\rho\lambda}\right]\vartheta^{\rho-1}[(\vartheta^\rho)^\alpha\mathcal{G}'|(m^\rho)| + (1-\vartheta^\rho)^\alpha|\mathcal{G}'(n^\rho)|]d\vartheta$$
$$\leq |\mathcal{G}'(m^\rho)| \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\alpha(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\alpha(\vartheta^\rho)^\lambda]d\vartheta$$
$$+|\mathcal{G}'(n^\rho)| \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\alpha + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\alpha]d\vartheta$$
$$\leq |\mathcal{G}'(m^\rho)|\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]$$
$$+|\mathcal{G}'(n^\rho)|\left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right].$$

Hence, the inequalities (32) and (33) complete the proof.

The second case can be evaluated when $q > 1$. Using the Hölder's inequality and generalized convexity of $|\mathcal{G}'|$, for $p = \frac{q}{q-1}$, we obtain

$$\begin{aligned}
&\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right] \vartheta^{\rho-1} \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta\right| \leq \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right] \vartheta^{\rho-1}\right)^{1-\frac{1}{q}} \\
&\times \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right] \vartheta^{\rho-1} \left|\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right] \vartheta^{\rho-1}\right)^{1-\frac{1}{q}} \\
&\times \left(|\mathcal{G}'(m^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\alpha (1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda (\vartheta^\rho)^\alpha] d\vartheta\right. \\
&\left. + |\mathcal{G}'(n^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\lambda (1-\vartheta^\rho)^\alpha + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda (1-\vartheta^\rho)^\alpha] d\vartheta\right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}} \\
&\times \left(|\mathcal{G}'(m^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]\right. \\
&\left. + |\mathcal{G}'(n^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]\right)^{\frac{1}{q}}.
\end{aligned} \tag{33}$$

The inequalities (32) and (33) complete the proof. □

Other special cases related to Theorem 7 are stated in the following corollary. This would estimate the difference between trapezoid-type and the integral of $\mathcal{G}$.

Corollary 2. *Consider inequality (31) of the Theorem 7,*

1. *If $\lambda = \alpha = \frac{1}{3}$ and $\rho = 1$, we have the trapezoid inequality:*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m) + \mathcal{G}(n)}{2} - \frac{\lambda\Gamma(\lambda+1)}{2(n-m)^\lambda}\left[I_{m+}^\lambda \mathcal{G}(n) + I_{n-}^\lambda \mathcal{G}(m)\right]\right| &\leq \frac{n-m}{2}\left(\frac{3}{4}\right)^{1-\frac{1}{q}} \left[\beta\left(\frac{4}{3},\frac{4}{3}\right) + \frac{3}{5}\right]^{\frac{1}{q}} \\
&\times \left(|\mathcal{G}'(m)|^q + |\mathcal{G}'(n)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{34}$$

2. *For $\lambda = \alpha = \frac{3}{5}$, we have*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I_{m+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n-}^\lambda \mathcal{G}(m^\rho)\right]\right| &\leq \frac{n^\rho-m^\rho}{2}\left(\frac{5}{8\rho}\right)^{1-\frac{1}{q}} \left[\frac{\beta_\rho(\frac{8}{5},\frac{8}{5})}{\rho} + \frac{5}{11\rho}\right]^{\frac{1}{q}} \\
&\times \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{35}$$

Theorem 8. *Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ be a differentiable function on (m^ρ, n^ρ), and $\mathcal{G}' \in L^1[m,n]$ with $0 \leq a < b$. If $|\mathcal{G}'|^q$is generalized convex on $[m^\rho, n^\rho]$ for $q \geq 1$, we obtain*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I_{m+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n-}^\lambda \mathcal{G}(m^\rho)\right]\right| &\leq \frac{n^\rho-m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]^{\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{36}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2(n^\rho-m^\rho)^\lambda}\left[{}^\rho I_{m+}^\lambda \mathcal{G}(n^\rho) + {}^\rho I_{n-}^\lambda \mathcal{G}(m^\rho)\right]\right| &\leq \frac{n^\rho-m^\rho}{2} \\
&\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right] \vartheta^{\rho-1} \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta\right|.
\end{aligned} \tag{37}$$

Using the Hölder's inequality and generalized convexity of $|\mathcal{G}'|$, we obtain

$$\begin{aligned}
\left|\int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda}\right] \vartheta^{\rho-1}\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right) d\vartheta\right| &\le \left(\int_0^1 \left(\vartheta^{\rho-1}\right)^p d\vartheta\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda\right] \vartheta^{\rho-1} \left|\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \\
&\le \left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times \left(|\mathcal{G}'(m^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\alpha(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(\vartheta^\rho)^\alpha] d\vartheta\right. \\
&\left. + |\mathcal{G}'(n^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\alpha(1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda(1-\vartheta^\rho)^\alpha] d\vartheta\right)^{\frac{1}{q}} \\
&\le \left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times \left(|\mathcal{G}'(m^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]\right. \\
&\left. + |\mathcal{G}'(n^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

□

In order to simplify Theorem 8, we consider some special cases related to inequality (36), when $\lambda = \alpha = \frac{1}{2}$ and $\lambda = \alpha = \frac{4}{9}$.

Corollary 3. *Considering inequality (36) of Theorem 8, we have the following trapezoid inequality*

1. *For $\lambda = \alpha = \frac{1}{2}$, we get*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2\,(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| &\le \frac{n^\rho - m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times \left[\frac{\beta_\rho(\frac{3}{2},\frac{3}{2})}{\rho} + \frac{1}{2\rho}\right]^{\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

2. *If $\lambda = \alpha = \frac{4}{9}$, we have*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2\,(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| &\le \frac{n^\rho - m^\rho}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \\
&\times \left[\frac{\beta_\rho(\frac{13}{9},\frac{13}{9})}{\rho} + \frac{9}{7\rho}\right]^{\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 9. *Let $\lambda > 0$ and $\rho > 0$. Let $\mathcal{G} : [m^\rho, n^\rho] \subset \mathbb{R}_+ \to \mathbb{R}^\alpha$ $(0 < \alpha \le 1)$ be a differentiable function on (m^ρ, n^ρ), and $\mathcal{G}' \in L^1[m,n]$ with $0 \le m < n$. If $|\mathcal{G}'|^q$is generalized convex on $[m^\rho, n^\rho]$ for $q \ge 1$, we obtain*

$$\begin{aligned}
\left|\frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda\rho^\lambda\Gamma(\lambda+1)}{2\,(n^\rho - m^\rho)^\lambda}\left[{}^\rho I^\lambda_{m+}\mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-}\mathcal{G}(m^\rho)\right]\right| &\le \frac{n^\rho - m^\rho}{2}\left(\frac{1}{\rho}\right)^{1-\frac{1}{q}} \\
&\times \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]^{\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{38}$$

Proof. Using the fact $|\mathcal{G}'|^q$, a generalized convex on $[m^\rho, n^\rho]$ with $q \ge 1$, we obtain

$$\mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho)\, n^\rho\right) \le (\vartheta^\rho)^\alpha\, \mathcal{G}'(m^\rho) + (1-\vartheta^\rho)^\alpha\, \mathcal{G}'(n^\rho). \tag{39}$$

Applying inequality (39), together with the power mean inequality, on (37), we have

$$\left| \int_0^1 \left[(1-\vartheta^\rho)^\lambda - \vartheta^{\rho\lambda} \right] \vartheta^{\rho-1} \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) d\vartheta \right| \leq \left(\int_0^1 \vartheta^{\rho-1} d\vartheta \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[(1-\vartheta^\rho)^\lambda + (\vartheta^\rho)^\lambda \right] \vartheta^{\rho-1} \left| \mathcal{G}'\left(\vartheta^\rho m^\rho + (1-\vartheta^\rho) n^\rho\right) \right|^q d\vartheta \right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(\vartheta^\rho)^\alpha (1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\alpha (\vartheta^\rho)^\lambda] d\vartheta \right.$$
$$\left. + |\mathcal{G}'(n^\rho)|^q \int_0^1 [\vartheta^{\rho-1}(1-\vartheta^\rho)^\alpha (1-\vartheta^\rho)^\lambda + \vartheta^{\rho-1}(\vartheta^\rho)^\lambda (1-\vartheta^\rho)^\alpha] d\vartheta \right)^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)} \right] \right.$$
$$\left. + |\mathcal{G}'(n^\rho)|^q \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)} \right] \right)^{\frac{1}{q}}.$$

□

The following corollary is given to simplify inequality (38) in Theorem (9).

Corollary 4. *Considering inequality (38) of Theorem 9, for $\lambda = \alpha = \frac{3}{7}$, we get*

$$\left| \frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2(n^\rho - m^\rho)^\lambda} \left[{}^\rho I^\lambda_{m+} \mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-} \mathcal{G}(m^\rho) \right] \right| \leq \frac{n^\rho - m^\rho}{2} \left(\frac{1}{\rho}\right)^{1-\frac{1}{q}}$$
$$\times \left[\frac{\beta_\rho(\frac{10}{3}, \frac{10}{3})}{\rho} + \frac{7}{13\rho} \right]^{\frac{1}{q}} \left(|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q \right)^{\frac{1}{q}}.$$

Corollary 5. *From Theorems 7, 8 and 9 for $q > 1$, we obtain the following:*

$$\left| \frac{\mathcal{G}(m^\rho) + \mathcal{G}(n^\rho)}{2} - \frac{\lambda \rho^\lambda \Gamma(\alpha+1)}{2(n^\rho - m^\rho)^\lambda} \left[{}^\rho I^\lambda_{m+} \mathcal{G}(n^\rho) + {}^\rho I^\lambda_{n-} \mathcal{G}(m^\rho) \right] \right| \leq \min\{S_1, S_2, S_3\} \frac{n^\rho - m^\rho}{2},$$

where,

$$S_1 = \left(\frac{1}{\rho(\lambda+1)} \right)^{1-\frac{1}{q}} \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)} \right]^{\frac{1}{q}} (|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q)^{\frac{1}{q}},$$

$$S_2 = \left(\frac{1}{p(\rho-1)+1} \right)^{\frac{1}{p}} \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)} \right]^{\frac{1}{q}} (|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q)^{\frac{1}{q}},$$

and

$$S_3 = \left(\frac{1}{\rho} \right)^{1-\frac{1}{q}} \left[\frac{\beta_\rho(\lambda+1,\alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)} \right]^{\frac{1}{q}} (|\mathcal{G}'(m^\rho)|^q + |\mathcal{G}'(n^\rho)|^q)^{\frac{1}{q}}.$$

4. Applications to Special Means

In this section, some generalized inequalities connected to the special means are obtained to serve as an application of our results, as in [2]. Thus,

i. The arithmetic mean:
$A(m,n) = \frac{m+n}{2}$; $m, n \in \mathbb{R}$, with $m, n > 0$.

ii. The generalized log-mean:
$L_i(m,n) = \left[\frac{n^{i+1} - m^{i+1}}{(i+1)(n-m)} \right]^{\frac{1}{i}}$; $i \in \mathbb{Z}\{-1, 0\}$, $m, n \in \mathbb{R}$, with $m, n > 0$.

Proposition 1. *Let $i \in \mathbb{Z}$, $|i| \geq 2$ and $m, n \in \mathbb{R}$ where $0 < m < n$. For $\rho > 0$, $\lambda > 0$, $0 < \alpha < 1$ and $q \geq 1$, we obtain the following:*

$$\left|A\left(m^i, n^i\right) - L_i^i(m,n)\right| \leq \frac{|i|(n-m)}{8} A^{\frac{1}{q}}(|m|^{q(i-1)}, |n|^{q(i-1)}).$$

Proof. Applying $\mathcal{G}(m) = m^i$ in inequality (31) of Theorem 7, we have

$$\left|\frac{m^{\rho i} + n^{\rho i}}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2\left(n^\rho - m^\rho\right)^\lambda}\left[{}^\rho I_{m^+}^\lambda\left(n^{i\rho}\right) + {}^\rho I_{n^-}^\lambda\left(m^{i\rho}\right)\right]\right| \leq \frac{i(n^\rho - m^\rho)}{2}\left(\frac{1}{\rho(\lambda+1)}\right)^{1-\frac{1}{q}}\left[\frac{\beta_\rho(\lambda+1, \alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]^{\frac{1}{q}} \times (|m^\rho|^{q(i-1)} + |n^\rho|^{q(i-1)})^{\frac{1}{q}}. \tag{40}$$

Choosing $\rho = 1$, $\alpha = 1$ and $\lambda = 1$, in inequality (40) gives the required result. □

Proposition 2. *Let $m, n \in \mathbb{R}$ where $0 < m < n$. For $i \in \mathbb{Z}$, $|i| \geq 2$, $\rho > 0$, $\lambda > 0$, $0 < \alpha < 1$ and $q \geq 1$, we obtain the following:*

$$\left|A\left(m^i, n^i\right) - L_i^i(m,n)\right| \leq \frac{|i|(n-m)}{2}\left(\frac{1}{2}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|m|^{q(i-1)}, |n|^{q(i-1)}).$$

Proof. From inequality (36) of Theorem 8, when applying $\mathcal{G}(m) = m^i$, we have

$$\left|\frac{m^{\rho i} + n^{\rho i}}{2} - \frac{\lambda \rho^\lambda \Gamma(\lambda+1)}{2\left(n^\rho - m^\rho\right)^\lambda}\left[{}^\rho I_{m^+}^\lambda\left(n^{i\rho}\right) + {}^\rho I_{n^-}^\lambda\left(m^{i\rho}\right)\right]\right| \leq \frac{i(n^\rho - m^\rho)}{2}\left(\frac{1}{p(\rho-1)+1}\right)^{\frac{1}{p}} \times \left[\frac{\beta_\rho(\lambda+1, \alpha+1)}{\rho} + \frac{1}{\rho(\lambda+\alpha+1)}\right]^{\frac{1}{q}} \times (|m^\rho|^{q(i-1)} + |n^\rho|^{q(i-1)})^{\frac{1}{q}}. \tag{41}$$

Considering $\rho = 1$ and $\lambda = \alpha = 1$ in inequality (41) gives the required result. □

5. Conclusions

In this paper, we defined a new identity for the generalized fractional integrals. Connected to this, the new integral inequality for a differentiable convex function is derived. We obtained the generalization of Theorem 2 introduced by Chen and Katugampola. In addition, the trapezoid and mid-point type inequalities are studied, along with generalized Hermite-Hadamard inequality, for Katugampola fractional integrals.

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