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On the Oscillatory Behavior of a Class of Fourth-Order Nonlinear Differential Equation

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Abstract: In this work, we study the oscillatory behavior of a class of fourth-order differential equations. New oscillation criteria were obtained by employing a refinement of the Riccati transformations. The new theorems complement and improve a number of results reported in the literature. An example is provided to illustrate the main results.

Keywords: Oscillatory solutions; fourth-order; delay differential equations

1. Introduction

In this paper, we are concerned with the oscillation and the asymptotic behavior of solutions of the fourth-order nonlinear differential equation

$$\left(r(t)\left(x'''(t)\right)^{\alpha}\right)' + q(t)\, x^{\beta}(\sigma(t)) = 0, \tag{1}$$

where α and β are quotient of odd positive integers, $r \in C^1([t_0,\infty))$, $q \in C([t_0,\infty))$, $r(t) > 0$, $q(t) > 0$, $r'(t) \geq 0$, $\sigma(t) \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) \leq t$, $\lim_{t\to\infty}\sigma(t) = \infty$. Moreover, we study (1) under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = \infty. \tag{2}$$

We intend to a solution of (1) a function $x(t) : [t_x,\infty) \to \mathbb{R}$, $t_x \geq t_0$ such that $x(t)$ and $r(t)(x'''(t))^{\alpha}$ are continuously differentiable for all $t \in [t_x,\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. We assume that (1) possesses such a solution. A solution y is said to be non-oscillatory if it is eventually positive or eventually negative; otherwise, it is said to be oscillatory. (1) is said to be oscillatory if all its solutions are oscillatory. The equation itself is called oscillatory if all of its solutions are oscillatory.

The reliance on the past shows up normally in various applications in biology, electrical engineering or physiology. A basic model in nature is reforestation. A cut timberland, in the wake of replanting, will take in any event 20 years before arriving at any sort of development. Consequently, any scientific model of backwoods gathering and recovery plainly should have time defers incorporated with it. Another model happens because of the way that creatures must set aside some effort to process their nourishment before further exercises and reactions occur. Consequently, any model of species dynamics without delays is an approximation at best, see [1].

For several decades, an growing interest in studying the oscillation and non-oscillation criteria of different classes and different orders of differential equations with delay has been observed; see, for instance, the monographs [2,3], the papers [4–26], and the references cited therein.

The purpose of this paper is to give new sufficient conditions for the oscillatory behavior of (1). In Section 2, we will provide some auxiliary lemmas that will help us to prove our oscillation criteria. In Section 3, by employing a refinement of the Riccati transformations, we establish new oscillation criteria of (1).

2. Auxiliary Lemmas

Notation 1. *Here, we introduce Riccati substitutions*

$$\omega_1(t) := \frac{r(t)\left(x'''(t)\right)^{\alpha}}{x^{\alpha}(t)}$$

and

$$\omega_2(t) := \frac{x'(t)}{x(t)}.$$

Moreover, for convenience, we denote that

$$Q(t) := M_1^{\beta-\alpha} q(t) \frac{\sigma^{3\alpha}(t)}{t^{3\alpha}}, \quad R_1(t) := \frac{\alpha\mu}{2} \frac{t^2}{r^{1/\alpha}(t)},$$

and

$$\widetilde{R}(t) := \lambda^{\beta/\alpha} M_2^{\beta-\alpha} \int_t^{\infty} \left(\frac{1}{r(u)} \int_u^{\infty} q(s) \left(\frac{\sigma(s)}{s} \right)^{\beta} \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u,$$

where $\mu, \lambda \in (0,1)$ and M_1, M_2 are positive constants.

All functional inequalities are assumed to hold eventually, that is, they are assumed to be satisfied for all t sufficiently large. We begin with the following lemmas that can be found in [2,8,16,18], respectively.

Lemma 1. *Let $h \in C^n([t_0, \infty))$ and $h(t) > 0$. Suppose that $h^{(n)}(t)$ is of a fixed sign, on $[t_0, \infty)$, $h^{(n)}(t)$ not identically zero and that there exists a $t_1 \geq t_0$ such that, for all $t \geq t_1$,*

$$h^{(n-1)}(t)\, h^{(n)}(t) \leq 0.$$

If we have $\lim_{t\to\infty} h(t) \neq 0$, then there exists $t_\lambda \geq t_0$ such that

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| h^{(n-1)}(t) \right|,$$

for every $\lambda \in (0,1)$ and $t \geq t_\lambda$.

Lemma 2. *If the function x satisfies $x^{(i)}(t) > 0$, $i = 0, 1, \ldots, n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 3. *Assume that α is a quotient of odd positive integers. Then*

$$Uy - Vy^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} U^{\alpha+1} V^{-\alpha}, \quad V > 0. \tag{3}$$

Lemma 4. *Assume that (2) is satisfied and let $x(t)$ be an eventually positive solution of (1). Then, there exists a sufficiently large $t_1 \geq t_0$ such that for all $t \geq t_1$, either*

$$(S_1) \;\; x^{(\kappa)}(t) > 0 \;\; \textit{for } \kappa = 0, 1, 2, 3;$$

or

$$(S_2) \;\; x^{(\kappa)}(t) > 0 \;\; \textit{for } \kappa = 0, 1, 3, \; \textit{and } x''(t) < 0,$$

is holds.

Lemma 5. *Let $x(t)$ is an eventually positive solution of Equation (1).*

($\mathbf{i}_1$) *If x satisfies (S_1), then*
$$\omega_1'(t) + Q(t) + R_1(t)\,\omega_1^{1+1/\alpha}(t) \leq 0; \tag{4}$$

($\mathbf{i}_2$) *If x satisfies (S_2), then*
$$\omega_2'(t) + \omega_2^2(t) + B^{\beta-\alpha}\widetilde{R}(t) \leq 0. \tag{5}$$

Proof. Let that $x(t)$ is an eventually positive solution of Equation (1). From Lemmas 4, there exist two possible cases (S_1) and (S_2) for $t \geq t_1$ large enough.

Let (S_1) holds. Then, taking Lemma 1 and 2 into account, we arrive at

$$x'(t) \geq \frac{\mu}{2} t^2 x'''(t) \tag{6}$$

and $x(t) \geq \frac{1}{3} t x'(t)$. Hence,

$$x(\sigma(t)) \geq \frac{\sigma^3(t)}{t^3} x(t). \tag{7}$$

Differentiating ω_1 and using (1), (6) and (7), we obtain

$$\omega_1'(t) \leq -q(t)\frac{\sigma^{3\alpha}(t)}{t^{3\alpha}} x^{\beta-\alpha}(\sigma(t)) - \frac{\alpha\mu}{2}\frac{t^2}{r^{1/\alpha}(t)}\omega_1^{1+1/\alpha}(t).$$

Since $x'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $B > 0$ such that $x(t) > B$, for all $t \geq t_2$. Thus, we see that

$$\omega_1'(t) \leq -q(t)\frac{\sigma^{3\alpha}(t)}{t^{3\alpha}} B^{\beta-\alpha}(\sigma(t)) - \frac{\alpha\mu}{2}\frac{t^2}{r^{1/\alpha}(t)}\omega_1^{1+1/\alpha}(t),$$

Thus, (4) is satisfied.

Let (S_2) holds. Integrating (1) from t to l, we have

$$r(l)\left(x'''(l)\right)^{\alpha} = r(t)\left(x'''(t)\right)^{\alpha} - \int_t^l q(s)\, x^{\beta}(\sigma(s))\, \mathrm{d}s. \tag{8}$$

Taking Lemma 2 into account, we arrive at

$$x(t) \geq t x'(t). \tag{9}$$

Thus, $x(\sigma(t)) \geq (\sigma(t)/t)\, x(t)$, which with (8) and the fact that $x'(t) > 0$ gives

$$r(l)\left(x'''(l)\right)^{\alpha} - r(t)\left(x'''(t)\right)^{\alpha} + x^{\beta}(t)\int_t^l q(s)\left(\frac{\sigma(s)}{s}\right)^{\beta} \mathrm{d}s \leq 0.$$

Letting $l \to \infty$, we obtain

$$x'''(t) \geq \frac{\lambda^{\beta/\alpha}}{r^{1/\alpha}(t)} x^{\beta/\alpha}(t) \left(\int_t^\infty q(s) \left(\frac{\sigma(s)}{s} \right)^\beta \mathrm{d}s \right)^{1/\alpha}.$$

Integrating the above inequality from t to ∞, we obtain

$$\begin{aligned} x''(t) &\leq -\lambda^{\beta/\alpha} x^{\beta/\alpha}(t) \int_t^\infty \left(\frac{1}{r(u)} \int_u^\infty q(s) \left(\frac{\sigma(s)}{s} \right)^\beta \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u \\ &\leq -\widetilde{R}(t)\, x^{\beta/\alpha}(t). \end{aligned} \tag{10}$$

Differentiating ω_2 and using (10), we get

$$\omega_2'(t) + \omega_2^2(t) + B^{\beta-\alpha}\widetilde{R}(t) \leq 0.$$

Thus, the proof is complete. □

3. Oscillation Criteria

Theorem 1. *If*

$$\int_{t_0}^\infty Q(s)\,\mathrm{d}s = \infty \tag{11}$$

and

$$\int_{t_0}^\infty \widetilde{R}(s)\,\mathrm{d}s = \infty, \tag{12}$$

then (1) is oscillatory.

Proof. Assume to the contrary that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we can assume that $x(t) > 0$. From Lemma 4 that there exist two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

For case (S_1), from Lemma 5, we see that (4) holds, which yields

$$\omega_1'(t) + Q(t) \leq 0. \tag{13}$$

Integrating (13) from t_2 to t and using (11),we obtain

$$\omega_1(t) \leq \omega_1(t_2) - \int_{t_2}^t Q(s)\,ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $\omega_1(t) > 0$.

Similarly, in the case where (S_2) holds, we get a contradicts with (12), which is omitted here for convenience. Therefore, the proof is complete. □

Definition 1. *The sequence of functions* $\{y_n(t)\}_{n=0}^\infty$ *and* $\{z_n(t)\}_{n=0}^\infty$ *define as*

$$y_n(t) = y_0(t) + \int_t^\infty R_1(s)\, y_{n-1}^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s \tag{14}$$

and

$$z_n(t) = z_0(t) + \int_t^\infty z_{n-1}^2(s)\,\mathrm{d}s, \tag{15}$$

where

$$y_0(t) = \int_t^\infty Q(s)\,\mathrm{d}s$$

and

$$z_0(t) = \int_t^\infty \widetilde{R}(s)\,\mathrm{d}s.$$

Theorem 2. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{y_0(t)} \int_t^\infty R_1(s)\, y_0^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} \tag{16}$$

and

$$\liminf_{t\to\infty} \frac{1}{z_0(t)} \int_t^\infty z_0^2(s)\,\mathrm{d}s > \frac{1}{4}. \tag{17}$$

Then, (1) is oscillatory.

Proof. Assume to the contrary that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we can assume that $x(t) > 0$. From Lemma 4 that there exist two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Let case (S_1) holds. By using Lemma 5, we obtain (4). Integrating (4) from t to l, we get

$$\omega_1(l) - \omega_1(t) + \int_t^l Q(s)\,\mathrm{d}s + \int_t^l R_1(s)\,\omega_2^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s \leq 0. \tag{18}$$

From (18), it is obvious that

$$\omega_1(l) - \omega_1(t) + \int_t^l R_1(s)\,\omega_1(s)\,\mathrm{d}s \leq 0. \tag{19}$$

Then we conclude from (19) that

$$\int_t^\infty R_1(s)\,\omega_1(s)\,\mathrm{d}s < \infty, \text{ for } t \geq T, \tag{20}$$

otherwise,

$$\omega_1(l) \leq \omega_1(t) - \int_t^l R_1(s)\,\omega_1(s)\,\mathrm{d}s \to -\infty \text{ as } l \to \infty,$$

which contradicts to the fact that $\omega_1(t) > 0$. Since $\omega_1(t)$ is positive and decreasing $\lim_{t\to\infty} \omega_1(t) = k \geq 0$. By virtue of (20), we have $k = 0$. Thus, from (18), we have

$$\omega_1(t) \geq \widetilde{Q}(t) + \int_t^\infty R_1(s)\,\omega_1(s)\,\mathrm{d}s = y_0(t) + \int_t^\infty R_1(s)\,\omega_1(s)\,\mathrm{d}s. \tag{21}$$

From (21), we have

$$\frac{\omega_1(t)}{y_0(t)} \geq 1 + \frac{1}{y_0(t)} \int_t^\infty R_1(s)\, y_0^{\frac{\alpha+1}{\alpha}}(s) \left(\frac{\omega_1(s)}{y_0(s)}\right)^{\frac{\alpha+1}{\alpha}} \mathrm{d}s,\ t \geq T. \tag{22}$$

If we set $\delta = \inf_{t\geq T} \omega_1(t)/y_0(t)$, then obviously $\delta \geq 1$. Hence, from (16) and (22) we see that

$$\delta \geq 1 + \alpha \left(\frac{\delta}{\alpha+1}\right)^{(\alpha+1)/\alpha}$$

or

$$\frac{\delta}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1} \left(\frac{\delta}{\alpha+1}\right)^{(\alpha+1)/\alpha}$$

which contradicts the admissible value of δ and α.

Similarly, in case (S_2), if we set $\delta_1 = \inf_{t \geq T_1} \omega(t)/z_0(t)$ and taking (17) into account, then we arrive at a contradicts with the admissible value of δ_1. Therefore, the proof is complete. □

Theorem 3. *Assume that there exist some y_n and z_n such that*

$$\limsup_{t\to\infty} y_n(t) \left(\frac{\mu}{2} t^2 \int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s \right)^{\alpha} > 1 \tag{23}$$

and

$$\limsup_{t\to\infty} t z_n(t) > 1, \tag{24}$$

hold. Then (1) is oscillatory.

Proof. Assume to the contrary that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we can assume that $x(t) > 0$. From Lemma 4 that there exist two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Let case (S_1) holds. Taking Lemma 1 into account, we arrive at

$$x(t) \geq \frac{\mu}{6} t^3 x'''(t). \tag{25}$$

From the definition of ω and (25), we have

$$\begin{aligned} \frac{1}{\omega_1(t)} &= \frac{1}{r(t)} \left(\frac{x(t)}{x'''(t)} \right)^{\alpha} \\ &\geq \frac{1}{r(t)} \left(\frac{\mu}{6} t^3 \right)^{\alpha} \end{aligned}$$

Thus,

$$\omega_1(t) \frac{1}{r(t)} \left(\frac{\mu}{6} t^3 \right)^{\alpha} \leq 1 \tag{26}$$

and

$$\limsup_{t\to\infty} \omega_1(t) \left(\frac{\mu t^3}{6 r^{1/\alpha}(t)} \right)^{\alpha} \leq 1,$$

which contradicts (23).

Similarly, in case (S_2), we arrive at a contradicts with (24). Therefore, the proof is complete. □

Corollary 1. *If there exist y_n and z_n such that*

$$\int_T^t Q(s) \exp\left(\int_T^s R_1(u)\, y_n^{1/\alpha}(u)\,\mathrm{d}u \right) \mathrm{d}s = \infty \tag{27}$$

and

$$\int_T^t \widetilde{R}(s) \exp\left(\int_T^s z_n(u)\,\mathrm{d}u \right) \mathrm{d}s = \infty, \tag{28}$$

hold, then (1) is oscillatory.

Proof. Assume to the contrary that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we can assume that $x(t) > 0$. From Lemma 4 that there exist two possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Let case (S_1) hold. Proceeding as in the proof of Theorem 2, we obtain (21). From (21), we have

$$\omega_1(t) \geq y_0(t).$$

Moreover, by induction we can also see that $\omega_1(t) \geq y_n(t)$ for $t \geq t_0$, $n = 1, 2, 3, \ldots$. Thus, since the sequence $\{y_n(t)\}_0^\infty$ monotone increasing and bounded above, it converges to $y(t)$. Letting $n \to \infty$ in (14) and using Lebesgues monotone convergence theorem, we obtain

$$y(t) = y_0(t) + \int_t^\infty R_1(s)\, y^{\frac{\alpha+1}{\alpha}}(s)\, \mathrm{d}s. \tag{29}$$

From (29), we have that

$$y'(t) = -R_1(t)\, y^{\frac{\alpha+1}{\alpha}}(t) - Q(t). \tag{30}$$

Since $y_n(t) \leq y(t)$, it follows from (30) that

$$y'(t) \leq -R_1(t)\, y_n^{1/\alpha}(t)\, y(t) - Q(t).$$

Hence, we get

$$y(t) \leq \exp\left(-\int_T^t R_1(s)\, y_n^{1/\alpha}(s)\, \mathrm{d}s\right)\left(y(T) - \int_T^t Q(s) \exp\left(\int_T^s R_1(u)\, y_n^{1/\alpha}(u)\, \mathrm{d}u\right) \mathrm{d}s\right).$$

The above inequality follows

$$\int_T^t Q(s) \exp\left(\int_T^s R_1(u)\, y_n^{1/\alpha}(u)\, \mathrm{d}u\right) \mathrm{d}s \leq y(T) < \infty,$$

which contradicts (27).

Similarly, in case (S_2), we arrive at a contradiction with (28). Therefore, the proof is complete. □

Example 1. *Consider the equation*

$$x^{(4)}(t) + \frac{q_0}{t^4} x\left(\frac{1}{2}t\right) = 0, \tag{31}$$

where $q_0 > 0$. We note that $\alpha = \beta = 1$, $r(t) = 1$, $\sigma(t) = t/2$ and $q(t) = q_0/t^4$. Hence, it is easy to see that

$$y_0 = \frac{q_0}{24t}$$

and

$$z_0(t) = \frac{q_0}{2t}.$$

Thus, by using Theorem 2, Equation (31) is oscillatory if $q_0 > 36$. However, we note that $\int^\infty Q(s)\, \mathrm{d}s \neq \infty$, and hence, Theorem 1 fails.

Remark 1. *Theorem 1 introduces a criterion in traditional form $\int^\infty (\cdot)\, \mathrm{d}s = \infty$. Howoever, Theorem 2 provides a better criterion which can be applied to different equations. While, we can use Theorem 3 if Theorem 2 fails.*

Remark 2. *Agarwal et al. [27] studied the oscillation properties of the higher-order equation*

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-1} x^{(n-1)}(t)\right)' + q(t)\, f(x(\sigma(t))) = 0,$$

under the condition (2). From Theorem 2.1 in [27], Equation (31) is oscillatory if $q_0 > 96$. Thus, our results improve the results in [27].

4. Conclusions

New criteria for oscillation of fourth-order delay differential equations are established. By employing a refinement of the Riccati transformations, we obtain new oscillation criteria that improve some related results and can be used in cases where known theorems fail to apply. By applying our results to an example, we show that our results improved the results in [27]. Furthermore, in future work, we can try to study the oscillation properties of the neutral case by the same approach as that used in this work.

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