symmetry

MDPI

Article

A New Hilbert-Type Inequality with Positive Homogeneous Kernel and Its Equivalent Forms

Bicheng Yang [1], Shanhe Wu [2,*] and Aizhen Wang [3]

1 Institute of Applied Mathematics, Longyan University, Longyan 364012, Fujian, China; bcyang@gdei.edu.cn
2 Department of Mathematics, Longyan University, Longyan 364012, Fujian, China
3 Department of Mathematics, Guangdong University of Education, Guangzhou 510303, Guangdong, China; ershimath@163.com
* Correspondence: shanhewu@gmail.com

Received: 1 February 2020; Accepted: 24 February 2020; Published: 28 February 2020

Abstract: We establish a new inequality of Hilbert-type containing positive homogeneous kernel $(\min\{m,n\})^{\lambda}$ and derive its equivalent forms. Based on the obtained Hilbert-type inequality, we discuss its equivalent forms and give the operator expressions in some particular cases.

Keywords: Hilbert-type inequality; homogeneous kernel; equivalent statement; operator expression; Euler–Maclaurin summation formula

MSC: 26D15; 26D10; 26A42

1. Introduction

If $a_m \geq 0,\ b_n \geq 0,\ 0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\Big(\sum_{m=1}^{\infty} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is the celebrated Hilbert's inequality (see [1]). Inequality (1) was generalized by Hardy as follows:

If $p > 1,\ \frac{1}{p}+\frac{1}{q} = 1,\ a_m,\ b_n \geq 0,\ 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\Big(\sum_{m=1}^{\infty} a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} b_n^q\Big)^{\frac{1}{q}} \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy–Hilbert's inequality (c.f. [1], Theorem 315).

The following analogue of Hardy–Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\Big(\sum_{m=1}^{\infty} a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} b_n^q\Big)^{\frac{1}{q}} \tag{3}$$

is known in the literature as Hardy–Littlewood–Polya's inequality, and the constant factor pq in (3) is the best possible (c.f. [1], Theorem 341).

In 2006, Krnić and Pečarić [2] presented an extension of inequality (1) by introducing parameters λ_1 and λ_2 as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}} \tag{4}$$

where $\lambda_i \in (0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$,

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\ (u,v>0)$$

is the beta function, in (4) the constant factor $B(\lambda_1,\lambda_2)$ is the best possible.

For $\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (4) reduces to inequality (2); for $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, inequality (4) reduces to Yang's inequality given in [3]. It is well known that inequalities (1–3) and their integral analogues play an important role in analysis and its applications (see [4–14]).

Recently, by applying inequality (3), Adiyasuren, Batbold and Azar [15] gave a new Hilber-type inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ and partial sums.

In 2016, Hong and Wen [16] studied the equivalent statements of the extended inequalities (1) and (2), and estimated the best possible constant factor for several parameters.

The results proposed in [2,15,16] have greatly attracted our interest. In 2019, Yang, Wu and Wang [17] established the following Hardy–Hilbert-type inequality and discussed its equivalent forms

$$\begin{aligned}&\int_0^{\infty}\sum_{n=1}^{\infty}\frac{f(x)a_n}{(x+n)^{\lambda}}dx > B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)\\ &\times\Big\{\int_0^{\infty}(1-\rho_{\sigma}(x))x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\Big\}^{\frac{1}{q}},\end{aligned} \tag{5}$$

where $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,5]$, $\sigma\in(0,2]\cap(0,\lambda)$, $\mu\in(0,\lambda)$,

$$\rho_{\sigma}(x)=\frac{(1+\theta_x)^{-\lambda}}{\sigma B(\sigma,\lambda-\sigma)}\frac{1}{x^{\sigma}}=O(\frac{1}{x^{\sigma}})\in(0,1)\ (\theta_x\in(0,\frac{1}{x});x>0),$$

$$f(x)\geq 0,\ x\in(0,\infty),\ a_n\geq 0$$

In a recent paper [18], Yang, Wu and Liao gave an extension of Hardy–Hilbert's inequality for $k_{\lambda}(\lambda_i)=\frac{\pi}{\lambda\sin(\pi\lambda_i/\lambda)}$ (i $=1,2$), as follows:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\\ &\times\{\sum_{m=1}^{\infty}m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\}^{\frac{1}{q}},\end{aligned} \tag{6}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,\frac{5}{2}]$, $\lambda_i\in(0,\frac{5}{4}]\cap(0,\lambda)$ $(i=1,2)$ $a_m,\ b_n\geq 0$.

For more results related to the extensions of inequalities (1) and (2) and their equivalent statements, we refer the reader to [19–24] and references cited therein.

Motivated by the ideas of [2] and [16], in the present paper we deal with a new Hilbert-type inequality containing positive homogeneous kernel $(\min\{m,n\})^{\lambda}$ and deduce its equivalent forms. Furthermore, we discuss the equivalent statements relating to the best possible constant factor, based on the obtained Hilbert-type inequality.

2. Some Lemmas

In what follows, we suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0, \frac{34}{11}]$, $\lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda)$ $(i = 1, 2)$, a_m, $b_n \geq 0$ $(m, n \in \mathrm{N} = \{1, 2, \dots\})$ such that

$$0 < \sum_{m=1}^{\infty} m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q < \infty.$$

Lemma 1. *Define the weight coefficient with positive homogeneous kernel*

$$\varpi_\lambda(\lambda_1, m) := \frac{1}{m^{\lambda_1}} \sum_{n=1}^{\infty} \frac{(\min\{m, n\})^\lambda}{n^{\lambda-\lambda_1+1}} \ (m \in \mathrm{N})$$

Then, we have the following inequalities

$$k_\lambda(\lambda_1)(1 - \frac{\lambda - \lambda_1}{\lambda m^{\lambda_1}}) < \varpi_\lambda(\lambda_1, m) < k_\lambda(\lambda_1) := \frac{\lambda}{\lambda_1(\lambda - \lambda_1)} \ (m \in \mathrm{N}) \tag{7}$$

Proof. For fixed $m \in \mathrm{N}$, we define a function $g_m(t) := \frac{(\min\{m,t\})^\lambda}{t^{\lambda-\lambda_1+1}} (t > 0)$, and obtain

$$g_m(t) = \begin{cases} t^{\lambda_1-1}, \ 0 < t < m, \\ \frac{m^\lambda}{t^{\lambda-\lambda_1+1}}, \ t \geq m \end{cases}, \quad g'_m(t) = \begin{cases} (\lambda_1 - 1)t^{\lambda_1-2}, \quad 0 < t < m, \\ -(\lambda - \lambda_1 + 1)\frac{m^\lambda}{t^{\lambda-\lambda_1+2}}, \quad t > m \end{cases},$$

$g_m(1) = 1$, and $\int_0^1 g_m(t)dt = \int_0^1 \frac{t^\lambda}{t^{\lambda-\lambda_1+1}}dt = \frac{1}{\lambda_1}$.

To prove the inequalities in (7), we consider two cases below:

(i) For $\lambda_1 \in (0, 1] \cap (0, \lambda)$, it is easy to observe that $g_m(t)$ is decreasing in $(0, \infty)$, and strictly decreasing in $[m, \infty)$. By following the decreasing property of the series, we find

$$\varpi_\lambda(\lambda_1, m) < \frac{1}{m^{\lambda_1}} \int_0^\infty \frac{(\min\{m, t\})^\lambda dt}{t^{\lambda-\lambda_1+1}} = \frac{1}{m^{\lambda_1}}[\int_0^m \frac{t^\lambda dt}{t^{\lambda-\lambda_1+1}} + \int_m^\infty \frac{m^\lambda dt}{t^{\lambda-\lambda_1+1}}] = k_\lambda(\lambda_1),$$

$$\begin{aligned} \varpi_\lambda(\lambda_1, m) \ &> \frac{1}{m^{\lambda_1}} \int_1^\infty \frac{(\min\{m,t\})^\lambda dt}{t^{\lambda-\lambda_1+1}} = \frac{1}{m^{\lambda_1}} \int_0^\infty \frac{(\min\{m,t\})^\lambda dt}{t^{\lambda-\lambda_1+1}} - \frac{1}{m^{\lambda_1}} \int_0^1 \frac{t^\lambda dt}{t^{\lambda-\lambda_1+1}} \\ &= k_\lambda(\lambda_1) - \frac{1}{\lambda_1 m^{\lambda_1}} = k_\lambda(\lambda_1)(1 - \frac{\lambda-\lambda_1}{\lambda m^{\lambda_1}}), \end{aligned}$$

which implies the required inequalities in (7).

(ii) For $\lambda_1 \in (1, \frac{11}{8}] \cap (0, \lambda)$, by using the Euler–Maclaurin summation formula (c.f. [2,3]) with the Bernoulli function of 1-order $\rho(t) := t - [t] - \frac{1}{2}$, we obtain

$$\begin{aligned} \sum_{n=2}^{m} g_m(n) \ &= \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m + \int_1^m \rho(t)g'_m(t)dt \\ &= \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m + (\lambda_1 - 1)\int_1^m \rho(t)t^{\lambda_1-2}dt \\ &= \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m + (\lambda_1 - 1)\tfrac{\widetilde{\varepsilon}}{12}t^{\lambda_1-2}|_1^m \\ &\leq \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m (1 < \lambda_1 < 2, 0 < \widetilde{\varepsilon} < 1), \end{aligned}$$

$$\begin{aligned} \sum_{n=m+1}^{\infty} g_m(n) \ &= \int_m^\infty g_m(t)dt + \tfrac{1}{2}g_m(t)|_m^\infty + \int_m^\infty \rho_1(t)g'_m(t)dt \\ &= \int_m^\infty g_m(t)dt + \tfrac{1}{2}g_m(t)|_m^\infty + \tfrac{\lambda_1-\lambda-1}{12}m^\lambda \varepsilon_1 t^{\lambda_1-\lambda-2}|_m^\infty \\ &< \int_m^\infty g_m(t)dt + \tfrac{1}{2}g_m(t)|_m^\infty + \tfrac{\lambda-\lambda_1+1}{12m^{2-\lambda_1}} \ (\lambda_1 < \lambda, 0 < \varepsilon_1 < 1), \end{aligned}$$

and then one has

$$\begin{aligned}\sum_{n=1}^{\infty} g_m(n) &< \int_1^{\infty} g_m(t)dt + \tfrac{1}{2}g_m(1) + \tfrac{\lambda-\lambda_1+1}{12m^{2-\lambda_1}} \\ &= \int_0^{\infty} g_m(t)dt - h_m(\lambda,\lambda_1),\end{aligned}$$

where $h(\lambda_1) := 12-(7+\lambda)\lambda_1+\lambda_1^2$ and

$$\begin{aligned}h_m(\lambda,\lambda_1) := \int_0^1 g_m(t)dt - \tfrac{1}{2}g_m(1) - \tfrac{\lambda-\lambda_1+1}{12m^{2-\lambda_1}} = \tfrac{1}{\lambda_1} - \tfrac{1}{2} - \tfrac{\lambda-\lambda_1+1}{12m^{2-\lambda_1}} \geq \tfrac{1}{\lambda_1} - \tfrac{1}{2} - \tfrac{\lambda-\lambda_1+1}{12} \\ = \tfrac{h(\lambda_1)}{12\lambda_1}.\end{aligned}$$

Since $h'(\lambda_1) = -(7+\lambda)+2\lambda_1 < 0$ $(\lambda_1 \in (1, \frac{11}{8}],\ \lambda \in (0, \frac{34}{11}])$, it follows that

$$h_m(\lambda,\lambda_1) > \frac{h(\lambda_1)}{12\lambda_1} \geq \frac{12-(7+\lambda)\times(\frac{11}{8})+(\frac{11}{8})^2}{12\lambda_1} = \frac{273-88\lambda}{768\lambda_1} > 0 \ \ (\lambda \in (0, \frac{34}{11}])$$

Thus, we get

$$\varpi_\lambda(\lambda_1,m) = \frac{1}{m^{\lambda_1}}\sum_{n=1}^{\infty} g_m(n) < \frac{1}{m^{\lambda_1}}\int_0^{\infty} g_m(t)dt = k_\lambda(\lambda_1) = \frac{\lambda}{\lambda_1(\lambda-\lambda_1)}.$$

On the other hand, we have

$$\begin{aligned}\sum_{n=2}^{m} g_m(n) &= \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m + (\lambda_1-1)\tfrac{\widetilde{\varepsilon}}{12}t^{\lambda_1-2}|_1^m \\ &\geq \int_1^m g_m(t)dt + \tfrac{1}{2}g_m(t)|_1^m + \tfrac{\lambda_1-1}{12}(m^{\lambda_1-2}-1),\end{aligned}$$

$$\begin{aligned}\sum_{n=m+1}^{\infty} g_m(n) &= \int_m^{\infty} g_m(t)dt + \tfrac{1}{2}g_m(t)|_m^{\infty} + \tfrac{\lambda_1-\lambda-1}{12}m^{\lambda}\varepsilon_1 t^{\lambda_1-\lambda-2}|_m^{\infty} \\ &> \int_m^{\infty} g_m(t)dt + \tfrac{1}{2}g_m(t)|_m^{\infty},\end{aligned}$$

and then by $\frac{1}{2} - \frac{\lambda_1-1}{12} > \frac{1}{2} - \frac{1}{12} > 0$ $(\lambda_1 < 2)$, we find

$$\begin{aligned}\sum_{n=1}^{\infty} g_m(n) &> \int_1^{\infty} g_m(t)dt + \tfrac{1}{2}g_m(1) + \tfrac{\lambda_1-1}{12}(m^{\lambda_1-2}-1) \\ &> \int_1^{\infty} g_m(t)dt + (\tfrac{1}{2} - \tfrac{\lambda_1-1}{12}) > \int_0^{\infty} g_m(t)dt - \int_0^1 g_m(t)dt.\end{aligned}$$

Hence, from the expression $g_m(t)$ we deduce the inequalities in (7). The proof of Lemma 1 is thus complete. □

Next, we shall establish a new inequality of Hilbert type for positive homogeneous kernel.

Lemma 2. *The following Hilbert-type inequality holds true:*

$$\begin{aligned}I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\min\{m,n\})^{\lambda} a_m b_n < k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) \\ \times [\sum_{m=1}^{\infty} m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q]^{\frac{1}{q}}\end{aligned} \tag{8}$$

Proof. Following the pattern in which the proof of Lemma 1 was obtained, for $n \in \mathbf{N}$, $\lambda \in (0, \frac{34}{11}]$, $\lambda_2 \in (0, \frac{11}{8}] \cap (0, \lambda)$, we have the following inequality:

$$k_\lambda(\lambda_2)(1 - \frac{\lambda - \lambda_2}{\lambda n^{\lambda_2}}) < \omega(\lambda_2, n) := \frac{1}{n^{\lambda_2}} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^\lambda}{m^{\lambda - \lambda_2 + 1}} < k_\lambda(\lambda_2). \tag{9}$$

Using the Hölder's inequality (see [25]), we obtain

$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m,n\})^\lambda [\frac{m^{(\lambda-\lambda_2+1)/q}}{n^{(\lambda-\lambda_1+1)/p}} a_m][\frac{n^{(\lambda-\lambda_1+1)/p}}{m^{(\lambda-\lambda_2+1)/q}} b_n]$$

$$\leq [\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\min\{m,n\})^\lambda \frac{m^{(\lambda-\lambda_2+1)(p-1)}}{n^{\lambda-\lambda_1+1}} a_m^p]^{\frac{1}{p}}$$

$$\times [\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m,n\})^\lambda \frac{n^{(\lambda-\lambda_1+1)(q-1)}}{m^{\lambda-\lambda_2+1}} b_n^q]^{\frac{1}{q}}$$

$$= [\sum_{m=1}^{\infty} \varpi(\lambda_1, m) m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} w(\lambda_2, n) n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q]^{\frac{1}{q}}.$$

Hence, by using the inequalities in (7) and (9), we derive inequality (8). This completes the proof of Lemma 2. □

As a consequence of Lemma 2, we can deduce the following Hilbert-type inequality for the positive homogeneous kernel.

Remark 1. *By inequality (8), for* $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}](\subset (0, \frac{34}{11}])$, $\lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda)$ $(i = 1, 2)$ *we obtain*

$$0 < \sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} a_m^p < \infty, \ 0 < \sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} b_n^q < \infty$$

and the following inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m,n\})^\lambda a_m b_n < \frac{\lambda}{\lambda_1 \lambda_2} [\sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} b_n^q]^{\frac{1}{q}} \tag{10}$$

In Lemma 3 below, we show that the constant factor given in (10) is the best possible.

Lemma 3. *For* $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}]$, $\lambda_i \subset (0, \frac{11}{8}] \cap (0, \lambda)$ $(i = 1, 2)$, *the constant factor* $\frac{\lambda}{\lambda_1 \lambda_2}$ *in (10) is the best possible.*

Proof. For any $0 < \varepsilon < q\lambda_1$, we set

$$\widetilde{a}_m := m^{-\lambda_1 - \frac{\varepsilon}{p} - 1}, \ \widetilde{b}_n := n^{-\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (m, n \in \mathrm{N})$$

If there exists a constant $M \leq \frac{\lambda}{\lambda_1 \lambda_2}$ such that (10) is valid when replacing $\frac{\lambda}{\lambda_1 \lambda_2}$ by M, then in particular, by substitution of $a_m = \widetilde{a}_m$ *and* $b_n = \widetilde{b}_n$ in (10), we have

$$\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m,n\})^\lambda \widetilde{a}_m \widetilde{b}_n < M[\sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} \widetilde{a}_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} \widetilde{b}_n^q]^{\frac{1}{q}} \tag{11}$$

In the following, we shall prove that $M \geq \frac{\lambda}{\lambda_1\lambda_2}$, which would reveal that $M = \frac{\lambda}{\lambda_1\lambda_2}$ is the best possible constant factor in (10).

By inequality (11) and the decreasing property of the series, we obtain

$$\begin{aligned}\widetilde{I} &< M[\sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} m^{-p\lambda_1-\varepsilon-p}]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1+\lambda_2)-1} n^{-q\lambda_2-\varepsilon-q}]^{\frac{1}{q}} \\ &= M(1+\sum_{m=2}^{\infty} m^{-\varepsilon-1})^{\frac{1}{p}} (1+\sum_{n=2}^{\infty} n^{-\varepsilon-1})^{\frac{1}{q}} \\ &< M(1+\int_1^{\infty} x^{-\varepsilon-1}dx)^{\frac{1}{p}} (1+\int_1^{\infty} y^{-\varepsilon-1}dy)^{\frac{1}{q}} = \frac{M}{\varepsilon}(\varepsilon+1).\end{aligned}$$

By inequalities in (9) and setting

$$\hat{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{q} \in (0, \frac{11}{8}) \cap (0,\lambda)\ (0 < \hat{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{q} = \lambda - \hat{\lambda}_1 < \lambda),$$

we obtain

$$\begin{aligned}\widetilde{I} &= \sum_{m=1}^{\infty} [m^{-(\lambda_1-\frac{\varepsilon}{q})} \sum_{n=1}^{\infty} (\min\{m,n\})^{\lambda} n^{-(\lambda_2+\frac{\varepsilon}{q})-1}] m^{-\varepsilon-1} \\ &= \sum_{m=1}^{\infty} \varpi(\hat{\lambda}_1, m) m^{-\varepsilon-1} > \frac{\lambda}{\hat{\lambda}_1\hat{\lambda}_2} \sum_{m=1}^{\infty} (1 - \frac{\hat{\lambda}_2}{\lambda m^{\hat{\lambda}_1}}) m^{-\varepsilon-1} \\ &= \frac{\lambda}{\hat{\lambda}_1\hat{\lambda}_2} (\sum_{m=1}^{\infty} m^{-\varepsilon-1} - \frac{\hat{\lambda}_2}{\lambda} \sum_{m=1}^{\infty} \frac{1}{m^{\lambda_1+\frac{\varepsilon}{p}+1}}) > \frac{\lambda}{\hat{\lambda}_1,\hat{\lambda}_2} (\int_1^{\infty} x^{-\varepsilon-1}dx - O(1)) \\ &= \frac{\lambda}{\varepsilon\hat{\lambda}_1\hat{\lambda}_2}(1 - \varepsilon O(1)).\end{aligned}$$

Then, we have

$$\frac{\lambda}{(\lambda_1 - \frac{\varepsilon}{q})(\lambda_2 + \frac{\varepsilon}{q})}(1 - \varepsilon O(1)) < \varepsilon \widetilde{I} < M(\varepsilon + 1).$$

Taking $\varepsilon \to 0^+$, we deduce that $\frac{\lambda}{\lambda_1\lambda_2} \leq M$. Hence, $M = \frac{\lambda}{\lambda_1,\lambda_2}$ is the best possible constant factor in (10). Lemma 3 is thus proven. □

Setting $\widetilde{\lambda}_1 := \frac{\lambda_1}{p} + \frac{\lambda-\lambda_2}{q}$, $\widetilde{\lambda}_2 := \frac{\lambda_2}{q} + \frac{\lambda-\lambda_1}{p}$, we find $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, and then we can reduce inequality (8) to the following:

$$\begin{aligned}I = &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (\min\{m,n\})^{\lambda} a_m b_n < k_{\lambda}^{\frac{1}{p}}(\lambda_1) k_{\lambda}^{\frac{1}{q}}(\lambda_2) \\ &\times [\sum_{m=1}^{\infty} m^{p(1+\widetilde{\lambda}_1)-1} a_m^p]^{\frac{1}{p}} [\sum_{n=1}^{\infty} n^{q(1+\widetilde{\lambda}_2)-1} b_n^q]^{\frac{1}{q}}\end{aligned} \quad (12)$$

It is worth noting that inequality (12) is an analogue of the Hilbert-type inequality (8). In the following lemma, we present a relation between the parameters λ, λ_1 and λ_2 on the best possible constant factor in inequality (12).

Lemma 4. *If inequality (12) has the best possible constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_1) k_{\lambda}^{\frac{1}{q}}(\lambda_2)$ for various parameters, then $\lambda = \lambda_1 + \lambda_2$.*

Proof. From the assumption conditions of inequality (12), it follows that

$$\widetilde{\lambda}_1 = \frac{\lambda_1}{p} + \frac{\lambda - \lambda_2}{q} > 0,\ \widetilde{\lambda}_1 < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,\ 0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 < \lambda.$$

Hence, we have

$$k_{\lambda}(\widetilde{\lambda}_1) = \frac{\lambda}{\widetilde{\lambda}_1(\lambda - \widetilde{\lambda}_1)} = \frac{\lambda}{\widetilde{\lambda}_1\widetilde{\lambda}_2} \in \mathrm{R}_+ = (0, \infty)$$

If the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2)$ in (12) is the best possible, then in view of inequality (10), we have

$$k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2) \leq k_{\lambda}(\widetilde{\lambda}_1)$$

By Hölder's inequality with weight, we find

$$\begin{aligned} & k_{\lambda}(\widetilde{\lambda}_1) = k_{\lambda}(\tfrac{\lambda_1}{p} + \tfrac{\lambda-\lambda_2}{q}) \\ &= \int_0^{\infty} (\min\{1,u\})^{\lambda} u^{-(\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} du = \int_0^{\infty} (\min\{1,u\})^{\lambda} (u^{\frac{-\lambda_1-1}{p}})(u^{\frac{-\lambda+\lambda_2-1}{q}}) du \\ &\leq [\int_0^{\infty} (\min\{1,u\})^{\lambda} u^{-\lambda_1-1} du]^{\frac{1}{p}} [\int_0^{\infty} (\min\{1,u\})^{\lambda} u^{-\lambda+\lambda_2-1} du]^{\frac{1}{q}} \\ &= [\int_0^{\infty} (\min\{1,u\})^{\lambda} u^{-\lambda_1-1} du]^{\frac{1}{p}} [\int_0^{\infty} (\min\{1,v\})^{\lambda} v^{-\lambda_2-1} dv]^{\frac{1}{q}} \\ &= k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2). \end{aligned} \tag{13}$$

It follows that $k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2) = k_{\lambda}(\widetilde{\lambda}_1)$, and thus (13) keeps the form of equality.

It is easy to see that (13) keeps the form of equality if, and only if, there exist constants A and B (not all zero) such that (c.f. [25])

$$Au^{-\lambda_1-1} = Bu^{-(\lambda-\lambda_2)-1} \quad a.e. \quad \text{in } R_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ *a.e.* in R_+, and this yields $\lambda - \lambda_2 - \lambda_1 = 0$, hence $\lambda = \lambda_1 + \lambda_2$. The proof of Lemma 4 is thus complete. □

3. Main Results and Some Particular Cases

Theorem 1. *Inequality (8) is equivalent to the following inequality:*

$$\begin{aligned} J: &= \left\{ \sum_{n=1}^{\infty} n^{-p(\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} [\sum_{m=1}^{\infty} (\min\{m,n\})^{\lambda} a_m]^p \right\}^{\frac{1}{p}} \\ &< k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2)[\sum_{m=1}^{\infty} m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p]^{\frac{1}{p}}. \end{aligned} \tag{14}$$

If the constant factor in (8) is the best possible, then so is the constant factor in (14).

Proof. Suppose that inequality (14) is valid. By Hölder's inequality (c.f. [25]), we have

$$I = \sum_{n=1}^{\infty} [n^{\frac{-1}{p}-\frac{\lambda_2}{q}-\frac{\lambda-\lambda_1}{p}}) \sum_{m=1}^{\infty} (\min\{m,n\})^{\lambda} a_m][n^{\frac{1}{p}+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p}} b_n] \leq J[\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q]^{\frac{1}{q}} \tag{15}$$

Then, by using inequality (14), we obtain inequality (8).

On the other hand, assuming that inequality (8) is valid, we set

$$b_n := n^{-p(\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} [\sum_{m=1}^{\infty} (\min\{m,n\})^{\lambda} a_m]^{p-1}, n \in \mathbf{N}$$

If $J = 0$, then inequality (14) is naturally valid; if $J = \infty$, then it is impossible to make inequality (14) valid, which implies $J < \infty$. Suppose that $0 < J < \infty$. By inequality (8), we have

$$\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q = J^p = I < k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) \times [\sum_{m=1}^{\infty} m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p]^{\frac{1}{p}}$$
$$[\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q]^{\frac{1}{q}},$$

$$J = [\sum_{n=1}^{\infty} n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1} b_n^q]^{\frac{1}{p}} < k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) [\sum_{m=1}^{\infty} m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1} a_m^p]^{\frac{1}{p}}$$

Thus, inequality (14) follows, and we conclude that inequality (8) is equivalent to inequality (14).

Furthermore, we show that if the constant factor in (8) is the best possible, then the constant factor in (14) is also the best possible. Otherwise, from inequality (15) we would reach a contradiction, namely that the constant factor in (8) is not the best possible. The proof of Theorem 1 is thus completed. □

In the following theorem, we give some equivalent statements of the best possible constant factor related to several parameters.

Theorem 2. *The statements (i), (ii), (iii) and (iv) below are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ *is independent of* p, q*;*

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ *is expressible as a single integral*

$$k_\lambda(\hat{\lambda}_1) = \int_0^\infty (\min\{1,u\})^\lambda u^{-\hat{\lambda}_1-1} du \ (0 < \hat{\lambda}_1 < \lambda)$$

(iii) $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ *in (8) is the best possible constant factor;*

(iv) $\lambda = \lambda_1 + \lambda_2 \ \ (\in (0, \frac{11}{4}])$.

If the statement (iv) is valid, namely, $\lambda = \lambda_1 + \lambda_2 \in (0, \frac{11}{4}]$*, then we have inequality (10) and the following equivalent inequality with the best possible constant factor* $\frac{\lambda}{\lambda_1\lambda_2}$*:*

$$\left\{\sum_{n=1}^{\infty} \frac{1}{n^{p\lambda_2+1}} [\sum_{m=1}^{\infty} (\min\{m,n\})^\lambda a_m]^p\right\}^{\frac{1}{p}} < \frac{\lambda}{\lambda_1\lambda_2} [\sum_{m=1}^{\infty} m^{p(1+\lambda_1)-1} a_m^p]^{\frac{1}{p}} \tag{16}$$

Proof. (i) ⇒ (ii). By (i), we have

$$k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) = \lim_{p\to 1^+} \lim_{q\to\infty} k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) = k_\lambda(\lambda_1).$$

Namely, $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ is expressible as a single integral

$$k_\lambda(\lambda_1) = \int_0^\infty (\min\{1,u\})^\lambda u^{-\lambda_1-1} du \ (0 < \lambda_1 < \lambda)$$

(ii) ⇒ (iv). If $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ is expressible as a single integral

$$k_\lambda(\hat{\lambda}_1) = \int_0^\infty (\min\{1,u\})^\lambda u^{-\hat{\lambda}_1-1} du \ (0 < \hat{\lambda}_1 < \lambda)$$

then for $\hat{\lambda}_1 = \widetilde{\lambda}_1 \ (\in (0,\lambda))$, (13) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda = \lambda_1 + \lambda_2$.

(iv) ⇒ (i). If $\lambda = \lambda_1 + \lambda_2$, then $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) = k_\lambda(\lambda_1)$, which is independent of p, q. Thus, we deduce that (i) ⇔ (ii) ⇔ (iv).

(iii) ⇒ (iv). By Lemma 4, we get $\lambda = \lambda_1 + \lambda_2$.

(iv) ⇒ (iii). By Lemma 3, for $\lambda = \lambda_1 + \lambda_2$, $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ $(= \frac{\lambda}{\lambda_1 \lambda_2})$ is the best possible constant factor in (8). It follows that (iii) ⇔ (iv).

Therefore, we assert that the statements (i), (ii), (iii) and (iv) are equivalent. This completes the proof of Theorem 2. □

Now, we discuss some particular cases of the inequalities obtained above, from which we will derive some interesting inequalities.

Remark 2. *(i) Putting $\lambda = 1,\ \lambda_1 = \frac{1}{q},\ \lambda_2 = \frac{1}{p}$ in (10) and (16), we obtain the following equivalent inequalities with the best possible constant factor pq:*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min\{m,n\} a_m b_n < pq \Big[\sum_{m=1}^{\infty} m^{2(p-1)} a_m^p\Big]^{\frac{1}{p}} \Big[\sum_{n=1}^{\infty} n^{2(q-1)} b_n^q\Big]^{\frac{1}{q}} \tag{17}$$

$$\Big[\sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\sum_{m=1}^{\infty} \min\{m,n\} a_m\Big)^p\Big]^{\frac{1}{p}} < pq \Big[\sum_{m=1}^{\infty} m^{2(p-1)} a_m^p\Big]^{\frac{1}{p}} \tag{18}$$

(ii) Putting $\lambda = 1,\ \lambda_1 = \frac{1}{p},\ \lambda_2 = \frac{1}{q}$ in (10) and (16), we get the following equivalent inequalities with the best possible constant factor pq:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min\{m,n\} a_m b_n < pq \Big[\sum_{m=1}^{\infty} (m a_m)^p\Big]^{\frac{1}{p}} \Big[\sum_{n=1}^{\infty} (n b_n)^q\Big]^{\frac{1}{q}} \tag{19}$$

$$\Big[\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{m=1}^{\infty} \min\{m,n\} a_m\Big)^p\Big]^{\frac{1}{p}} < pq \Big[\sum_{m=1}^{\infty} (m a_m)^p\Big]^{\frac{1}{p}} \tag{20}$$

(iii) Setting $p = q = 2$, both (17) and (19) reduce to the inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \min\{m,n\} a_m b_n < 4 \Big[\sum_{m=1}^{\infty} (m a_m)^2 \sum_{n=1}^{\infty} (n b_n)^2\Big]^{\frac{1}{2}} \tag{21}$$

furthermore, both (18) and (20) reduce to the equivalent form of (21) as follows:

$$\Big[\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{m=1}^{\infty} \min\{m,n\} a_m\Big)^2\Big]^{\frac{1}{2}} < 4 \Big[\sum_{m=1}^{\infty} (m a_m)^2\Big]^{\frac{1}{2}} \tag{22}$$

(iv) Putting $\lambda = 2,\ \lambda_1 = \lambda_2 = 1$ in (10) and (16), we have the following equivalent inequalities with the best possible constant factor 2:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\min\{m,n\})^2 a_m b_n < 2 \Big(\sum_{m=1}^{\infty} m^{2p-1} a_m^p\Big)^{\frac{1}{p}} \Big(\sum_{n=1}^{\infty} n^{2q-1} b_n^q\Big)^{\frac{1}{q}} \tag{23}$$

$$\{\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}[\sum_{m=1}^{\infty}(\min\{m,n\})^{2}a_{m}]^{p}\}^{\frac{1}{p}}<2(\sum_{m=1}^{\infty}m^{2p-1}a_{m}^{p})^{\frac{1}{p}} \tag{24}$$

(v) Putting $\lambda = e\ (< \frac{11}{4} = 2.75)$, $\lambda_1 = \lambda_2 = \frac{e}{2}$ in (10) and (16), we have the following equivalent inequalities with the best possible constant factor $\frac{4}{e}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\min\{m,n\})^{e}a_{m}b_{n}<\frac{4}{e}[\sum_{m=1}^{\infty}m^{p(1+\frac{e}{2})-1}a_{m}^{p}]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1+\frac{e}{2})-1}b_{n}^{q}]^{\frac{1}{q}} \tag{25}$$

$$\left\{\sum_{n=1}^{\infty}\frac{1}{n^{\frac{e}{2}p+1}}[\sum_{m=1}^{\infty}(\min\{m,n\})^{e}a_{m}]^{p}\right\}^{\frac{1}{p}}<\frac{4}{e}[\sum_{m=1}^{\infty}m^{p(1+\frac{e}{2})-1}a_{m}^{p}]^{\frac{1}{p}} \tag{26}$$

4. Operator Expressions

We choose the functions

$$\phi(m) := m^{p(1+\frac{\lambda_1}{p}+\frac{\lambda-\lambda_2}{q})-1},\ \psi(n) := n^{q(1+\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1},$$

where from,

$$\psi^{1-p}(n) = n^{-p(\frac{\lambda_2}{q}+\frac{\lambda-\lambda_1}{p})-1}\quad (m,n\in \mathrm{N})$$

We define the following real normed spaces:

$$l_{p,\varphi} := \{a=\{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} := (\sum_{m=1}^{\infty}\varphi(m)|a_m|^{p})^{\frac{1}{p}}<\infty\},$$
$$l_{q,\psi} := \{b=\{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} := (\sum_{n=1}^{\infty}\psi(n)\big|b_n|^{q})^{\frac{1}{q}}<\infty\},$$
$$l_{p,\psi^{1-p}} := \{c=\{c_n\}_{n=1}^{\infty}; ||c||_{p,\psi^{1-p}} := (\sum_{n=1}^{\infty}\psi^{1-p}(n)|c_n|^{p})^{\frac{1}{p}}<\infty\}.$$

We let $a \in l_{p,\varphi}$, and set

$$c=\{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty}(\min\{m,n\})^{\lambda}a_m, n\in \mathrm{N}.$$

Then, we can rewrite inequality (14) as follows:

$$||c||_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_1)k_{\lambda}^{\frac{1}{q}}(\lambda_2)||a||_{p,\varphi}<\infty,\ \text{that is } c\in l_{p,\psi^{1-p}}.$$

Definition 1. *Define a Hilbert-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:*

$$(Ta,b) := \sum_{n=1}^{\infty}[\sum_{m=1}^{\infty}(\min\{m,n\})^{\lambda}a_m]b_n$$

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}$$

Then, by Theorems 1 and 2, we obtain the operator expressions of inequalities (8) and (14) as follows:

Theorem 3. *If* $a \in l_{p,\phi}$, $b \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, *then we have the following inequalities:*

$$(Ta, b) < k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) ||a||_{p,\varphi} ||b||_{q,\psi}, \tag{27}$$

$$||Ta||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2) ||a||_{p,\varphi} \tag{28}$$

Furthermore, $\lambda_1 + \lambda_2 = \lambda$ $(\in (0, \frac{11}{4}])$ *if, and only if, the constant factor* $k_\lambda^{\frac{1}{p}}(\lambda_1) k_\lambda^{\frac{1}{q}}(\lambda_2)$ *in (27) and (28) is the best possible, namely,*

$$\left\|T\right\| = k_\lambda(\lambda_1) = \frac{\lambda}{\lambda_1 \lambda_2} \ (\lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda),\ i = 1, 2) \tag{29}$$

5. Conclusions

In this paper, we give, with Lemma 2 and Theorem 1, respectively, a new inequality of the Hilbert-type containing positive homogeneous kernel and its equivalent forms. Based on the obtained Hilbert-type inequality, we discuss in Theorem 2 the equivalent statements of the best possible constant factor related to several parameters. As applications, the operator expressions of the obtained inequalities are given in Theorem 3, and some particular cases of the obtained inequalities (10) and (16) are considered in Remark 2. It is shown that the results obtained in Theorems 1 and 2 would generate more new inequalities of Hilbert-type.

Author Contributions: B.Y. carried out the mathematical studies and drafted the manuscript. S.W. and A.W. participated in the design of the study and performed the numerical analysis. All authors contributed equally and significantly in this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation (No. 61772140), and the Science and Technology Planning Project Item of Guangzhou City (No. 201707010229).

Acknowledgments: The authors are grateful to the reviewers for their valuable comments and suggestions to improve the quality of the manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1934.
2. Krnić, M.; Pečarić, J. Extension of Hilbert's inequality. *J. Math. Anal. Appl.* **2006**, *324*, 150–160. [CrossRef]
3. Yang, B.C. On a generalization of Hilbert double series theorem. *J. Nanjing Univ. Math.* **2001**, *18*, 145–152.
4. Yang, B.C. *The Norm of Operator and Hilbert-Type Inequalities*; Science Press: Beijing, China, 2009.
5. Krnić, M.; Pečarić, J. General Hilbert's and Hardy's inequalities. *Math. Inequalities Appl.* **2005**, *8*, 29–51. [CrossRef]
6. Perić, I.; Vuković, P. Multiple Hilbert's type inequalities with a homogeneous kernel. *Banach J. Math. Anal.* **2011**, *5*, 33–43. [CrossRef]
7. Huang, Q.L. A new extension of Hardy-Hilbert-type inequality. *J. Inequalities Appl.* **2015**, *2015*, 397. [CrossRef]
8. He, B. A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor. *J. Math. Anal. Appl.* **2015**, *431*, 889–902. [CrossRef]
9. Xu, J.S. Hardy-Hilbert's inequalities with two parameters. *Adv. Math.* **2007**, *36*, 63–76.
10. Xie, Z.T.; Zeng, Z.; Sun, Y.F. A new Hilbert-type inequality with the homogeneous kernel of degree −2. *Adv. Appl. Math. Sci.* **2013**, *12*, 391–401.
11. Zhen, Z.; Raja Rama Gandhi, K.; Xie, Z.T. A new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral. *Bull. Math. Sci. Appl.* **2014**, *7*, 9–17.
12. Xin, D.M. A Hilbert-type integral inequality with the homogeneous kernel of zero degree. *Math. Theory Appl.* **2010**, *30*, 70–74.
13. Azar, L.E. The connection between Hilbert and Hardy inequalities. *J. Inequalities Appl.* **2013**, *1*, 452. [CrossRef]

14. Adiyasuren, V.; Batbold, T.; Krnić, M. Hilbert–type inequalities involving differential operators, the best constants and applications. *Math. Inequalities Appl.* **2015**, *18*, 111–124. [CrossRef]
15. Adiyasuren, V.; Batbold, T.; Azar, L.E. A new discrete Hilbert-type inequality involving partial sums. *J. Inequalities Appl.* **2019**, *1*, 1–6. [CrossRef]
16. Hong, Y.; Wen, Y.M. A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor. *Ann. Math.* **2016**, *37*, 329–336.
17. Yang, B.C.; Wu, S.H.; Wang, A.Z. On a reverse half-discrete Hardy-Hilbert's inequality with parameters. *Mathematics* **2019**, *7*, 1054. [CrossRef]
18. Yang, B.C.; Wu, S.H.; Liao, J.Q. On a new extended Hardy-Hilbert's inequality with parameters. *Mathematics* **2020**, *8*, 73. [CrossRef]
19. Hong, Y. On the structure character of Hilbert's type integral inequality with homogeneous kernel and application. *J. Jilin Univ.* **2017**, *55*, 189–194.
20. Hong, Y.; Huang, Q.L.; Yang, B.C.; Liao, J.Q. The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non -homogeneous kernel and its applications. *J. Inequalities Appl.* **2017**, *1*, 316. [CrossRef]
21. Xin, D.M.; Yang, B.C.; Wang, A.Z. Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane. *J. Funct. Spaces* **2018**, *2018*, 2691816. [CrossRef]
22. Hong, Y.; He, B.; Yang, B.C. Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory. *J. Math. Inequalities* **2018**, *12*, 777–788. [CrossRef]
23. Huang, Z.X.; Yang, B.C. Equivalent property of a half-discrete Hilbert's inequality with parameters. *J. Inequalities Appl.* **2018**, *1*, 1–11. [CrossRef] [PubMed]
24. Wang, A.Z.; Yang, B.C.; Chen, Q. Equivalent properties of a reverse 's half-discret Hilbert's inequality. *J. Inequalities Appl.* **2019**, *2019*, 279. [CrossRef]
25. Kuang, J.C. *Applied Inequalities*; Shangdong Science and Technology Press: Jinan, China, 2004.