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New Multi-Parametrized Estimates Having *pth*-Order Differentiability in Fractional Calculus for Predominating $\hbar$-Convex Functions in Hilbert Space

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Abstract: In Hilbert space, we develop a novel framework to study for two new classes of convex function depending on arbitrary non-negative function, which is called a predominating $\hbar$-convex function and predominating quasiconvex function, with respect to η, are presented. To ensure the symmetry of data segmentation and with the discussion of special cases, it is shown that these classes capture other classes of η-convex functions, η-quasiconvex functions, strongly $\hbar$-convex functions of higher-order and strongly quasiconvex functions of a higher order, etc. Meanwhile, an auxiliary result is proved in the sense of κ-fractional integral operator to generate novel variants related to the Hermite–Hadamard type for *pth*-order differentiability. It is hoped that this research study will open new doors for in-depth investigation in convexity theory frameworks of a varying nature.

Keywords: convex function; η-convex functions; predominating convex functions; Hermite–Hadamard inequality; predominating η-quasiconvex functions

1. Introduction

The fractional behavior of real-life phenomenon is condensed by powerful tools such as fractional calculus (FC) in an accurate way. This characteristic is the principle of the expediency of derivatives with fractional-order versus integer-order models. FC has acquired a lot of interest for their utilities in distinct areas, for example, technology, porous media, image processing, and scientific demonstrating on the grounds that they are increasingly reasonable and sensible to portray numerous natural phenomena. As a consequence, FC has a solid possibility to regulate continuous issues with high proficiency. The objective of analyzing FC for the aforementioned, major analysis [1–5] had been carried out. Machado et al. [6] depicted a graph of the straightforward history of FC, especially with applications, and it has also been observed that FC can be beneficial and even proficient. Integral inequalities with applications that are nowadays very much popular among scientists for research is one of the perspectives. Inequalities have concrete application in fixed point theory and the existence of solutions for differential equations. Integral inequalities of fractional techniques appear much more commonly in several research areas and engineering applications. For instance, the nonlinear oscillation of earthquakes can be demonstrated with fractional operators [7], as well as the fluid-dynamic traffic model with fractional inequalities [8] that can dispense with the inadequacy emerging from the suppositions of continuum traffic flow.

The noteworthy scope of uses of the integral inequalities on convexity for both derivation and integration, while also maintaining the symmetry of sets and functions has been a subject of discourse for a long while. These variants had been progressed by means of various analysts [9–13]. Sarikaya et al. [14] utilized the concepts of fractional calculus for deriving a bulk of variants that essentially depend on Hermite–Hadamard inequality. Among them, most captivating inequality for a convex function is of a Hermite–Hadamard type, which can be stated as follows:

Let Λ be an interval in $\mathbb{R}$, $\mathcal{G} : \Lambda \to \mathbb{R}$ be a convex function on Λ, and $\sigma_1, \sigma_2 \in \Lambda, \sigma_1 < \sigma_2$, then we have

$$(\sigma_2 - \sigma_1)\mathcal{G}\Big(\frac{\sigma_1 + \sigma_2}{2}\Big) \leq \int\limits_{\sigma_1}^{\sigma_2} \mathcal{G}(x)dx \leq (\sigma_2 - \sigma_1)\frac{\mathcal{G}(\sigma_1) + \mathcal{G}(\sigma_2)}{2}.$$

We note that both the variants hold in the reversed direction if $\mathcal{G}$ is concave. These variants have considerable significance in the literature. Numerous researchers have broadly used the ideas of FC and attained many novel generalizations via convex functions and their refinements, see [15–17] and the references therein.

Following this tendency, we introduce two more general concepts of higher-order strongly η-convex functions which are known as the predominating $\hbar$-convex functions and predominating quasiconvex function. Several novel versions of Hermite–Hadamard inequality are established that can be utilized to describe the uniformly reflex Banach spaces. Taking into account the novel ideas, these variants are a connection of an auxiliary outcome dependent on identity which relates to FC. New outcomes are introduced and new theorems are derived. Additionally, our consequences for the new Definitions 3 and 7 in predominating $\hbar$-convex functions and predominating quasiconvex function are presented. The recently acquainted numerical estimation is used to comprehend the parallelogram laws for L^p-spaces. The new definitions are thought to open new doors of investigation toward convexity theory.

2. Related Work

The idea of strongly convex functions was contemplated and investigated by Polyak [18], which had a significant contribution to fitting most machine learning models that involve solving some sort of optimization problem and concerned areas. Strongly convex functions are helpful in determining the existence of a solution of nonlinear complementary problems, see [19]. Zu and Marcotte [20] investigated the convergence of the iterative techniques for solving variational inequalities and equilibrium problems by employing the idea of strongly convex functions. The novel and innovative application of the characterization of the inner product space was discovered by Nikodem and Pales in [21] with the help of strongly convex functions. The assembly of stochastic slope descent for the class of functions fulfilling the Polyak–Lojasiewicz condition that relies upon strongly-convex functions too as a wide scope of non-convex functions incorporating those utilized in machine learning applications [22]. Recently, Rashid et al. [23] proposed the concepts of differentiable higher-order strongly $\hbar$-convex functions. Kalsoom et al. [24] explored the higher-order strongly generalized preinvex function in a different way and presented several generalizations for two-variable quantum Simpson's-type inequalities. For more features and utilities of the strongly convex functions, see [25–32].

In [33], Varosanec discovered a class of convex functions unifies and modify numerous new concepts of classical convexity, comprising Breckner type convex functions [34], P-functions [35], Godunova–Levin type convex, and Q-functions [36,37]. We admit that this class plays a significant contribution to convexity theory and helps to define some new classes of a convex function. Therefore, a number of papers had been investigated for this class. For information, see [38,39].

3. Preliminaries

Firstly, suppose $\mathcal{K}$ be a nonempty set in a real Hilbert space $\mathcal{H}$. The inner product and norm are presented by $\langle .,. \rangle$ and $\|.\|$, respectively. Moreover, there is an arbitrary non-negative function $\hbar : (0,1) \to \mathbb{R}$ and a continuous bifunction $\eta(.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Definition 1. *([40]) A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be an η-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if*

$$\mathcal{G}(\tau x + (1-\tau)y) \leq \tau \mathcal{G}(x) + (1-\tau)\big[\mathcal{G}(x) + \eta\big(\mathcal{G}(y), \mathcal{G}(x)\big)\big] \tag{1}$$

for all $x, y \in \mathcal{K}$ and $\tau \in [0,1]$.

If $\eta(x,y) = x - y$, then the η-convex functions reduces to convex function.

Further, we mention the concept of η-convex functions which depend on arbitrary non-negative function $\hbar$. These concepts also explore several new classes of convex and η-convex functions under some specific conditions.

Definition 2. *([41]) Suppose $\hbar : J \to \mathbb{R}$ is a non-negative arbitrary function and a function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be $(\eta, \hbar)$-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if*

$$\mathcal{G}(\tau x + (1-\tau)y) \leq \hbar(\tau)\mathcal{G}(x) + \hbar(1-\tau)\big[\mathcal{G}(x) + \eta\big(\mathcal{G}(y), \mathcal{G}(x)\big)\big]$$

for all $x, y \in \mathcal{K}$ and $\tau \in [0,1]$.

Further, We demonstrate several novel classes of η-convex mappings considering arbitrary non-negative function.

Definition 3. *Suppose $\hbar : J \to \mathbb{R}$ is a non-negative arbitrary function and a function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be predominating $\hbar$-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if the inequality*

$$\mathcal{G}\big(\tau x + (1-\tau)y\big) \leq \hbar(\tau)\mathcal{G}(x) + \hbar(1-\tau)\big[\mathcal{G}(x) + \eta\big(\mathcal{G}(y), \mathcal{G}(x)\big)\big] + \mathbb{D}(x,y), \tag{2}$$

holds for all $x, y \in \mathcal{K}, \tau \in [0,1]$.

Some remarkable cases of Definition 3 are presented as follows:
(I). If we choose $\mathbb{D}(x,y) = -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho}$ for some $\mu \geq 0$ and $\varrho > 2$, then Definition 3 reduces to a new definition of a higher-order strongly η-convex function for a given arbitrary non-negative function $\hbar$.

Definition 4. *Suppose $\hbar : J \to \mathbb{R}$ be a non-negative arbitrary function and a function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be a higher-order strongly η-convex function in the sense of a continuous bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\mu \geq 0$ if the inequality*

$$\begin{aligned}\mathcal{G}\big(\tau x + (1-\tau)y\big) \quad &\leq \hbar(\tau)\mathcal{G}(x) + \hbar(1-\tau)\big[\mathcal{G}(x) + \eta\big(\mathcal{G}(y), \mathcal{G}(x)\big)\big] \\ &\quad -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho},\end{aligned}$$

holds for all $x, y \in \mathcal{K}, \tau \in [0,1]$.

(II). If we choose $\mathbb{D}(x,y) = -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho}$ along with $\hbar(\tau) = \tau$ for some $\mu \geq 0$ and $\varrho > 2$, then Definition 3 reduces to a new definition of higher-order strongly η-convex function.

Definition 5. *A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be higher-order strongly η-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ having $\mu \geq 0$ if the inequality*

$$\mathcal{G}\big(\tau x + (1-\tau)y\big) \leq \mathcal{G}(y) + \tau\eta\big(\mathcal{G}(x), \mathcal{G}(y)\big) - \mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho},$$

holds for all $x, y \in \mathcal{K}, \tau \in [0,1]$.

(III). If we choose $\mathbb{D}(x,y) = -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}(y-x)^{\varrho}, \hbar(\tau) = \tau$ along with $\varrho = 2$ for some $\mu \geq 0$ in Definition 3, then we get the definition of strongly η-convex function proposed by [27].

Definition 6. *([27]) A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ having $\mu \geq 0$ if the inequality*

$$\mathcal{G}\big(\tau x + (1-\tau)y\big) \leq \mathcal{G}(y) + \tau\eta\big(\mathcal{G}(x), \mathcal{G}(y)\big) - \mu\tau(1-\tau)\|y-x\|^{2},$$

holds for all $x, y \in \mathcal{K}, \tau \in [0,1]$.

We now introduce more a general version of strongly η-quasiconvex functions as follows:

Definition 7. *A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be predominating quasi-convex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if the inequality*

$$\mathcal{G}\big(\tau x + (1-\tau)y\big) \leq \max\Big\{\mathcal{G}(x), \big[\mathcal{G}(x) + \eta\big(\mathcal{G}(y), \mathcal{G}(x)\big)\big]\Big\} + \mathbb{D}(x,y),$$

holds for all $x, y \in \mathcal{K}$.

We now discuss some remarkable cases of Definition 7.

(I). If we choose $\mathbb{D}(x,y) = -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho}$ for some $\mu > 0$ and $\varrho > 2$, then Definition 7 reduces to a new definition of higher-order strongly η-quasiconvex function.

Definition 8. *A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be higher-order strongly η-quasiconvex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\mu \geq 0$ if the inequality*

$$\begin{aligned}\mathcal{G}\big(\tau x + (1-\tau)y\big) \leq & \max\Big\{\mathcal{G}(y), \big[\mathcal{G}(y) + \eta\big(\mathcal{G}(x), \mathcal{G}(y)\big)\big]\Big\} \\ & -\mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho},\end{aligned}$$

holds for all $x, y \in \mathcal{K}$, $\tau \in [0,1]$ and $\varrho > 2$.

Example 1. *The mapping $\mathcal{G}(x) = x^2$ is strongly η-quasiconvex in the sense of bifunction $\eta(x,y) = 2x + y$ and $\varrho = 2$ with $\mu = 1$. Observe that, let $\tau \in [0,1]$. Then*

$$\begin{aligned}&\max\Big\{\mathcal{G}(y), \big[\mathcal{G}(y) + \eta\big(\mathcal{G}(x), \mathcal{G}(y)\big)\big]\Big\} - \mu\{\tau^{\varrho}(1-\tau) + \tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho} \\ &\geq \mathcal{G}(y) + \eta\big(\mathcal{G}(x), \mathcal{G}(y)\big) - \tau(1-\tau)(y-x)^2 \\ &\geq y^2 + \tau(2x^2 + y^2) - \tau(1-\tau)(y-x)^2 \\ &= \tau^2x^2 + 2xy\tau(1-\tau) + (1-\tau)^2y^2 + \tau(x^2 + 2y^2) \\ &\geq \tau^2x^2 + 2xy\tau(1-\tau) + (1-\tau)^2y^2 \\ &= \mathcal{G}(\tau x + (1-\tau)y).\end{aligned}$$

(II). If we choose $\mathbb{D}(x,y) = -\mu\{\tau^{\varrho}(1-\tau)+\tau(1-\tau)^{\varrho}\}\|y-x\|^{\varrho}$ for some $\mu > 0$ and $\varrho = 2$, then Definition 7 reduces to strongly η-quasiconvex function introduced by [27].

Definition 9. *([27]) A function $\mathcal{G} : \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ is said to be higher-order strongly η-quasiconvex function in the sense of $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\mu \geq 0$ if the inequality*

$$\mathcal{G}\big(\tau x + (1-\tau)y\big) \qquad \leq \max\Big\{\mathcal{G}(y), \big[\mathcal{G}(y) + \eta\big(\mathcal{G}(x), \mathcal{G}(y)\big)\big]\Big\} - \mu\tau(1-\tau)\|y-x\|^2,$$

holds for all $x, y \in \mathcal{K}$, and $\tau \in [0,1]$.

We close this segment by presenting a notable κ-fractional integral operators in the literature presented by [42].

Definition 10. *([42])For $\zeta > 0$ and let $\Psi \in L_1[\sigma_1, \sigma_2]$, then the κ-fractional integrals $J^{\zeta,\kappa}_{\sigma_1^+}$ and $J^{\zeta,\kappa}_{\sigma_2^-}$ are defined as*

$$J^{\zeta,\kappa}_{\sigma_1^+}\Psi(x) = \frac{1}{\kappa\Gamma_\kappa(\zeta)} \int_{\sigma_1}^{x} (x-\lambda)^{\frac{\zeta}{\kappa}-1}\Psi(\lambda)d\lambda, \quad x < \sigma_1 \tag{3}$$

and

$$J^{\zeta,\kappa}_{\sigma_2^-}\Psi(x) = \frac{1}{\kappa\Gamma_\kappa(\zeta)} \int_{x}^{\sigma_2} (\lambda-x)^{\frac{\zeta}{\kappa}-1}\Psi(\lambda)d\lambda, \quad x > \sigma_2, \tag{4}$$

respectively, where $\kappa > 0$, and $\Gamma_\kappa(x) := \int_0^\infty \lambda^{x-1}e^{-\frac{\lambda^\kappa}{\kappa}}d\lambda$, $\Re(x) > 0$, is the κ-Gamma function, with the condition that $\Gamma_\kappa(x+\kappa) = x\Gamma_\kappa(x)$ and $\Gamma_\kappa(\kappa) = 1$.

The incomplete Beta function is defined as follows:

$$\mathbb{B}_x(\sigma_1, \sigma_2) = \int_0^x \tau^{a_1-1}(1-\tau)^{a_2-1}d\tau, \quad a_1, a_2 > 0, 0 < x < 1.$$

Remark 1. *Observe that for exceptional and appropriate selections of function $\hbar(.)$, i.e., $\hbar(\tau) = \tau, \tau^s, \tau^{-s}, \tau^{-1}$, and $\hbar(\tau) = 1$, in Definitions 3, 4, 5, and 6, we can acquire several other versions of predominating convex, predominating s-convex of Breckner type, predominating s-convex of Godunova–Levin type, predominating P-convex function, higher-order strongly η-convex, higher-order strongly (η, s)-convex of Breckner type, higher-order strongly (η, s)-convex of Godunova–Levin type, and higher-order strongly η-P-convex function, respectively. Moreover, if we take $\eta(y, x) = y - x$, then all above cases can be reduced to classical higher-order strongly convex and classical strongly convex functions.*

4. Auxiliary Result

The following lemma assumes a key job in setting up the principle consequences of this paper. The distinguishing proof is expressed as follows.

Lemma 1. *For $\zeta > 0$, $n, p \in \mathbb{N}$, there is a pth-order differentiable function $\Psi : \Lambda \to \mathbb{R}$ such that σ_1, $\sigma_2 \in \Lambda$ with $\sigma_2 >, \sigma_1$ and $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$(the Lebesgue space). Then*

$$\begin{aligned}
&\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi) \\
&= n^{\frac{\zeta+\kappa p}{\kappa}} \Gamma(\zeta + \kappa p) \sum_{\theta=1}^{p} \frac{\left[(-1)^{\theta-1} - 1\right]}{\Gamma(\zeta + \kappa(p - \theta + 1))} \left(\frac{2}{\sigma_2 - \sigma_1}\right)^{\theta} \Psi^{(p-\theta)}\left(\frac{\sigma_1 + \sigma_2}{2}\right) \\
&+ \frac{\Gamma(\zeta + \kappa p)\Gamma_\kappa(\frac{\zeta}{\kappa})}{\kappa^{p-1}\Gamma(\zeta)} \left(\frac{2}{\sigma_2 - \sigma_1}\right)^{\frac{\zeta+\kappa p}{\kappa}} \left[J^{\zeta,\kappa}_{(\frac{\sigma_1+\sigma_2}{2})^-} \Psi(\sigma_1) + (-1)^p J^{\zeta,\kappa}_{(\frac{\sigma_1+\sigma_2}{2})^+} \Psi(\sigma_1)\right],
\end{aligned}$$

where

$$\begin{aligned}
&\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi) \\
&= \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \left[\Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) + \Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right)\right] d\tau.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \left[\Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) + \Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right)\right] d\tau. \\
&= \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau \\
&+ \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau \\
&= I_1 + I_2.
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau \\
&= -\frac{2n}{\sigma_2 - \sigma_1} (n - \tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p-1)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right)\Big|_0^n \\
&- \frac{2n(\frac{\zeta}{\kappa} + p - 1)}{(\sigma_2 - \sigma_1)} \int_0^n (n - \tau)^{\zeta+k-2} \Psi^{(p-1)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau \\
&= \frac{2n^{\frac{\zeta}{\kappa}+p}}{\sigma_2 - \sigma_1} \Psi^{(p-1)}\left(\frac{\sigma_1 + \sigma_2}{2}\right) \\
&- \frac{2n(\frac{\zeta}{\kappa} + p - 1)}{\sigma_2 - \sigma_1} \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-2} \Psi^{(p-1)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau.
\end{aligned}$$

Again, by the integration by parts, we have

$$\begin{aligned}
I_1 &= \frac{2n^{\frac{\zeta}{\kappa}+p}}{\sigma_2 - \sigma_1} \Psi^{(p-1)}\left(\frac{\sigma_1 + \sigma_2}{2}\right) - \frac{2^2 n^{\frac{\zeta}{\kappa}+p}(\frac{\zeta}{\kappa} + p - 1)}{(\sigma_2 - \sigma_1)^2} \Psi^{(k-2)}\left(\frac{\sigma_1 + \sigma_2}{2}\right) \\
&+ \frac{2^2 n^2(\frac{\zeta}{\kappa} + p - 1)(\frac{\zeta}{\kappa} + p - 2)}{(\sigma_2 - \sigma_1)^2} \int_0^n (n - \tau)^{\frac{\zeta}{\kappa}+p-3} \Psi^{(k-3)}\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau.
\end{aligned}$$

Applying successive integration by parts up to κ-times, we get

$$
\begin{aligned}
I_1 \quad &= n^{\frac{\zeta+\kappa p}{\kappa}} \sum_{\theta=1}^{p} \frac{(-1)^{\theta-1}}{(\zeta+\kappa p)\kappa^{\theta-1}} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\theta} \prod_{q=0}^{\theta-1} \left(\zeta+\kappa(p-q)\right) \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{(-1)^p}{(\zeta+\kappa p)\kappa^p} \left(\frac{2n}{\sigma_2-\sigma_1}\right)^{p} \prod_{q=0}^{p} (\zeta+\kappa(p-q)) \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}-1} \Psi\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau \\
&= n^{\frac{\zeta+\kappa p}{\kappa}} \sum_{\theta=1}^{p} \frac{(-1)^{\theta-1}}{(\zeta+\kappa p)\kappa^{\theta-1}} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\theta} \prod_{q=0}^{\theta-1} \left(\zeta+\kappa(p-q)\right) \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{(-1)^p \Gamma(\zeta+\kappa p)}{\kappa^p \Gamma(\zeta)} \left(\frac{2n}{\sigma_2-\sigma_1}\right)^{p} \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}-1} \Psi\left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2\right) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= n^{\frac{\zeta+\kappa p}{\kappa}} \sum_{\theta=1}^{p} \frac{(-1)^{\theta-1}\Gamma(\zeta+\kappa p)}{\kappa^{\theta-1}\Gamma(\zeta+\kappa(p-\theta+1))} \left(\frac{2}{(\sigma_2-\sigma_1)}\right)^{\theta} \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{(-1)^p \Gamma(\zeta+\kappa p)\Gamma_\kappa(\frac{\zeta}{\kappa})}{\kappa^{p-1}\Gamma(\zeta)} \left(\frac{2n}{\sigma_2-\sigma_1}\right)^{\frac{\kappa p+\zeta}{\kappa}} J^{\zeta,\kappa}_{\left(\frac{\sigma_1+\sigma_2}{2}\right)^-} \Psi(\sigma_1).
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
I_2 \quad &= \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau \\
&= \frac{2n}{\sigma_2-\sigma_1} (n-\tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p-1)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right)\Bigg|_0^n \\
&+ \frac{2n(\frac{\zeta}{\kappa}+p-1)}{\sigma_2-\sigma_1} \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-2} \Psi^{(p-1)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau \\
&= \frac{-2n^{\frac{\zeta+\kappa p}{\kappa}}}{\sigma_2-\sigma_1} \Psi^{(p-1)}\left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{2^2 n^{\frac{\zeta}{\kappa}+p}(\frac{\zeta}{\kappa}+p-1)}{(\sigma_2-\sigma_1)^2} \Psi^{(p-2)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{2^2 n^2(\frac{\zeta}{\kappa}+p-1)(\frac{\zeta}{\kappa}+p-2)}{(\sigma_2-\sigma_1)^2} \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-3} \Psi^{(p-3)}\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
&= -n^{\frac{\zeta+p}{\kappa}} \sum_{\theta=1}^{p} \frac{1}{\kappa^{\theta-1}(\zeta+\kappa p)} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\theta} \prod_{q=0}^{\theta-1} (\zeta+\kappa(p-q)) \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{1}{\kappa^p(\zeta+\kappa p)} \left(\frac{2n}{\sigma_2-\sigma_1}\right)^{p} \prod_{p=0}^{p} (\zeta+\kappa(p-p)) \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}-1} \Psi\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau \\
&= -n^{\frac{\zeta+p}{\kappa}} \sum_{\theta=1}^{p} \frac{1}{(\zeta+\kappa p)\kappa^{\theta-1}} \left(\frac{2}{\sigma_2-\sigma_1}\right) \prod_{q=0}^{\theta-1} (\zeta+\kappa(p-q))^i \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right) \\
&+ \frac{\Gamma(\zeta+\kappa p)}{\kappa^p \Gamma(\zeta)} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{p} \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}-1} \Psi\left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2\right) d\tau
\end{aligned}
$$

$$= -n^{\frac{\zeta+\kappa p}{\kappa}} \sum_{\theta=1}^{p} \frac{\Gamma(\zeta+\kappa p)}{\kappa^{\theta-1}\Gamma(\zeta+\kappa(p-\theta+1))} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\theta} \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right)$$
$$+\frac{\Gamma(\zeta+\kappa p)\Gamma_\kappa(\frac{\zeta}{\kappa})}{\kappa^{p-1}\Gamma(\zeta)} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\frac{\zeta+\kappa p}{\kappa}} J^{\zeta,\kappa}_{(\frac{\sigma_1+\sigma_2}{2})^+}\Psi(\sigma_2).$$

Summing up I_1 and I_2, we have

$$I_1+I_2 = n^{\frac{\zeta+\kappa p}{\kappa}}\Gamma(\zeta+\kappa p) \sum_{\theta=1}^{p} \frac{\left[(-1)^{\theta-1}-1\right]}{\Gamma(\zeta+\kappa(p-\theta+1))} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\theta} \Psi^{(p-\theta)}\left(\frac{\sigma_1+\sigma_2}{2}\right)$$
$$+\frac{\Gamma(\zeta+\kappa p)\Gamma_\kappa(\frac{\zeta}{\kappa})}{\kappa^{p-1}\Gamma(\zeta)} \left(\frac{2}{\sigma_2-\sigma_1}\right)^{\frac{\zeta+\kappa p}{\kappa}} \left[J^{\zeta,\kappa}_{(\frac{\sigma_1+\sigma_2}{2})^-}\Psi(\sigma_1) + (-1)^p J^{\zeta,\kappa}_{(\frac{\sigma_1+\sigma_2}{2})^+}\Psi(\sigma_1)\right].$$

□

5. Some New Results for Predominating $\hbar$-Convex Functions in Settings of *pth*-Order Differentiable Functions

Let Λ be an interval in real line $\mathbb{R}$, and there is a differentiable mapping $\Psi : \Lambda = [\sigma_1, \sigma_2] \subset \mathbb{R} \to \mathbb{R}$ on the interior Λ° of Λ, also let $\eta(.,.) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous bifunction.

Theorem 1. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|$ is a predominating $\hbar$-convex function on Λ, then*

$$|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|$$
$$\leq \mathcal{Y}_1(p,n,\zeta,\kappa)\left[2|\Psi^{(p)}(\sigma_1)| + \eta\left(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\right)\right] + \frac{2\mathbb{D}(\sigma_1,\sigma_2)\kappa n^{\frac{\zeta+p\kappa}{\kappa}}}{\zeta+p\kappa},$$

where

$$\mathcal{Y}_1(p,n,\zeta,\kappa) = \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\hbar\left(\frac{n+\tau}{2n}\right) + \hbar\left(\frac{n-\tau}{2n}\right)\right]d\tau.$$

Proof. By the given supposition, utilizing Lemma 1 and the modulus property, we have

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\tfrac{n+\tau}{2n}\sigma_1 + \tfrac{n-\tau}{2n}\sigma_2\right)d\tau \\
&\quad + \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\tfrac{n-\tau}{2n}\sigma_1 + \tfrac{n+\tau}{2n}\sigma_2\right)d\tau \\
&\leq \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\hbar\left(\tfrac{n+\tau}{2n}\right)|\Psi^{(p)}(\sigma_1)| + \hbar\left(\tfrac{n-\tau}{2n}\right)\left[|\Psi^{(p)}(\sigma_1)| + \eta\left(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\right)\right] + \mathbb{D}(\sigma_1,\sigma_2)\right]d\tau \\
&\quad + \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\hbar\left(\tfrac{n-\tau}{2n}\right)|\Psi^{(p)}(\sigma_1)| + \hbar\left(\tfrac{n+\tau}{2n}\right)\left[|\Psi^{(p)}(\sigma_1)| + \eta\left(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\right)\right] + \mathbb{D}(\sigma_1,\sigma_2)\right] \\
&\leq \left[2|\Psi^{(p)}(\sigma_1)| + \eta\left(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\right)\right]\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\hbar\left(\tfrac{n+\tau}{2n}\right) + \hbar\left(\tfrac{n-\tau}{2n}\right)\right]d\tau \\
&\quad + 2\mathbb{D}(\sigma_1,\sigma_2)\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}d\tau \qquad (5)\\
&= \mathcal{Y}_1(p,n,\zeta,\kappa)\left[2|\Psi^{(p)}(\sigma_1)| + \eta\left(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\right)\right] + 2\mathbb{D}(\sigma_1,\sigma_2)\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}d\tau,
\end{aligned}$$

where

$$\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1} d\tau = \frac{\kappa n^{\frac{\zeta+p\kappa}{\kappa}}}{\zeta+p\kappa}. \tag{6}$$

Substituting Equation (6) in Equation (5), we get the desired inequality of Equation (5). □

Now we shall discuss some remarkable cases of Theorem 1.

(I) If we choose $\hbar(\tau)=\tau$, then we get a new result for predominating convex functions.

Corollary 1. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a predominating η-convex function on Λ, then*

$$\begin{aligned}&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\ &\leq \frac{2\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{(\zeta+\kappa p)}\Big[\Big[2|\Psi^{(p)}(\sigma_1)|+\eta\big(|\Psi^{(p)}(\sigma_2),|\Psi^{(p)}(\sigma_1)\big)\Big]+\mathbb{D}(\sigma_1,\sigma_2)\Big].\end{aligned}$$

(II) If we choose $\hbar(\tau)=\tau^s$, then we get Breckner type predominating s-convex functions.

Corollary 2. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a Breckner type predominating s-convex function on Λ, then*

$$\begin{aligned}&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\ &\leq \mathcal{Y}_2(p,n,\zeta,\kappa)\Big[2|\Psi^{(p)}(\sigma_1)|+\eta\big(|\Psi^{(p)}(\sigma_2),|\Psi^{(p)}(\sigma_1)\big)\Big]+\frac{2\mathbb{D}(\sigma_1,\sigma_2)\kappa n^{\frac{\zeta+p\kappa}{\kappa}}}{\zeta+p\kappa},\end{aligned}$$

where

$$\begin{aligned}\mathcal{Y}_2(p,n,\zeta,\kappa) &= \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Big[\Big(\frac{n+\tau}{2n}\Big)^s+\Big(\frac{n-\tau}{2n}\Big)^s\Big]d\tau\\ &= \frac{1}{(2n)^s}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+s)}{\kappa}}}{\zeta+\kappa(p+s)}+(2n)^{\frac{\zeta+\kappa(p+s)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\big(\frac{\zeta+\kappa p}{\kappa},s+1\big)\Big].\end{aligned}$$

(III) If we choose $\hbar(\tau)=\tau^{-s}$, then we get Godunova–Levin predominating s-convex functions.

Corollary 3. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is Godunova–Levin type predominating s-convex function on Λ, then*

$$\begin{aligned}&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\ &\leq \mathcal{Y}_3(p,n,\zeta,\kappa)\Big[2|\Psi^{(p)}(\sigma_1)|+\eta\big(|\Psi^{(p)}(\sigma_2),|\Psi^{(p)}(\sigma_1)\big)\Big]+\frac{2\mathbb{D}(\sigma_1,\sigma_2)\kappa n^{\frac{\zeta+p\kappa}{\kappa}}}{\zeta+p\kappa},\end{aligned}$$

where

$$\mathcal{Y}_3(p,n,\zeta,\kappa) = \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Big[\Big(\frac{n+\tau}{2n}\Big)^{-s} + \Big(\frac{n-\tau}{2n}\Big)^{-s}\Big]d\tau$$
$$= (2n)^s\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p-s)}{\kappa}}}{\zeta+\kappa(p-s)} + (2n)^{\frac{\zeta+\kappa(p-s)}{\kappa}}\mathbb{B}_{\frac{1}{2}}(\frac{\zeta+\kappa p}{\kappa}, -s+1)\Big].$$

(IV) If we choose $\hbar(\tau) = 1$, then we get predominating P-η-convex functions.

Corollary 4. *For $n,p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a predominating P-convex function on Λ, then*

$$|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|$$
$$\leq \frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+p\kappa}\Big[\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big] + 2\mathbb{D}(\sigma_1,\sigma_2)\Big].$$

(V) If we choose $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-convex function for a given arbitrary non-negative function $\hbar$.

Corollary 5. *For $n,p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a higher-order strongly η-convex function for a given arbitrary non-negative function $\hbar$ on Λ, then*

$$|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|$$
$$\leq \mathcal{Y}_1(p,n,\zeta,\kappa)\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big]$$
$$-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{\zeta+\kappa(p+\varrho+1)} + (2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\Big(\frac{\zeta+\kappa(p+1)}{\kappa}, \varrho+1\Big)\Big],$$

where

$$\mathcal{Y}_1(p,n,\zeta,\kappa) = \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Big[\hbar\Big(\frac{n+\tau}{2n}\Big) + \hbar\Big(\frac{n-\tau}{2n}\Big)\Big]d\tau.$$

(VI) If we choose $\hbar(\tau) = \tau$ along with $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-convex functions.

Corollary 6. *For $n,p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a higher-order strongly η-convex function on Λ, then*

$$|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|$$
$$\leq \frac{2\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{(\zeta+\kappa p)}\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big]$$
$$-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{\zeta+\kappa(p+\varrho+1)} + (2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\Big(\frac{\zeta+\kappa(p+1)}{\kappa}, \varrho+1\Big)\Big].$$

(VII) If we choose $\hbar(\tau) = \tau^s$ along with $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get a Breckner type of a higher-order strongly (η,s)-convex function.

Corollary 7. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|$ is a Breckner type of a higher-order strongly (η, s)-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \leq \frac{1}{(2n)^s}\left[\frac{\kappa n^{\frac{\zeta+\kappa(p+s)}{\kappa}}}{\zeta+\kappa(p+s)} + (2n)^{\frac{\zeta+\kappa(p+s)}{\kappa}}\mathbb{B}_{\frac{1}{2}}(\frac{\zeta+\kappa p}{\kappa}, s+1)\right] \\
&\times\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big] \\
&-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{\zeta+\kappa(p+\varrho+1)} + (2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\Big(\frac{\zeta+\kappa(p+1)}{\kappa}, \varrho+1\Big)\Big].
\end{aligned}
$$

$(VIII)$ If we choose $\hbar(\tau) = \tau^{-s}$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get Godunova–Levin type of a higher-order strongly (η, s)-convex function.

Corollary 8. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|$ is a Godunova–Levin of a higher-order strongly (η, s)-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \leq (2n)^s\left[\frac{\kappa n^{\frac{\zeta+\kappa(p-s)}{\kappa}}}{\zeta+\kappa(p-s)} + (2n)^{\frac{\zeta+\kappa(p-s)}{\kappa}}\mathbb{B}_{\frac{1}{2}}(\frac{\zeta+\kappa p}{\kappa}, -s+1)\right] \\
&\times\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big] \\
&-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{\zeta+\kappa(p+\varrho+1)} + (2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\Big(\frac{\zeta+\kappa(p+1)}{\kappa}, \varrho+1\Big)\Big].
\end{aligned}
$$

(IX) If we choose $\hbar(\tau) = 1$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-P-convex function.

Corollary 9. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|$ is a higher-order strongly η-P-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \leq \frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+p\kappa}\Big[2|\Psi^{(p)}(\sigma_1)| + \eta\big(|\Psi^{(p)}(\sigma_2), |\Psi^{(p)}(\sigma_1)\big)\Big] \\
&-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\Big[\frac{\kappa n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{\zeta+\kappa(p+\varrho+1)} + (2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\Big(\frac{\zeta+\kappa(p+1)}{\kappa}, \varrho+1\Big)\Big].
\end{aligned}
$$

Theorem 2. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}$, $\delta_1 > 1$ is a predominating $\hbar$-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa} \right)^{\frac{1}{\delta_1}} \left[\left\{ \int_0^n \left(\hbar\left(\frac{n-\tau}{2n}\right) |\Psi^{(p)}(\sigma_1)|^{\delta_2} \right.\right.\right. \\
&\left.\left. + \hbar\left(\frac{n+\tau}{2n}\right) \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) d\tau + n\mathbb{D}(\sigma_1, \sigma_2) \right\}^{\frac{1}{\delta_2}} \\
&+ \left\{ \int_0^n \left(\hbar\left(\frac{n+\tau}{2n}\right) |\Psi^{(p)}(\sigma_1)|^{\delta_2} + \hbar\left(\frac{n-\tau}{2n}\right) \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) d\tau \right. \\
&\left.\left. + n\mathbb{D}(\sigma_1, \sigma_2) \right\}^{\frac{1}{\delta_2}} \right].
\end{aligned}
$$

Proof. Since $|\Psi^{(p)}|^{\delta_1}$ is a predominating $\hbar$-convex function on Λ, utilizing Lemma 1 and the well-known H*ö*lder inequality, we have

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} d\tau \right)^{\frac{1}{\delta_1}} \left(\int_0^n \left| \Psi^{(p)} \left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2 \right) \right|^{\delta_2} d\tau \right)^{\frac{1}{\delta_2}} \\
&+ \left(\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} d\tau \right)^{\frac{1}{\delta_1}} \left(\int_0^n \left| \Psi^{(p)} \left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2 \right) \right|^{\delta_2} d\tau \right)^{\frac{1}{\delta_2}} \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa} \right)^{\frac{1}{\delta_1}} \left[\left\{ \int_0^n \left(\hbar\left(\frac{n-\tau}{2n}\right) |\Psi^{(p)}(\sigma_1)|^{\delta_2} \right.\right.\right. \\
&\left.\left. + \hbar\left(\frac{n+\tau}{2n}\right) \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) d\tau + n\mathbb{D}(\sigma_1, \sigma_2) \right\}^{\frac{1}{\delta_2}} \\
&+ \left\{ \int_0^n \left(\hbar\left(\frac{n+\tau}{2n}\right) |\Psi^{(p)}(\sigma_1)|^{\delta_2} + \hbar\left(\frac{n-\tau}{2n}\right) \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) d\tau \right. \\
&\left.\left. + n\mathbb{D}(\sigma_1, \sigma_2) \right\}^{\frac{1}{\delta_2}} \right],
\end{aligned}
$$

the required result. □

Now we shall discuss some remarkable cases of Theorem 2.

(I) If we choose $\hbar(\tau) = \tau$, then we get predominating convex functions.

Corollary 10. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$ and let $\Psi : \Lambda \to \mathbb{R}$ be a predominating convex function such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a predominating convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \left[\left(\frac{n}{4}|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right. \\
&\left.\left.+\frac{3n}{4}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) + n\mathbb{D}(\sigma_1, \sigma_2)\right\}^{\frac{1}{\delta_2}} \\
&+\left\{\frac{3n}{4}|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{n}{4}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) \\
&\left.+n\mathbb{D}(\sigma_1, \sigma_2)\right\}^{\frac{1}{\delta_2}}\Bigg].
\end{aligned}
$$

(II) If we choose $\hbar(\tau) = \tau^s$, then we get Breckner type predominating s-convex functions.

Corollary 11. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a Breckner type predominating s-convex functions on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \left[\left\{\frac{n}{2^s(s+1)}|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right. \\
&\left.+\frac{n(2^{s+1}-1)}{2^s(s+1)}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) + n\mathbb{D}(\sigma_1, \sigma_2)\right\}^{\frac{1}{\delta_2}} \\
&+\left\{\left(\frac{n(2^{s+1}-1)}{2^s(s+1)}|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{n}{2^s(s+1)}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right)\right. \\
&\left.+n\mathbb{D}(\sigma_1, \sigma_2)\right\}^{\frac{1}{\delta_2}}\Bigg].
\end{aligned}
$$

(III) If we choose $\hbar(\tau) = \tau^{-s}$, then we get Godunova–Levin type predominating s-convex functions.

Corollary 12. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a Godunova–Levin type predominating s-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \Bigg[\Bigg\{\frac{2^s n}{1-s}|\Psi^{(p)}(\sigma_1)|^{\delta_2} \\
&+\frac{2^s n(2^{1-s}-1)}{1-s}\Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\big)\Big]\Big)+n\mathbb{D}(\sigma_1,\sigma_2)\Bigg\}^{\frac{1}{\delta_2}} \\
&+\Bigg\{\frac{2^s n(2^{1-s}-1)}{1-s}|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\frac{2^s n}{1-s}\Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\big)\Big]\Big) \\
&+n\mathbb{D}(\sigma_1,\sigma_2)\Bigg\}^{\frac{1}{\delta_2}}\Bigg].
\end{aligned}
$$

(IV) If we choose $\hbar(\tau) = 1$, then we get predominating P-convex functions.

Corollary 13. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a predominating P-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq n^{\frac{1}{\delta_2}}\left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \Bigg[\Bigg\{\Big(|\Psi^{(p)}(\sigma_1)|^{\delta_2} \\
&+\Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\big)\Big]\Big)+n\mathbb{D}(\sigma_1,\sigma_2)\Bigg\}^{\frac{1}{\delta_2}} \\
&+\Bigg\{|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\big)\Big]\Big) \\
&+n\mathbb{D}(\sigma_1,\sigma_2)\Bigg\}^{\frac{1}{\delta_2}}\Bigg].
\end{aligned}
$$

(V) If we choose $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2 - \sigma_1)^{\varrho}$, then we get higher-order strongly η-convex function for a given arbitrary non-negative function $\hbar$.

Corollary 14. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a higher-order strongly η-convex function for a given arbitrary non-negative function $\hbar$ on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta + \kappa(p-1)) + \kappa} \right)^{\frac{1}{\delta_1}} \left[\left(\frac{n}{4} |\Psi^{(p)}(\sigma_1)|^{\delta_2} \right. \right. \\
&\quad \left. + \frac{3n}{4} \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) - \left. \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \right\}^{\frac{1}{\delta_2}} \\
&\quad + \left\{ \frac{3n}{4} |\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{n}{4} \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) \\
&\quad \left. \left. - \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \right\}^{\frac{1}{\delta_2}} \right].
\end{aligned}$$

(VI) If we choose $\hbar(\tau) = \tau$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}} \{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2 - \sigma_1)^{\varrho}$, then we get higher-order strongly η-convex functions.

Corollary 15. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a higher-order strongly η-convex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta + \kappa(p-1)) + \kappa} \right)^{\frac{1}{\delta_1}} \left[\left(\frac{n}{4} |\Psi^{(p)}(\sigma_1)|^{\delta_2} \right. \right. \\
&\quad \left. + \frac{3n}{4} \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) - \left. \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \right\}^{\frac{1}{\delta_2}} \\
&\quad + \left\{ \frac{3n}{4} |\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{n}{4} \left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right) \right] \right) \\
&\quad \left. \left. - \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \right\}^{\frac{1}{\delta_2}} \right].
\end{aligned}$$

(VII) If we choose $\hbar(\tau) = \tau^s$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}} \{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2 - \sigma_1)^{\varrho}$, then we get Breckner type of a higher-order strongly (η, s)-convex function.

Corollary 16. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a Breckner type of a higher-order strongly (η, s)-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \left[\left\{\frac{n}{2^s(s+1)}|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right. \\
&\left. + \frac{n(2^{s+1}-1)}{2^s(s+1)}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) - \frac{2n\mu(\sigma_1-\sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)}\Bigg\}^{\frac{1}{\delta_2}} \\
&+ \left\{\left(\frac{n(2^{s+1}-1)}{2^s(s+1)}|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{n}{2^s(s+1)}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right)\right. \\
&\left.\left. - \frac{2n\mu(\sigma_1-\sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)}\right\}^{\frac{1}{\delta_2}}\right].
\end{aligned}
$$

$(VIII)$ If we choose $\hbar(\tau) = \tau^{-s}$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get Godunova–Levin type of a higher-order strongly (η, s)-convex function.

Corollary 17. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a Godunova–Levin type of a higher-order strongly (η, s)-convex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right)^{\frac{1}{\delta_1}} \left[\left\{\frac{2^s n}{1-s}|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right. \\
&\left. + \frac{2^s n(2^{1-s}-1)}{1-s}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) - \frac{2n\mu(\sigma_1-\sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)}\Bigg\}^{\frac{1}{\delta_2}} \\
&+ \left\{\frac{2^s n(2^{1-s}-1)}{1-s}|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \frac{2^s n}{1-s}\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) \\
&\left.\left. - \frac{2n\mu(\sigma_1-\sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)}\right\}^{\frac{1}{\delta_2}}\right].
\end{aligned}
$$

(IX) If we choose $\hbar(\tau) = 1$ along with $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2 - \sigma_1)^{\varrho}$, then we get higher-order strongly η-P-convex function.

Corollary 18. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}, \delta_1 > 1$ is a higher-order strongly η-P-convex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq n^{\frac{1}{\delta_2}} \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta + \kappa(p-1)) + \kappa} \right)^{\frac{1}{\delta_1}} \Bigg[\Bigg\{ \Big(|\Psi^{(p)}(\sigma_1)|^{\delta_2} \\
&+ \Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\big) \Big] \Big) - \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \Bigg\}^{\frac{1}{\delta_2}} \\
&+ \Bigg\{ |\Psi^{(p)}(\sigma_1)|^{\delta_2} + \Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\big) \Big] \Big) \\
&- \frac{2n\mu(\sigma_1 - \sigma_2)^{\varrho}}{(\varrho+1)(\varrho+2)} \Bigg\}^{\frac{1}{\delta_2}} \Bigg].
\end{aligned}$$

Theorem 3. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_1}$ is a predominating $\hbar$-convex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq \Bigg[\Bigg\{ \mathcal{Y}_1^*(\zeta, n, \kappa, p)|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \mathcal{Y}_2^*(\zeta, n, \kappa, p) \Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\big) \Big] \Big) \\
&+ \frac{\kappa \mathbb{D}(\sigma_1, \sigma_2) n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta + \kappa(p-1)) + \kappa} \Bigg\}^{\frac{1}{\delta_2}} \\
&+ \Bigg\{ \mathcal{Y}_2^*(\zeta, n, \kappa, p)|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \mathcal{Y}_1^*(\zeta, n, \kappa, p) \Big[|\Psi^{(p)}(\sigma_1)|^{\delta_2} + \eta\big(\Psi^{(p)}(\sigma_2)|^{\delta_2}, \Psi^{(p)}(\sigma_1)|^{\delta_2}\big) \Big] \Big) \\
&+ \frac{\kappa \mathbb{D}(\sigma_1, \sigma_2) n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta + \kappa(p-1)) + \kappa} \Bigg\}^{\frac{1}{\delta_2}} \Bigg],
\end{aligned}$$

where

$$\mathcal{Y}_1^*(\zeta, n, \kappa, p) := \int_0^n (n-\tau)^{\frac{\delta_1(\zeta+\kappa(p-1))}{\kappa}} \hbar\Big(\frac{n-\tau}{2n}\Big) d\tau$$

and

$$\mathcal{Y}_2^*(\zeta, n, \kappa, p) := \int_0^n (n-\tau)^{\frac{\delta_1(\zeta+\kappa(p-1))}{\kappa}} \hbar\Big(\frac{n+\tau}{2n}\Big) d\tau.$$

Proof. Since $|\Psi^{(p)}|^{\delta_1}$ is a predominating $\hbar$-convex function on Λ, utilizing Lemma 1 and the well-known Hölder inequality, we have

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq \left(\int_0^n \frac{1}{n} d\tau\right)^{\frac{1}{\delta_1}} \left(\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} \left|\Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1+\frac{n+\tau}{2n}\sigma_2\right)\right|^{\delta_2} d\tau\right)^{\frac{1}{\delta_2}} \\
&+\left(\int_0^n \frac{1}{n} d\tau\right)^{\frac{1}{\delta_1}} \left(\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} \left|\Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1+\frac{n-\tau}{2n}\sigma_2\right)\right|^{\delta_2} d\tau\right)^{\frac{1}{\delta_2}} \\
&\leq \left[\left\{\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} \left(\hbar\left(\frac{n-\tau}{2n}\right)|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right.\right. \\
&\left.\left.+\hbar\left(\frac{n+\tau}{2n}\right)\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right)d\tau+\frac{\kappa\mathbb{D}(\sigma_1,\sigma_2)n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right\}^{\frac{1}{\delta_2}} \\
&+\left\{\int_0^n (n+\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} \left(\hbar\left(\frac{n-\tau}{2n}\right)|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right.\right. \\
&\left.\left.\left.+\hbar\left(\frac{n-\tau}{2n}\right)\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right)d\tau+\frac{\kappa\mathbb{D}(\sigma_1,\sigma_2)n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right\}^{\frac{1}{\delta_2}}\right] \\
&=\left[\left\{\mathcal{Y}_1^*(\zeta,n,\kappa,p)|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\mathcal{Y}_2^*(\zeta,n,\kappa,p)\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right)\right. \\
&\left.+\frac{\kappa\mathbb{D}(\sigma_1,\sigma_2)n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right\}^{\frac{1}{\delta_2}} \\
&+\left\{\mathcal{Y}_2^*(\zeta,n,\kappa,p)|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\mathcal{Y}_1^*(\zeta,n,\kappa,p)\left[|\Psi^{(p)}(\sigma_1)|^{\delta_2}+\eta\left(\Psi^{(p)}(\sigma_2)|^{\delta_2},\Psi^{(p)}(\sigma_1)|^{\delta_2}\right)\right]\right) \\
&\left.+\frac{\kappa\mathbb{D}(\sigma_1,\sigma_2)n^{\frac{\delta_1(\zeta+\kappa(p-1))+\kappa}{\kappa}}}{\delta_1(\zeta+\kappa(p-1))+\kappa}\right\}^{\frac{1}{\delta_2}}\right],
\end{aligned}
$$

the required result. □

Remark 2. *The similar cases can be obtained easily from Theorem 3 by adopting the same technique as we have done for Theorem 1 and Theorem 2 by utilizing the assumptions of predominating $\hbar$-convex functions and suitable choices of function $\hbar(.)$.*

6. New Generalizations for Predominating Quasiconvex Functions for *pth*-Order Differentiable Function

In this section, we discuss the main results of predominating quasiconvex functions via *pth*-order differentiability by employing Definitions 7, 9, and Lemma 1.

Theorem 4. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a predominating quasiconvex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq 2\left(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+\kappa p}\right)\left[\max\left\{|\Psi^{(p)}(\sigma_2)|,\left[|\Psi^{(p)}(\sigma_2)|+\eta\left(|\Psi^{(p)}(\sigma_2),|\Psi^{(p)}(\sigma_1)\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right].
\end{aligned}
$$

Proof. Since $|\Psi^{(p)}|$ is a predominating quasiconvex function on Λ, utilizing Lemma 1 and the modulus property, we have

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\frac{n+\tau}{2n}\sigma_1+\frac{n-\tau}{2n}\sigma_2\right)d\tau+\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\frac{n-\tau}{2n}\sigma_1+\frac{n+\tau}{2n}\sigma_2\right)d\tau \\
&\leq \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\max\left\{|\Psi^{(p)}(\sigma_2)|,\left[|\Psi^{(p)}(\sigma_2)|+\eta\left(|\Psi^{(p)}(\sigma_1),|\Psi^{(p)}(\sigma_2)\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right]d\tau \\
&\quad+\int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1}\left[\max\left\{|\Psi^{(p)}(\sigma_2)|,\left[|\Psi^{(p)}(\sigma_2)|+\eta\left(|\Psi^{(p)}(\sigma_1),|\Psi^{(p)}(\sigma_2)\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right]d\tau \\
&=2\left(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+\kappa p}\right)\left[\max\left\{|\Psi^{(p)}(\sigma_2)|,\left[|\Psi^{(p)}(\sigma_2)|+\eta\left(|\Psi^{(p)}(\sigma_1),|\Psi^{(p)}(\sigma_2)\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right],
\end{aligned}$$

the requuired result. □

Some special cases of Theorem 4 can be discussed as follows.
(I) If we choose $\mathbb{D}(\sigma_1,\sigma_2)=-\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau)+(n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-quasiconvex function.

Corollary 19. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a higher-order strongly η-quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq 2\left(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+\kappa p}\right)\left[\max\left\{|\Psi^{(p)}(\sigma_2)|,\left[|\Psi^{(p)}(\sigma_2)|+\eta\left(|\Psi^{(p)}(\sigma_1),|\Psi^{(p)}(\sigma_2)\right)\right]\right\}\right] \\
&\quad-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\left\{\frac{\kappa(\zeta+\kappa(p+\varrho+2))n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{(\zeta+\kappa(p+\varrho+1))(\zeta+\kappa(p+\varrho))}+(2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\left(\frac{\zeta+\kappa(p+1)}{\kappa},\varrho+1\right)\right\}.
\end{aligned}$$

(II) If we choose $\mathbb{D}(\sigma_1,\sigma_2)=-\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau)+(n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$ along with $\eta\left(\Psi(\sigma_2),\Psi(\sigma_1)\right)=\Psi(\sigma_2)-\Psi(\sigma_1)$, then we get higher-order strongly quasiconvex function.

Corollary 20. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|$ is a higher-order strongly quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq 2\left(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+\kappa p}\right)\left[\left\{|\Psi^{(p)}(\sigma_1)|+|\Psi^{(p)}(\sigma_2)\right\}\right] \\
&\quad-\frac{2\mu}{(2n)^{\varrho+1}}(\sigma_2-\sigma_1)^{\varrho}\left\{\frac{\kappa(\zeta+\kappa(p+\varrho+2))n^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}}{(\zeta+\kappa(p+\varrho+1))(\zeta+\kappa(p+\varrho))}+(2n)^{\frac{\zeta+\kappa(p+\varrho+1)}{\kappa}}\mathbb{B}_{\frac{1}{2}}\left(\frac{\zeta+\kappa(p+1)}{\kappa},\varrho+1\right)\right\}.
\end{aligned}$$

Theorem 5. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0, \delta_2 > 1$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a predominating quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq 2n^{\frac{1}{\delta_2}} \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1)+\kappa)}{\kappa}}}{\delta_1(\zeta+\kappa(p-1)+\kappa)} \right)^{\frac{1}{\delta_1}} \\
&\quad \times \left(\max\left\{ |\Psi^{(p)}(\sigma_2)|^{\delta_2}, \left[|\Psi^{(p)}(\sigma_2)|^{\delta_2} + \eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2}, |\Psi^{(p)}(\sigma_2)|^{\delta_2} \right) \right] \right\} + \mathbb{D}(\sigma_1,\sigma_2) \right)^{\frac{1}{\delta_2}}.
\end{aligned}$$

Proof. Since $|\Psi^{(p)}|^{\delta_2}$ is a predominating quasiconvex function on Λ, utilizing Lemma 1 and the well-known Hölder inequality, we have

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)} \left(\frac{n+\tau}{2n}\sigma_1 + \frac{n-\tau}{2n}\sigma_2 \right) d\tau + \int_0^n (n-\tau)^{\frac{\zeta}{\kappa}+p-1} \Psi^{(p)} \left(\frac{n-\tau}{2n}\sigma_1 + \frac{n+\tau}{2n}\sigma_2 \right) d\tau \\
&\leq 2 \left(\int_0^n (n-\tau)^{\delta_1(\frac{\zeta}{\kappa}+p-1)} d\tau \right)^{\frac{1}{\delta_1}} \\
&\quad \times \left(\int_0^n \left(\max\left\{ |\Psi^{(p)}(\sigma_2)|^{\delta_2}, \left[|\Psi^{(p)}(\sigma_2)|^{\delta_2} + \eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2}, |\Psi^{(p)}(\sigma_2)|^{\delta_2} \right) \right] \right\} + \mathbb{D}(\sigma_1,\sigma_2) \right) d\tau \right)^{\frac{1}{\delta_2}} \\
&= 2n^{\frac{1}{\delta_2}} \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1)+\kappa)}{\kappa}}}{\delta_1(\zeta+\kappa(p-1)+\kappa)} \right)^{\frac{1}{\delta_1}} \\
&\quad \times \left(\max\left\{ |\Psi^{(p)}(\sigma_2)|^{\delta_2}, \left[|\Psi^{(p)}(\sigma_2)|^{\delta_2} + \eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2}, |\Psi^{(p)}(\sigma_2)|^{\delta_2} \right) \right] \right\} + \mathbb{D}(\sigma_1,\sigma_2) \right)^{\frac{1}{\delta_2}},
\end{aligned}$$

the required result. □

Some special cases of Theorem 6 can be discussed as follows.
(I) If we choose $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-quasiconvex function.

Corollary 21. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0, \delta_2 > 1$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a higher-order strongly η-quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)| \\
&\leq 2n^{\frac{1}{\delta_2}} \left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1)+\kappa)}{\kappa}}}{\delta_1(\zeta+\kappa(p-1)+\kappa)} \right)^{\frac{1}{\delta_1}} \\
&\quad \times \left(\max\left\{ |\Psi^{(p)}(\sigma_2)|^{\delta_2}, \left[|\Psi^{(p)}(\sigma_2)|^{\delta_2} + \eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2}, |\Psi^{(p)}(\sigma_2)|^{\delta_2} \right) \right] \right\} \right. \\
&\quad \left. - \frac{4n\mu(\sigma_2-\sigma_1)^{\varrho}}{(\varrho+1)(\varrho+2)} \right)^{\frac{1}{\delta_2}}.
\end{aligned}$$

(II) If we choose $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau)+(n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$ along with $\eta\big(\Psi(\sigma_2),\Psi(\sigma_1)\big) = \Psi(\sigma_2)-\Psi(\sigma_1)$, then we get higher-order strongly quasiconvex function.

Corollary 22. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0,\delta_2>1$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a higher-order strongly quasiconvex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\
&\leq 2n^{\frac{1}{\delta_2}}\left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1)+\kappa)}{\kappa}}}{\delta_1(\zeta+\kappa(p-1)+\kappa)}\right)^{\frac{1}{\delta_1}}\\
&\quad\times\left(\left(|\Psi^{(p)}(\sigma_2)|^{\delta_2}+|\Psi^{(p)}(\sigma_1)|^{\delta_2}\right)-\frac{4n\mu(\sigma_2-\sigma_1)^{\varrho}}{(\varrho+1)(\varrho+2)}\right)^{\frac{1}{\delta_2}}.
\end{aligned}
$$

Theorem 6. *For $n,p\in\mathbb{N},\kappa>0,\zeta>0,\delta_2>1$, and let there be a differentiable mapping $\Psi:\Lambda\to\mathbb{R}$ such that $\sigma_1,\sigma_2\in\Lambda$ with $\sigma_2>\sigma_1$. If $\Psi^{(p)}\in L_1([\sigma_1,\sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a predominating quasiconvex function on Λ, then*

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\
&\leq 2n^{\frac{1}{\delta_2}}\left(\frac{\kappa n^{\frac{\delta_1(\zeta+\kappa(p-1)+\kappa)}{\kappa}}}{\delta_1(\zeta+\kappa(p-1)+\kappa)}\right)^{\frac{1}{\delta_1}}\\
&\quad\times\left(\max\left\{|\Psi^{(p)}(\sigma_2)|^{\delta_2},\left[|\Psi^{(p)}(\sigma_2)|^{\delta_2}+\eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2},|\Psi^{(p)}(\sigma_2)|^{\delta_2}\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right)^{\frac{1}{\delta_2}}.
\end{aligned}
$$

Proof. Since $|\Psi^{(p)}|^{\delta_2}$ is a predominating quasiconvex function on Λ, utilizing Lemma 1 and the the well-known Hölder inequality, we have

$$
\begin{aligned}
&|\Upsilon(p,n,\zeta,\kappa;\sigma_1,\sigma_2)(\Psi)|\\
&\leq\int_0^n(n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\tfrac{n+\tau}{2n}\sigma_1+\tfrac{n-\tau}{2n}\sigma_2\right)d\tau+\int_0^n(n-\tau)^{\frac{\zeta}{\kappa}+p-1}\Psi^{(p)}\left(\tfrac{n-\tau}{2n}\sigma_1+\tfrac{n+\tau}{2n}\sigma_2\right)d\tau\\
&\leq 2\left(\int_0^n(n-\tau)^{(\frac{\zeta}{\kappa}+p-1)}d\tau\right)^{1-\frac{1}{\delta_2}}\\
&\quad\times\left(\int_0^n|(n-\tau)^{(\frac{\zeta}{\kappa}+p-1)}|\left(\max\left\{|\Psi^{(p)}(\sigma_2)|^{\delta_2},\left[|\Psi^{(p)}(\sigma_2)|^{\delta_2}+\eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2},|\Psi^{(p)}(\sigma_2)|^{\delta_2}\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right)d\tau\right)^{\frac{1}{\delta_2}}\\
&=2\left(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta+\kappa p}\right)^{1-\frac{1}{\delta_2}}\\
&\quad\times\left(\max\left\{|\Psi^{(p)}(\sigma_2)|^{\delta_2},\left[|\Psi^{(p)}(\sigma_2)|^{\delta_2}+\eta\left(|\Psi^{(p)}(\sigma_1)|^{\delta_2},|\Psi^{(p)}(\sigma_2)|^{\delta_2}\right)\right]\right\}+\mathbb{D}(\sigma_1,\sigma_2)\right)^{\frac{1}{\delta_2}},
\end{aligned}
$$

the required result. □

Some special cases of Theorem 6 can be discussed as follows.
(I) If we choose $\mathbb{D}(\sigma_1,\sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau)+(n-\tau)(n+\tau)^{\varrho}\}(\sigma_2-\sigma_1)^{\varrho}$, then we get higher-order strongly η-quasiconvex function.

Corollary 23. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0, \delta_2 > 1$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a higher-order strongly η-quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq 2\Big(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta + \kappa p}\Big)^{1-\frac{1}{\delta_2}} \Bigg(\max\Big\{ |\Psi^{(p)}(\sigma_2)|^{\delta_2}, \big[|\Psi^{(p)}(\sigma_2)|^{\delta_2} + \eta\big(|\Psi^{(p)}(\sigma_1)|^{\delta_2}, |\Psi^{(p)}(\sigma_2)|^{\delta_2}\big)\big] \Big\} \\
&\quad - \frac{4n\mu(\sigma_2 - \sigma_1)^{\varrho}}{(\varrho+1)(\varrho+2)} \Bigg)^{\frac{1}{\delta_2}}.
\end{aligned}$$

(II) If we choose $\mathbb{D}(\sigma_1, \sigma_2) = -\frac{\mu}{(2n)^{\varrho+1}}\{(n-\tau)^{\varrho}(n+\tau) + (n-\tau)(n+\tau)^{\varrho}\}(\sigma_2 - \sigma_1)^{\varrho}$ along with $\eta\big(\Psi(\sigma_2), \Psi(\sigma_1)\big) = \Psi(\sigma_2) - \Psi(\sigma_1)$, then we get higher-order strongly quasiconvex function.

Corollary 24. *For $n, p \in \mathbb{N}, \kappa > 0, \zeta > 0, \delta_2 > 1$, and let there be a differentiable mapping $\Psi : \Lambda \to \mathbb{R}$ such that $\sigma_1, \sigma_2 \in \Lambda$ with $\sigma_2 > \sigma_1$. If $\Psi^{(p)} \in L_1([\sigma_1, \sigma_2])$ and $|\Psi^{(p)}|^{\delta_2}$ is a higher-order strongly quasiconvex function on Λ, then*

$$\begin{aligned}
&|\Upsilon(p, n, \zeta, \kappa; \sigma_1, \sigma_2)(\Psi)| \\
&\leq 2\Big(\frac{\kappa n^{\frac{\zeta+\kappa p}{\kappa}}}{\zeta + \kappa p}\Big)^{1-\frac{1}{\delta_2}} \Bigg(\big|\Psi^{(p)}(\sigma_2)|^{\delta_2} + |\Psi^{(p)}(\sigma_1)|^{\delta_2}\big) - \frac{4n\mu(\sigma_2 - \sigma_1)^{\varrho}}{(\varrho+1)(\varrho+2)} \Bigg)^{\frac{1}{\delta_2}}.
\end{aligned}$$

7. Conclusions

A new concept of predominating $\hbar$-convex function with respect to η with different kinds of convexities is presented. Meanwhile, we established an auxiliary result for *pth*-order differentiable functions. Moreover, we established numerous novel outcomes for predominating $\hbar$-convex function for *pth*-order differentiability and predominating quasiconvex functions. Here, we accentuate that all the determined results in the present paper endured preserving for higher-order strongly η-convex functions that can be perceived by the one of a kind estimations of ϱ and μ. The newly introduced numerical approximation will use to solve for parallelogram law in Banach space. We expect that these innovative techniques of this article will stimulate the specialists studying in functional analysis (uniform smoothness of norms in Banach space) in [43–45]. This is a new path for futuristic research.

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