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Oscillation Criteria for a Class of Third-Order Damped Neutral Differential Equations

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Abstract: In this paper, we study the asymptotic and oscillatory properties of a certain class of third-order neutral delay differential equations with middle term. We obtain new characterizations of oscillation of the third-order neutral equation in terms of oscillation of a related, well-studied, second-order linear equation without damping. An Example is provided to illustrate the main results.

Keywords: third-order differential equations; delay; oscillation criteria

1. Introduction

In this paper, we consider the third-order nonlinear damped neutral differential equation of the form

$$\left(r_2\left(r_1\left(y'\right)^{\alpha}\right)'\right)'(t)+b(t)\left(y'(t)\right)^{\alpha}+q(t)f(x(\sigma(t)))=0,\ \text{for } t\geq t_0, \tag{1}$$

where $y(t)=x(t)+p(t)x(\tau(t))$, α is a ratio of positive odd integers and $f\in C(\mathbb{R},\mathbb{R})$ satisfies

$$f(x)\geq kx^{\alpha}\ \text{for } x\neq 0.$$

Throughout this paper, we assume the following conditions:

($\mathbf{I}_1$) $r_1,r_2\in C([t_0,\infty),(0,\infty))$

$$\int_{t_1}^{\infty}r_1^{-1/\alpha}(s)\,\mathrm{d}s=\infty \text{ and } \int_{t_1}^{\infty}r_2^{-1}(s)\,\mathrm{d}s=\infty,\ t_1\geq t_0,\ \ t_1\in[t_0,\infty);$$

($\mathbf{I}_2$) $p,q\in C([t_0,\infty),[0,\infty))$, $p(t)\leq p_0<\infty$, q does not vanish identically;

($\mathbf{I}_3$) $\sigma,\tau\in C^1([t_0,\infty),\mathbb{R})$, $\sigma(t)<t$, $\tau(t)<t$, $\tau'\geq\tau_0$, $\sigma\circ\tau=\tau\circ\sigma$ and $\lim_{t\to\infty}\sigma(t)=\lim_{t\to\infty}\tau(t)=\infty$.

A solution of (1), we mean $x\in C([T_x,\infty),[0,\mathbb{R}))$ with $T_x\geq t_0$, which satisfies the property $y',r_2\left(r_1\left(y'\right)^{\alpha}\right)'\in C^1([T_x,\infty),\mathbb{R}))$ and moreover satisfies (1) on $[T_x,\infty)$. We consider the nontrivial solutions of (1) existing on some half-line $[T_x,\infty)$ and satisfying the condition $\sup\{|x(t)|:T\leq t<\infty\}>0$ for any $T\geq T_x$.

Moreover, throughout our results, we need an assumption:

(A) There exists a nonoscillatory a solution of

$$\left(r_2 z'\right)'(t) + \left(\frac{p(t)}{r_1(t)}\right) z(t) = 0. \tag{2}$$

A solution x of (1) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, it is easy to notice the growing interest in studying the qualitative properties of solutions to fractional/functional differential and difference equations, see [1–4]. The third-order differential equations have an important applications in many problems for instance, economy, physics, biology and population dynamics, see [5–7]. Although importance of those kind of equations in applications they had been realized very early.

In the last three decades, a few results asymptotic behavior of oscillation of third-order have been studied in the literatures. But even-order differential equations have been deeply studied, see [8–14]. In the early twentieth century, it have been appeared the basic interested paper in asymptotic behavior of third-order differential equations [15]. Recently, a study has developed, especially oscillatory or nonoscillatory of solutions studying by various techniques, see [16–23].

For the sake of brevity, we define the operators

$$£_1 y(t) := r_1 \left(y'\right)^{\alpha}, \quad £_2 y(t) := r_2 \left(r_1 \left(y'\right)^{\alpha}\right)', \quad £_3 y(t) := \left(r_2 \left(r_1 \left(y'\right)^{\alpha}\right)'\right)'.$$

From Equation (1) and assumption for $f(x)$, we obtain the inequality

$$£_3 y(t) + \frac{b(t)}{r_1(t)} £_1 y(t) + kq(t) x^{\alpha}(\sigma(t)) \leq 0. \tag{3}$$

Through this paper, we will use the following notation:

$$\eta_1(t, t_1) := \int_{t_1}^{t} (r_1(s))^{-\frac{1}{\alpha}} \, ds, \quad \eta_2(t, t_1) := \int_{t_1}^{t} (r_2(s))^{-1} \, ds,$$

$$\widetilde{\eta}_2(t, t_1) := \left(\frac{\eta_2(t, t_1)}{r_1(t)}\right)^{1/\alpha}, \quad \widehat{\eta}_2(t, t_1) := \int_{t_1}^{t} \widetilde{\eta}_2(s, t_1) \, ds.$$

Lemma 1. *[23] Assume that $c_1, c_2 \in [0, \infty)$ and $\gamma > 0$. Then*

$$(c_1 + c_2)^{\gamma} \leq \mu \left(c_1^{\gamma} + c_2^{\gamma}\right),$$

where

$$\mu := \begin{cases} 1 & \text{if } \gamma \leq 1 \\ 2^{\gamma - 1} & \text{if } \gamma > 1. \end{cases}$$

2. Results and Proofs

Lemma 2. *Assume that (A) holds. If x is a nonoscillatory solution of (1), then there are two possible classes for y :*

$$\begin{aligned} \mathbf{N}_1 &= \{y(t) : £_1 y(t) > 0,\ £_2 y(t) > 0\}; \\ \mathbf{N}_2 &= \{y(t) : £_1 y(t) < 0,\ £_2 y(t) > 0\}. \end{aligned}$$

Proof. Let x be a positive solution of (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$. By (3), it is easy to see that

$$\left(r_2 z'(t)\right)'(t) + \frac{b(t)}{r_1(t)} z(t) > 0, \tag{4}$$

where $z(t) = -\pounds_1 y(t)$. Let $u(t) > 0$ be a solution of (2) for $t \geq t_1 \geq t_0$. Assume that $z > 0$ is oscillatory. Hence z has consecutive zeros at a and b $(t_1 < a < b)$ such that $z'(a) \geq 0,\ z'(b) \leq 0$ for $t \in (a, b)$. This implies that

$$\begin{aligned}
0 &< \int_a^b \left(\left(r_2 z'(t)\right)'(t) + \frac{b(t)}{r_1(t)} z(t) \right) u(t)\,\mathrm{d}t \\
&= r_2(t) z'(t) u(t)\big|_a^b - \int_a^b r_2(t) z'(t) u'(t)\,\mathrm{d}t + \int_a^b \frac{b(t)}{r_1(t)} z(t) u(t)\,\mathrm{d}t \\
&= r_2(t) z'(t) u(t)\big|_a^b - r_2(t) z(t) u'(t)\big|_a^b + \int_a^b \left(r_2(t) u'(t)\right)' z(t)\,\mathrm{d}t + \int_a^b \frac{b(t)}{r_1(t)} z(t) u(t)\,\mathrm{d}t \\
&= r_2(t) z'(t) u(t)\big|_a^b + \int_a^b \left(\left(r_2(t) u'(t)\right)' + \frac{b(t)}{r_1(t)} u(t) \right) z(t)\,\mathrm{d}t = r_2(t) z'(t) u(t)\big|_a^b \leq 0.
\end{aligned}$$

This contradiction completes the proof. □

Lemma 3. *Assume x is a nonoscillatory solution of (1) with $y \in N_1$. Then*

$$\pounds_1 y(t) \geq \eta_2(t, t_1) \pounds_2 y(t) \tag{5}$$

and

$$y(t) \geq \widehat{\eta}_2(t, t_1) \left(\pounds_2 y(t)\right)^{1/\alpha}. \tag{6}$$

Proof. Let x be a positive solution and $y \in N_1$ be a solution of (1). Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. From (1), we see that $\pounds_3 y(t) \leq 0$. Thus

$$\begin{aligned}
\pounds_1 y(t) &= \int_{t_1}^t \left(\pounds_1 y(s)\right)' \mathrm{d}s + \pounds_1 y(t_1) \geq \int_{t_1}^t \frac{1}{r_2(s)} \pounds_2 y(s)\,\mathrm{d}s \geq \pounds_2 y(t) \int_{t_1}^t \frac{1}{r_2(s)} \mathrm{d}s \\
&= \eta_2(t, t_1) \pounds_2 y(t).
\end{aligned}$$

That is,

$$y'(t) \geq \left(\frac{\eta_2(t, t_1)}{r_1(t)} \right)^{1/\alpha} \left(\pounds_2 y(t)\right)^{1/\alpha} = \widetilde{\eta}_2(t, t_1) \left(\pounds_2 y(t)\right)^{1/\alpha}. \tag{7}$$

Now, integrating (7) from t_1 to t, we get

$$\begin{aligned}
y(t) &\geq \int_{t_1}^t \widetilde{\eta}_2(s, t_1) \left(\pounds_2 y(s)\right)^{1/\alpha} \mathrm{d}s \geq \left(\pounds_2 y(t)\right)^{1/\alpha} \int_{t_1}^t \widetilde{\eta}_2(s, t_1)\,\mathrm{d}s \\
&= \widehat{\eta}_2(t, t_1) \left(\pounds_2 y(t)\right)^{1/\alpha}.
\end{aligned}$$

Thus, the proof is complete. □

Lemma 4. *Assume x is nonoscillatory solution of (1) with $y \in N_2$. Then*

$$y(u) \geq \left(-\pounds_1 y(v)\right)^{1/\alpha} \eta_1(v, u), \ \textit{for } v \geq u \geq t. \tag{8}$$

Proof. Since $£_1 y(t)$ is nondecreasing, for $v \geq u \geq t$, we have

$$\begin{aligned} y(u) &= y(v) - \int_u^v \frac{1}{r_1^{\frac{1}{\alpha}}(\vartheta)} (£_1 y(\vartheta))^{1/\alpha} \mathrm{d}\vartheta \\ &\geq -(£_1 y(v))^{1/\alpha} \int_u^v \frac{1}{r_1^{\frac{1}{\alpha}}(\vartheta)} \mathrm{d}\vartheta \\ &= (-£_1 y(v))^{1/\alpha} \eta_1(v,u), \end{aligned}$$

i.e.,

$$y(u) \geq (-£_1 y(v))^{1/\alpha} \eta_1(v,u).$$

The proof of the lemma is complete. □

Lemma 5. *Assume that x is a positive a solution and $y \in N_1$ and $p(t) \in (0,1)$. Then*

$$£_3 y(t) + \frac{b(t)}{r_1(t)} £_1 y(t) + kQ(t) y^\alpha(\sigma(t)) \leq 0, \tag{9}$$

where

$$Q(t) = q(t)(1 - p(\sigma(t)))^\alpha.$$

Proof. Let x be a positive a solution of (1) and $y \in N_1$. Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. The corresponding $y(t)$ satisfies

$$\begin{aligned} x(t) &= y(t) - p(t) x(\tau(t)) \geq y(t) - p(t) y(\tau(t)) \\ &\geq y(t) - p(t) y(t) = y(t)(1 - p(t)). \end{aligned}$$

That is

$$x^\alpha(\sigma(t)) \geq y^\alpha(\sigma(t))(1 - p(\sigma(t)))^\alpha. \tag{10}$$

Combining (3) and (10), we have

$$£_3 y(t) + \frac{b(t)}{r_1(t)} £_1 y(t) + kQ(t) y^\alpha(\sigma(t)) \leq 0.$$

The proof of the lemma completed. □

Lemma 6. *Assume x is a positive a solution of (1) and $y \in N_2$. Then*

$$\left(1 + \frac{p_0^\alpha}{\tau_0}\right) £_3 y(t) + \left(\frac{b(t)}{r_1(t)} £_1 y(t) + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))} £_1 y(\tau(t))\right) + \frac{k}{\mu} \widetilde{q}(t) y^\alpha(\sigma(t)) \leq 0, \tag{11}$$

where $\widetilde{q}(t) := \min\{q(t), q(\tau(t))\}$.

Proof. Let x be a positive a solution of (1) and $y \in N_2$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$. From Lemma 1, we obtain

$$y^\alpha(t) \leq \mu(x^\alpha(t) + p_0^\alpha x^\alpha(\sigma(t))). \tag{12}$$

Now, from (3) and ($\mathbf{I}_3$), we have

$$0 \geq \frac{p_0^\alpha}{\tau_0} £_3 y(\tau(t)) + p_0^\alpha \frac{b(\tau(t))}{(\tau_0)^{\frac{1}{\alpha}} r_1(\tau(t))} £_1 y(\tau(t)) + p_0^\alpha kq(\tau(t)) x^\alpha(\tau(\sigma(t))). \tag{13}$$

Combining (3) along with (13), we get

$$\begin{aligned} 0 &\geq \pounds_3 y(t) + \frac{p_0^\alpha}{\tau_0}\pounds_3 y(\tau(t)) + \left(\frac{b(t)}{r_1(t)}\pounds_1 y(t) + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}}\frac{b(\tau(t))}{r_1(\tau(t))}\pounds_1 y(\tau(t))\right) \\ &\quad + k\widetilde{q}(t)\left(x^\alpha(\sigma(t)) + p_0^\alpha x^\alpha(\tau(\sigma(t)))\right). \end{aligned}$$

By virtue of (12) and using $\pounds_3 y(t) \leq 0$, we have

$$0 \geq \left(1 + \frac{p_0^\alpha}{\tau_0}\right)\pounds_3 y(t) + \left(\frac{b(t)}{r_1(t)}\pounds_1 y(t) + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}}\frac{b(\tau(t))}{r_1(\tau(t))}\pounds_1 y(\tau(t))\right) + \frac{k}{\mu}\widetilde{q}(t)\, y^\alpha(\sigma(t)).$$

The proof of the lemma completed. □

Theorem 1. *Assume (A) holds, $\alpha \geq 1$ and $\sigma'(t) > 0$. If there exist a function $\delta(t) \in C^1([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1 \geq t_0$, there is a $t_2 \geq t_1$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{k}{\mu}Q(t)\delta(s) - \frac{(\delta'(t)r_1(t) - \delta(t)b(t))^2}{4\alpha\delta(t)(r_1(t))^2\sigma'(t)\widetilde{\eta}_2(\sigma(t),t_1)(\widehat{\eta}_2(\sigma(t),t_1))^{\alpha-1}}\right]\mathrm{d}s = \infty, \tag{14}$$

then, $N_1 = \varnothing$.

Proof. Let x be a positive a solution of (1) and $y \in N_1$. Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. By (1), we see that $\pounds_3 y(t) \leq 0$

$$\begin{aligned} \pounds_1 y(t) &= \int_{t_2}^{t}(\pounds_1 y(s))'\,\mathrm{d}s + \pounds_1 y(t_2) \leq \int_{t_2}^{t}\frac{1}{r_2(s)}\pounds_2 y(s)\,\mathrm{d}s + \pounds_1 y(t_2) \leq \pounds_2 y(t)\int_{t_2}^{t}\frac{1}{r_2(s)}\mathrm{d}s \\ &= \pounds_1 y(t_2) + \eta_2(t,t_2)\pounds_2 y(t), \end{aligned}$$

for $t_2 \geq t_1$, that is $\pounds_2 y(t) > 0$ otherwise $\lim_{t\to\infty}\pounds_1 y(t) = -\infty$, a contradiction. Define a positive function by

$$w(t) = \delta(t)\frac{\pounds_2 y(t)}{y^\alpha(\sigma(t))}. \tag{15}$$

Using (7), we have

$$y'(\sigma(t)) \geq \left(\frac{\eta_2(\sigma(t),t_1)}{r_1(\sigma(t))}\right)^{1/\alpha}(\pounds_2 y(\sigma(t)))^{1/\alpha} \geq \left(\frac{\eta_2(\sigma(t),t_1)}{r_1(\sigma(t))}\right)^{1/\alpha}(\pounds_2 y(t))^{1/\alpha},$$

hence,

$$\begin{aligned} \frac{y'(\sigma(t))}{y(\sigma(t))} &\geq \left(\frac{\eta_2(\sigma(t),t_1)}{r_1(\sigma(t))\delta(t)}\right)^{1/\alpha}\frac{\delta^{1/\alpha}(t)(\pounds_2 y(t))^{1/\alpha}}{y(\sigma(t))} \\ &= \left(\frac{\eta_2(\sigma(t),t_1)}{\delta(t)r_1(\sigma(t))}\right)^{1/\alpha}w^{1/\alpha}(t). \end{aligned} \tag{16}$$

Also by (6), it is easy to see that

$$w(t) = \delta(t)\frac{\pounds_2 y(t)}{y^\alpha(\sigma(t))} \leq \delta(t)\frac{\pounds_2 y(\sigma(t))}{y^\alpha(\sigma(t))} \leq \frac{\delta(t)}{(\widehat{\eta}_2(\sigma(t),t_1))^\alpha},$$

hence

$$w(t)^{(1/\alpha)-1} \leq \frac{(\delta(t))^{(1/\alpha)-1}}{(\widehat{\eta}_2(\sigma(t),t_1))^{1-\alpha}}. \tag{17}$$

Now, by differentiating (15), we get

$$w'(t) = \delta'(t)\frac{\pounds_2 y(t)}{y^{\alpha}(\sigma(t))} + \delta(t)\frac{\pounds_3 y(t)}{y^{\alpha}(\sigma(t))} - \frac{\alpha\delta(t)\, y^{\alpha-1}(\sigma(t))\, y'(\sigma(t))\,\sigma'(t)\,\pounds_2 y(t)}{y^{2\alpha}(\sigma(t))}. \tag{18}$$

Using (15) and (9), we obtain

$$\begin{aligned} w'(t) &= \frac{\delta'(t)}{\delta(t)}w(t) + \frac{\pounds_3 y(t)}{\pounds_2 y(t)}w(t) - \frac{\alpha\sigma'(t)\, y'(\sigma(t))}{y(\sigma(t))}w(t) \\ &\leq \frac{\delta'(t)}{\delta(t)}w(t) - \frac{\frac{b(t)}{r_1(t)}\pounds_1 y(t) - kQ(t)\, y^{\alpha}(\sigma(t))}{\pounds_2 y(t)}w(t) - \frac{\alpha\sigma'(t)\, y'(\sigma(t))}{y(\sigma(t))}w(t). \end{aligned}$$

It follows from (15) and (5) that

$$\begin{aligned} w'(t) &\leq \frac{\delta'(t)}{\delta(t)}w(t) - \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\, w(t) - kQ(t)\,\delta(t) - \frac{\alpha\sigma'(t)\, y'(\sigma(t))}{y(\sigma(t))}w(t) \\ &\leq \left(\frac{\delta'(t)}{\delta(t)} - \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\right)w(t) - kQ(t)\,\delta(t) - \alpha\sigma'(t)\frac{y'(\sigma(t))}{y(\sigma(t))}w(t). \end{aligned} \tag{19}$$

From (16), we get

$$w'(t) \leq \left(\frac{\delta'(t)}{\delta(t)} - \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\right)w(t) - kQ(t)\,\delta(t) - \alpha\sigma'(t)\left(\frac{\eta_2(\sigma(t),t_1)}{\delta(t)\, r_1(\sigma(t))}\right)^{1/\alpha} w^{(1/\alpha)-1}(t)\, w^2(t).$$

By (17), we have

$$\begin{aligned} w'(t) &\leq -kQ(t)\,\delta(t) + \left(\frac{\delta'(t)}{\delta(t)} - \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\right)w(t) \\ &\quad - \frac{\alpha\sigma'(t)}{\delta(t)}\widetilde{\eta}_2(\sigma(t),t_1)\left(\widehat{\eta}_2(\sigma(t),t_1)\right)^{\alpha-1} w^2. \end{aligned}$$

Applying the inequality

$$Au - Bu^2 \leq \frac{A^2}{4B},$$

with

$$A = \left(\frac{\delta'(t)}{\delta(t)} - \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\right),\quad B = \frac{\alpha\sigma'(t)}{\delta(t)}\widetilde{\eta}_2(\sigma(t),t_1)\left(\widehat{\eta}_2(\sigma(t),t_1)\right)^{\alpha-1}.$$

Thus,

$$\begin{aligned} w'(t) &\leq -\frac{k}{\mu}Q(t)\,\delta(t) + \frac{\left(\delta'(t)\, r_1(t) - \delta(t)\, b(t)\right)^2}{\delta(t)^2\left(r_1(t)\right)^2} \\ &\quad \times \frac{\delta(t)}{4\alpha\sigma'(t)\,\widetilde{\eta}_2(\sigma(t),t_1)\left(\widehat{\eta}_2(\sigma(t),t_1)\right)^{\alpha-1}}, \end{aligned}$$

that is

$$\begin{aligned} w'(t) &\leq -\frac{k}{\mu}Q(t)\,\delta(t) \\ &\quad + \frac{\left(\delta'(t)\, r_1(t) - \delta(t)\, b(t)\right)^2}{4\alpha\delta(t)\left(r_1(t)\right)^2\sigma'(t)\,\widetilde{\eta}_2(\sigma(t),t_1)\left(\widehat{\eta}_2(\sigma(t),t_1)\right)^{\alpha-1}}. \end{aligned} \tag{20}$$

Integrating (20) from t_2 to t, we obtain

$$\int_{t_2}^{t} \left[\frac{k}{\mu} Q(t)\,\delta(s) - \frac{(\delta'(t)\,r_1(t) - \delta(t)\,b(t))^2}{4\alpha\delta(t)\,(r_1(t))^2\,\sigma'(t)\,\widetilde{\eta}_2(\sigma(t),t_1)\,(\widehat{\eta}_2(\sigma(t),t_1))^{\alpha-1}} \right] \mathrm{d}s \le w(t_2).$$

The proof of the lemma is complete. □

Now, let

$$P_1(t) = \frac{b(t)}{r_1(t)}\eta_2(t,t_1)^{\alpha} \quad P_2(t) = kQ(t)\,(\widehat{\eta}_2(t,t_1))^{\alpha}$$

and

$$\mu(t) = \exp\left(\int_{t_1}^{t} P_1(s)\,\mathrm{d}s\right).$$

We present the following theorem.

Theorem 2. *Assume (A) holds. If every a solution of the first-order equation*

$$z'(t) + P_2(t)\,z(\sigma(t)) = 0. \tag{21}$$

or $£_2 y(t)$ *is oscillatory, then* $N_1 = \emptyset$.

Proof. Let x be a positive a solution of (1) and $y \in N_1$. Then there exists $t_1 \ge t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. As in Theorem 1, we have $£_2 y(t) > 0$. Hence by (9), we obtain

$$0 \ge £_3 y(t) + \frac{b(t)}{r_1(t)} £_1 y(t) + kQ(t)\,y^{\alpha}(\sigma(t)).$$

Using (5) and (6), we have

$$0 \ge £_3 y(t) + \frac{b(t)}{r_1(t)}\eta_2(t,t_1)\,£_2 y(t) + kQ(t)\,(\widehat{\eta}_2(\sigma(t),t_1))^{\alpha}\,£_2 y(\sigma(t)).$$

Now, set $\varpi(t) = £_2 y(t)$, we get

$$\varpi'(t) + P_1(t)\,\varpi(t) + P_2(t)\,\varpi(\sigma(t)) \le 0. \tag{22}$$

Multiplying (22) by $\mu(t)$, we have

$$(\mu\varpi)'(t) + \mu(t)\,P_2(t)\,\varpi(\sigma(t)) \le 0.$$

Now, setting the positive function $z = \mu\varpi$ and taking into account $\mu(t)$ is increasing function, we obtain

$$z'(t) + \frac{\mu(t)}{\mu(\sigma(t))} P_2(t)\,z(\sigma(t)) \le 0.$$

That is

$$z'(t) + P_2(t)\,z(\sigma(t)) \le 0.$$

In view of [24](Theorem 1), we see that the first-order delay differential Equation (21) has a positive a solution, a contradiction. Then, the proof is complete. □

Corollary 1. *Assume (A) holds. If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} P_2(t)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{23}$$

then $N_1 = \emptyset$.

Theorem 3. *Assume* (2) *is a oscillatory, then* $N_1 = \emptyset$.

Proof. Let x be a positive a solution of (1) and $y \in N_1$, there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. Set $\chi = \pounds_1 y(t)$ in (9), we see that

$$\left(r_2 \chi'\right)'(t) + \frac{b(t)}{r_1(t)} \chi(t) \leq 0.$$

In view of [19] (Lemma 2.6), (2) has positive a solution, a contradiction. Then, the proof is complete. □

Now, we can extend Theorem 2 to

$$\left(r_2\left(r_1\left(y'\right)^{\alpha}\right)'\right)'(t) + b(t)\left(y'(h(t))\right)^{\alpha} + q(t) f(x(\sigma(t))) = 0, \tag{24}$$

where $h \in C^1([t_0, \infty), \mathbb{R})$ is such that $\sigma(t) \leq h(t) \leq t$ and $h'(t) \geq 0$.

Theorem 4. *If every solution of the first-order equation*

$$\left(r_2 \chi'\right)'(t) + \frac{b(t)}{r_1(h(t))} \chi(h(t)) = 0, \tag{25}$$

or y' *is oscillatory, then* $N_1 = \emptyset$.

Proof. Let x be a positive solution of (1) and $y \in N_1$. Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. Now, we can easily extend Lemma 5 to the equation

$$\pounds_3 y(t) + \frac{b(t)}{r_1(h(t))} \pounds_1 y(h(t)) + kQ(t) y^{\alpha}(\sigma(t)) \leq 0.$$

Set $\chi = \pounds_1 y(t)$, we see that

$$\left(r_2 \chi'\right)'(t) + \frac{b(t)}{r_1(h(t))} \chi(h(t)) \leq 0.$$

In view of [19] (Lemma 2.6), (25) has a positive solution, a contradiction. Then, the proof is complete. □

Theorem 5. *Assume (A) holds and* $\alpha \geq 1$, *and there exists a function* $h \in C^1(I, \mathbb{R})$ *such that* $\sigma(t) < h(t) < t$, $h'(t) \geq 0$. *If*

$$\limsup_{t \to \infty} \int_{h(t)}^{t} \widetilde{Q}(s) \eta_2(h(t), h(s)) \, \mathrm{d}s > 1 \tag{26}$$

holds with

$$\widetilde{Q}(t) = \frac{\tau_0}{\tau_0 + p_0^{\alpha}} \left(\frac{ck}{\mu} \widetilde{q}(t) \eta_1(h(t), \sigma(t)) - \frac{p_0^{\alpha}}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))} - \frac{b(t)}{r_1(t)} \right) \text{ for all } t \geq t_1,$$

where c is positive constant, then $N_2 = \emptyset$ *or* $\pounds_2 y(t)$ *is oscillatory.*

Proof. Let x be a positive solution of (1) and $y \in N_2$. Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. We consider $\pounds_2 y(t) \leq 0$ cannot hold for all large t, by a double integration of

$$y'(t) = \left(\frac{\pounds_1 y(t)}{r_1(t)}\right)^{1/\alpha} \leq \left(\frac{\pounds_1 y(t_2)}{r_1(t)}\right)^{1/\alpha}$$

that is $y' < 0$, a contradiction. Thus $£_2 y(t) \geq 0$. By using (8) with $u = \sigma(t)$ and $v = h(t)$, we get $(\sigma(t) < h(t) < \tau(t))$

$$y(\sigma(t)) \geq \eta_1(h(t), \sigma(t))(-£_1 y(h(t)))^{\frac{1}{\alpha}}. \tag{27}$$

Substituting (27) into (11), we obtain

$$0 \geq \left(1 + \frac{p_0^\alpha}{\tau_0}\right) £_3 y(t) + \left(\frac{b(t)}{r_1(t)} £_1 y(t) + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))} £_1 y(\tau(t))\right) + \frac{k}{\mu} \widetilde{q}(t) y^\alpha(\sigma(t)).$$

Since $h(t) < \tau(t) < t$, we find

$$\begin{aligned} 0 \quad &\geq \quad \left(1 + \frac{p_0^\alpha}{\tau_0}\right) £_3 y(t) + £_1 y(h(t)) \left(\frac{b(t)}{r_1(t)} + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))}\right) \\ &\quad + \frac{k}{\mu} \widetilde{q}(t) \eta_1(h(t), \sigma(t))(-£_1 y(h(t)))^{\frac{1}{\alpha}}. \end{aligned}$$

Set $\theta(t) = -£_1 y(t)$

$$\begin{aligned} 0 \quad &\leq \quad \left(1 + \frac{p_0^\alpha}{\tau_0}\right) (r_2(t) \theta'(t))' + \theta(h(t)) \left(\frac{b(t)}{r_1(t)} + \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))}\right) \\ &\quad - \frac{ck}{\mu} \widetilde{q}(t) \eta_1(h(t), \sigma(t))(\theta(h(t)))^{\frac{1}{\alpha}-1} \theta(h(t)). \end{aligned}$$

Taking into account that is $\theta'(t) \leq 0$ and $\alpha \geq 1$, there exists positive constant c such that $(\theta(h(t)))^{\frac{1}{\alpha}-1} \geq c$. Thus

$$(r_2(t) \theta'(t))' \geq \frac{\tau_0}{\tau_0 + p_0^\alpha} \left(\frac{ck}{\mu} \widetilde{q}(t) \eta_1(h(t), \sigma(t)) - \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}} \frac{b(\tau(t))}{r_1(\tau(t))} - \frac{b(t)}{r_1(t)}\right) \theta(h(t)).$$

This implies

$$(r_2(t) \theta'(t))' \geq \widetilde{Q}(t) \theta(h(t)). \tag{28}$$

From (28) we see that $r_2 \theta'$ is increasing, we get

$$\begin{aligned} \theta(t) \quad &= \quad \theta(t_2) + \int_{t_2}^{t} \frac{r_2(s) \theta'(s)}{r_2(s)} \mathrm{d}s > \theta(t_2) + r_2(t_2) \theta'(t_2) \int_{t_2}^{t} \frac{1}{r_2(s)} \mathrm{d}s \\ &= \quad \theta(t_2) + r_2(t_2) \theta'(t_2) \eta_2(t, t_1). \end{aligned}$$

Thus, $\theta'(t_2) < 0$ otherwise we imply $\lim_{t \to 0} \theta(t) = \infty$ a contradiction to the boundedness of θ. So for $t_2 \geq t$, we have

$$\theta > 0 \quad \theta' < 0 \quad (r_2 \theta')' > 0.$$

Therefor, for $v \geq u \geq t_1$, we find

$$\theta(u) > \theta(u) - \theta(v) = -\int_u^v \theta'(s) \mathrm{d}s = -\int_u^v \frac{r_2(s) \theta'(s)}{r_2(s)} \mathrm{d}s.$$

Since $r_2 \theta'$ is increasing

$$\theta(u) > -r_2(v) \theta'(v) \int_u^v \frac{1}{r_2(s)} \mathrm{d}s = -\eta_2(v, u) r_2(v) \theta'(v). \tag{29}$$

In (29), setting $u = h(s)$ and $v = h(t)$, we have

$$\theta(h(s)) > -\eta_2(h(t), h(s))\, r_2(h(t))\, \theta'(h(t)).$$

By Integrating (28) from $h(t) \geq t_1$ to t, we get

$$\begin{aligned} -r_2(h(t))\,\theta'(h(t)) &> \int_{h(t)}^{t} \widetilde{Q}(s)\,\theta(h(s))\,\mathrm{d}s \\ &> -r_2(h(t))\,\theta'(h(t)) \int_{h(t)}^{t} \widetilde{Q}(s)\,\eta_2(h(t), h(s))\, r_2(h(t))\,\theta'(h(t)). \end{aligned}$$

Thus,

$$\int_{h(t)}^{t} \widetilde{Q}(s)\,\eta_2(h(t), h(s))\,\mathrm{d}s < 1,$$

which contradicts (26). The proof is complete. □

Theorem 6. *Assume (A) holds and $\alpha \geq 1$, there is function $h \in C^1(I, \mathbb{R})$ such that $\sigma(t) < h(t) < t$, $h'(t) \geq 0$. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left(\frac{1}{r_2(u)} \int_{u}^{t} \widetilde{Q}(s)\,\mathrm{d}s \right) \mathrm{d}u > 1 \tag{30}$$

holds with $\widetilde{Q}(t)$ defined as in Theorem 5, then $N_2 = \emptyset$ or $£_2 y(t)$ is oscillatory.

Proof. Let x be a positive solution of (1) and $y \in N_2$. Then there exists $t_1 \geq t_0$ such that $x(\sigma(t)) > 0$ and $x(g(t)) > 0$. As in Theorem 5 ,we obtain (28) and

$$\theta > 0 \;\; \theta' < 0 \;\; (r_2\theta')' > 0.$$

Integrating (28) from u to t, we get

$$-r_2(u)\,\theta'(u) > \int_{u}^{t} \widetilde{Q}(s)\,\theta(h(s))\,\mathrm{d}s \geq \theta(h(t)) \int_{u}^{t} \widetilde{Q}(s)\,\mathrm{d}s,$$

that is

$$-\theta'(u) > \frac{1}{r_2(u)}\theta(h(t)) \int_{u}^{t} \widetilde{Q}(s)\,\mathrm{d}s.$$

Integrating from $h(t)$ to t, we have

$$\theta(h(t)) > \theta(h(t)) \int_{h(t)}^{t} \left(\frac{1}{r_2(u)} \int_{u}^{t} \widetilde{Q}(s)\,\mathrm{d}s \right) \mathrm{d}u.$$

Thus

$$\int_{h(t)}^{t} \left(\frac{1}{r_2(u)} \int_{u}^{t} \widetilde{Q}(s)\,\mathrm{d}s \right) \mathrm{d}u < 1,$$

which contradicts (30). The proof is complete. □

Note that the conditions (14), (23) and (25) eliminate solutions from the class N_1, while conditions (26) and (30) eliminate solutions from the class N_2. By combining condition eliminate solutions from the class N_1 and condition eliminate solutions from the class N_2, we ensure that the solutions of (1) are oscillatory. Therefore, we get the following theorem

Theorem 7. *Assume that* **(***A***)** *holds and there is a function $h \in C^1(I, \mathbb{R})$ such that $\sigma(t) < h(t) < t$ and $h'(t) \geq 0$. Let one of the following statements are true:*
(**a**) *$\alpha \leq 1$, $\sigma'(t) > 0$ and there exists a function $\delta(t) \in C^1([t_0, \infty), (0, \infty))$ such that (14) and (26) hold;*

(**b**) $\alpha \leq 1$, $\sigma'(t) > 0$ *and there exists a function* $\delta(t) \in C^1([t_0, \infty), (0, \infty))$ *such that* (14) *and* (30) *hold;*
(**c**) $\alpha \geq 1$, (26) *and* (23) *hold;*
(**d**) $\alpha \geq 1$, (30) *and* (23) *hold.*
Then every solution of (1), *or* $\pounds_2 y(t)$, *is oscillatory.*

Theorem 8. *Assume that* **(A)** *holds and there is a function* $h \in C^1(I, \mathbb{R})$ *such that* $\sigma(t) < h(t) < t$ *and* $h'(t) \geq 0$. *Let one of the following statements are true:*
(**a**) $\alpha \geq 1$, (2) *is oscillatory and* (26) *holds;*
(**b**) $\alpha \geq 1$, (2) *is oscillatory and* (30) *holds;*
(**c**) $\alpha \geq 1$, (25) *is oscillatory and* (26) *holds;*
(**d**) $\alpha \geq 1$, (25) *is oscillatory and* (30) *holds.*
Then every solution of (1), *or* $y'(t)$, *is oscillatory.*

Example 1. *Consider the damped neutral differential equation*

$$y'''(t) + \frac{1}{2}y'(t) + \frac{1}{2}x\left(t - \frac{3\pi}{2}\right) = 0,$$

where $y(t) = x(t) + p_0 x(\tau_0)$, $\alpha = 1$. *Let* $h(t) = (t - \pi)$, *we see that*

$$\sigma(t) < h(t) < t,\ h'(t) \geq 0,$$

also

$$\widetilde{Q}(t) = \frac{\tau_0}{\tau_0 + p_0^\alpha}\left(\frac{ck}{\mu}\frac{1}{2}\eta_1\left(t - \pi, t - \frac{3\pi}{2}\right) - \frac{p_0^\alpha}{(\tau_0)^{\frac{1}{\alpha}}}\frac{b(\tau(t))}{r_1(\tau(t))} - 1\right),$$

where

$$\eta_1\left(t - \pi, t - \frac{3\pi}{2}\right) = \frac{3\pi}{2} - \pi.$$

By Theorem 6, *condition* (30) *becomes*

$$\frac{c\tau_0\pi + 2p_0 + 4\tau_0\pi^2}{8(\tau_0 + p_0)} > 1. \tag{31}$$

If (31) *hold, then it is clear that all conditions of Theorem* 5 *are satisfied, and hence every solution of* (1), *or* $y'(t)$, *is oscillatory.*

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