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The Fourth Axiom of Similarity Measures

Chun-Hsiao Chu [1,*], Chih-Ping Yen [2] and Yi-Fong Lin [3]

[1] Department of Tourism, Aletheia University, New Taipei City 25103, Taiwan
[2] Department of Information Management, Central Police University, Taoyuan 33304, Taiwan; peter@mail.cpu.edu.tw
[3] Department of Fashion Industry Management, Hsing Wu University, New Taipei City 25103, Taiwan; 109287@mail.hwu.edu.tw
* Correspondence: shawchu@mail.au.edu.tw

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Abstract: In this research, the fourth axiom to improve the well-defined examination of similarity measures is studied, where the measures have a symmetric structure. We first provide a theoretic enhancement of three correlation coefficient similarity measures that were proposed by a source paper. Second, we use the same numerical example proposed by the source paper for pattern recognition problems to illustrate that the weighted correlation coefficient similarity measure is dependent on the set of weights. Finally, we demonstrate that the correlation coefficient similarity measure in the intuitionistic fuzzy set environment can address the issue of practical fault diagnosis when solving the turbine engine problems using similarity measures with symmetric characteristics.

Keywords: intuitionistic fuzzy set; correlation coefficient; similarity measure; pattern recognition

1. Introduction

Since Zadeh [1] developed fuzzy sets and Atanassov [2] constructed intuitionistic fuzzy sets (IFSs), numerous studies have examined fuzzy sets and IFSs to determine their theoretic evolution and devise applications to practical problems. Recently, motivated by Ye [3], for a correlation coefficient similarity measure, Zhang et al. [4] developed three new similarity measures, one for fuzzy sets and two for IFSs.

A research tendency has emerged to improve the mathematical approach of analytical methods and algebraic procedures in previously published papers. For example, a series of papers—Deng et al. [5], Tang et al. [6], Lan et al. [7], Yang et al. [8], Deng [9], Chang et al. [10], Jung et al. [11], and Deng et al. [12]—make revisions to existing proofs. Motivated by these articles, Zhang et al. [4] provided a new direction for similarity measures with correlation coefficient types, which is worthy of careful examination. Based on the detailed study of Zhang et al. [4], we found that there is a questionable result about their proof on a well-defined similarity measure. Specifically, Zhang et al. [4] only adopted the axioms of Gerstenkorn and Manko [13] to solve the problem. However, following a comprehensive study, we concluded that most researchers tend to include the fourth axiom [14] than to use Gerstenkorn and Manko [13] alone. For example, Ye [3] mentioned that the systems of axioms of both Gerstenkorn and Manko [13] and Li and Cheng [14] area well-defined similarity measure. However, in an examination of the satisfying axioms for well-defined similarity measures, Ye [3] only investigated the three axioms of Li and Cheng [14] and neglected their fourth proposed axiom. Hence, the first goal of this paper is to provide a revision to enhance the proof of Zhang et al. [4] on their similarity measures for the fourth axiom of Li and Cheng [14].

Moreover, we note that the third similarity measure proposed by Zhang et al. [4], which isa weighted correlation coefficient similarity measure, is dependent on the weights for elements in the universe of discourse.

Finally, we demonstrate that the second similarity measure proposed by Zhang et al. [4] addressed a practical pattern recognition problem of fault diagnosis for the turbine engine. If a turbine engine does not operate optimally, an engineer attempts to determine the cause of possible problems and may replace malfunctioning components. Potential explanations for the suboptimal performance can be treated as patterns and the engineer represents a sample facing pattern recognition problem. Using fuzzy sets or IFSs to address practical issues involving significant uncertainty and missing information can present a vague environment in a well-defined setting.

2. Brief Review of Similarity Measures with Intuitionistic Fuzzy Sets

Zadeh [1] was the first author to develop the fuzzy set theorem to deal with uncertain conditions. More than twenty thousand papers and hundreds of books have followed his approach to investigate complicated and dynamic real-world issues. One extension of the fuzzy set theorem is the proposal of IFSs by Atanassov [2], which have been used extensively in numerous variations to address the problem of uncertainty. In the following, we recall the definition of an intuitionistic fuzzy set and several related similarity measures.

Definition 1. (Atanassov [2]). *We assume that X is the universe of discourse; then, an intuitionistic fuzzy set on X is an object having the expression*

$$A = \{\langle \mathrm{x},\ \mu_{\mathrm{A}}(\mathrm{x})\rangle,\ \mathrm{v}_{\mathrm{A}}(\mathrm{x}) : \mathrm{x} \in X\}, \tag{1}$$

where $\mu_A(x) : X \to [0,1]$ *is the membership function and* $v_A(x) : X \to [0,1]$ *is the non-membership function with* $\mu_A(x) + v_A(x) \leq 1$.

$$\pi_{\mathrm{A}}(\mathrm{x}) = 1 - \mu_{\mathrm{A}}(\mathrm{x}) - \mathrm{v}_{\mathrm{A}}(\mathrm{x}) \tag{2}$$

is the hesitation degree with $\pi_A(x) : X \to [0,1]$.

Hundreds of similarity measures have been defined for intuitionistic fuzzy sets. Several are listedin the following.

Li and Cheng [14] assumed an auxiliary notation, $\varphi_{\mathrm{A}}(\mathrm{x})$ with

$$\varphi_{\mathrm{A}}(\mathrm{x}) = \frac{\mu_{\mathrm{A}}(\mathrm{x}) + 1 - \mathrm{v}_{\mathrm{A}}(\mathrm{x})}{2}, \tag{3}$$

Then, for two intuitionistic fuzzy sets, A and B, Li and Cheng [14] defined a similarity measure, $\mathrm{S}_{\mathrm{d}}^{\mathrm{p}}(A,B)$, as

$$\mathrm{S}_{\mathrm{d}}^{\mathrm{p}}(A,B) = 1 - \frac{1}{\sqrt[p]{\mathrm{n}}}\sqrt[p]{\sum_{\mathrm{i}=1}^{\mathrm{n}}(\varphi_{\mathrm{A}}(\mathrm{x_i}) - \varphi_{\mathrm{B}}(\mathrm{x_i}))^{\mathrm{p}}}, \tag{4}$$

where the universe of discourse is $X = \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}\}$.

For two intuitionistic fuzzy sets, A and B, Hung and Wang [15] considered a new similarity measure, $\mathrm{C}_{\mathrm{IFS}}^{\mathrm{new}}(A,B)$, as

$$\mathrm{C}_{\mathrm{IFS}}^{\mathrm{new}}(A,B) = \frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mu_{\mathrm{A}}(\mathrm{x_i})\mu_{\mathrm{B}}(\mathrm{x_i}) + \mathrm{v}_{\mathrm{A}}(\mathrm{x_i})\mathrm{v}_{\mathrm{B}}(\mathrm{x_i}) + \pi_{\mathrm{A}}(\mathrm{x_i})\pi_{\mathrm{B}}(\mathrm{x_i})}{\sqrt{\mu_{\mathrm{A}}^2(\mathrm{x_i}) + \mathrm{v}_{\mathrm{A}}^2(\mathrm{x_i}) + \pi_{\mathrm{A}}^2(\mathrm{x_i})}\sqrt{\mu_{\mathrm{B}}^2(\mathrm{x_i}) + \mathrm{v}_{\mathbf{B}}^2(\mathrm{x_i}) + \pi_{\mathrm{B}}^2(\mathrm{x_i})}}, \tag{5}$$

where the universe of discourse is $X = \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}\}$.

For two intuitionistic fuzzy sets, A and B, Hung et al. [16] developed a new similarity measure, $\mathrm{S}_{\lambda,\,\mathrm{W}}^{\mathrm{p}}(A,B)$,

$$S^{p}_{\lambda,W}(A,B) = 1 - \left(\sum_{i=1}^{n} w_i \left(\frac{|\mu_A(x_i) - \mu_B(x_i)| + \lambda|\pi_A(x_i) - \pi_B(x_i)|}{2}\right)^p\right)^{1/p}, \tag{6}$$

where $W = \{w_1, w_2, \ldots, w_n\}$ is the set of weights for elements in the universe of discourse, with $\sum_{i=1}^{n} w_i = 1$, $1 \le p < \infty$, and λ is the preference value for the decision-maker, with $0 \le \lambda \le 1$.

For two intuitionistic fuzzy sets, A and B, under a continuous domain, Julian et al. [17] assumed the following similarity measure, $S_{new,p}(A,B)$,

$$\begin{aligned} S_{new,p}(A,B) &= 1 - \left(\int w(x)|\mu_A(x) - \mu_B(x)|^p\right)^{1/p} \\ &-\left(\int w(x)|\phi_A(x) - \phi_B(x)|^p\right)^{1/p}, \end{aligned} \tag{7}$$

where $w(x)$ is the weight function, with $w(x) \ge 0$ and $\int w(x)dx = 1$, with $1 \le p < \infty$.

Chu and Guo [18] constructed a similarity measure for two intuitionistic fuzzy sets, A and B, as follows:

$$S(A,B) = \frac{1}{1 + \left(\sum_{i=1}^{n} w_i(\Delta_1 + \Delta_2 + \Delta_3)\right)^{1/\alpha}}, \tag{8}$$

where $\Delta_1 = \delta_1|\mu_A(x_i) - \mu_B(x_i)|^\alpha$, $\Delta_2 = \delta_2|v_A(x_i) - v_B(x_i)|^\alpha$, and $\Delta_3 = \delta_3|\pi_A(x_i) - \pi_B(x_i)|^\alpha$ are three abbreviations to simplify the expressions; w_i are the weights for elements in the universe of discourse, for $i = 1, 2, \ldots, n$, and δ_1, δ_2, and δ_3 are weights of the membership, non-membership, and hesitation functions; $\alpha \ge 1$ is a constant.

For two intuitionistic fuzzy sets, A and B, Yen et al. [19] constructed two similarity measures, $S_{q,\rho,w}(A,B)$ and $S_{q,\rho,w,M}(A,B)$,

$$S_{q,\rho,w}(A,B) = 1 - \left(\sum_{i=1}^{n} w_i \left(\frac{|\mu_A(x_i) - \mu_B(x_i)| + \rho|\pi_A(x_i) - \pi_B(x_i)|}{2}\right)^q\right)^{1/q}, \tag{9}$$

and

$$S_{q,\rho,w,M}(A,B) = 1 - \left(\sum_{i=1}^{n} w_i(\Omega_1 + \Omega_2)^q\right)^{1/q}, \tag{10}$$

where $\Omega_1 = \frac{|\mu_A(x_i)-\mu_B(x_i)|}{2\max\{\mu_A(x_i),\mu_B(x_i)\}}$ and $\Omega_2 = \frac{\rho|\pi_A(x_i)-\pi_B(x_i)|}{2\max\{\pi_A(x_i),\pi_B(x_i)\}}$ are two abbreviations to simplify the expressions and $W = \{w_1, w_2, \ldots, w_n\}$ is the set of weights for elements in the universe of discourse, with $\sum_{i=1}^{n} w_i = 1$, $1 \le q < \infty$, and ρ is the preferred rate for the decision-maker, with $0 \le \rho$.

3. Review of the Source Paper

Based on Gerstenkorn and Manko [13] and Ye [3], Zhang et al. [4] mentioned that the three axioms for a well-defined similarity measure denoted as (A1), (A2), and (A3) in the following $S : IFSs(X) \times IFSs(X) \to [0,1]$ should satisfy the following three requirements:

For three IFSs A, B and C in $IFSs(X)$,

(A1) $0 \le S(A,B) \le 1$;
(A2) If $A = B$, then $S(A,B) = 1$;
(A3) $S(A,B) = S(B,A)$.

Zhang et al. [4] developed three similarity measures. We cite them in the following.

For two FSs, $A = (\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n))$ and $B = (\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n))$ with the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, the first similarity measure is defined as

$$S^{FS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i)}. \tag{11}$$

For two *IFSs* A and B, the second similarity measure is defined as

$$S^{IFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2[\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)]}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)}. \tag{12}$$

For two *IFSs* A and B, the third similarity measure is defined as

$$WS^{IFS}(A,B) = \sum_{i=1}^{n} w_i \frac{2[\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)]}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)}, \tag{13}$$

where $0 \le w_i$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

4. Our Patchwork for the Fourth Axiom (A4) for the Source Paper

Li and Cheng [14] claimed that, besides the three axioms (A1), (A2), and (A3), a well-defined similarity measure should also satisfy the fourth axiom (A4) as cited below:

(A4) If $A \subseteq B \subseteq C$, then $S(A,C) \le S(A,B)$, and $S(A,C) \le S(B,C)$.

Up to now, 624 papers have cited Li and Cheng [14] in their references—for example, Hung and Lin [20] Julian et al. [17], Tung et al. [21] Hung and Lin [22], Yen et al. [19], Hung and Wang [15], Chu and Guo [18], Tung and Hopscotch [23], and Hung et al. [16]—to indicate that to include (A4) for a well-defined similarity measure isaccepted by the research community. We compare the above ten papers in Table 1.

Table 1. Comparisons among several papers citing Li and Cheng [14].

	Counterexample	Theoretical Improvement	New Measure	Check Axiom A4	Iterative Algorithm	Real Application
[14]			√	√		√
[20]	√		√			√
[17]	√	√	√			
[21]	√	√				
[22]		√				√
[19]	√		√	√	√	
[15]	√		√	√	√	√
[18]			√	√	√	√
[23]	√	√				
[16]	√	√	√	√	√	√

Especially, inthe last three years, 152 papers have cited Li and Cheng [14] in their references. We pay attention to those 17 papers which are related to decision sciences in the following. Aggarwal et al. [24] applied Hurwicz optimism–pessimism criterion to solve Atanassov's I-fuzzy linear programming problems by changing convex breakpoints into concave breakpoints on the lines with the indeterminacy factor resolution principle. Farhadinia and Xu [25] established a metrical T-norm-based similarity measure to compare with a metrical T-norm-based entropy measure for hesitant fuzzy sets. Fei et al. [26] defined a new vector-valued similarity measure for intuitionistic fuzzy sets that contain a similarity measure and an uncertainty measure to express all data in the universe of discourse that satisfy all axioms of intuitionistic fuzzy sets. Joshi and Kumar [27] considered a new approach to applying exponential hesitant fuzzy entropy in multiple attribute decision-making problems. They constructed two methods to derive criterion weight. Khanmohammadi et al. [28] constructed a new fuzzy logarithmic least squares method to rank the strategic objectives by the fuzzy similarity technique to improve efficiency and the significance level. Li and Liu [29] extended two classical distances with fuzzy sets to intuitionistic fuzzy sets that satisfy the approximation and continuity properties of a method while dealing with intuitionistic fuzzy reasoning. Lin [30] used the technique for order preference by similarity to the ideal solution method to solve a group multi-criteria decision-making problem with a new distance measure that satisfied axioms of distance measure.Mishra and Rani [31] developed an interval-valued intuitionistic fuzzy method to derive weights for attributes and experts for a reservoir flood control management policy. Rani et al. [32] applied the Shapley function to deal with interval-valued intuitionistic fuzzy methods and then addressed an investment problem with an incomplete and uncertain information environment. Rouyendegh [33] constructed a new intuitionistic fuzzy index of hesitation degree method to handle multi-criteria decision-making problems under incomplete information conditions. Shen et al. [34] generalized the technique for order preference by similarity to the ideal solution method by a new similarity measure under an intuitionistic fuzzy set environment that was applied to solve credit risk evaluation problems. Shokeen and Rana [35] provided a brief introduction for advanced fuzzy sets that is the generalization of fuzzy sets, rough sets (for incomplete data), interval-valued fuzzy sets (for uncertainty and vagueness), and soft sets (for insufficiency of parameterization).Wang et al. [36] developed two fuzzy aggregate operators to deal with multi-criteria decision-making problems with Pythagorean fuzzy linguistics that are generalizations for many previously existing operators. Wei [37] constructed new similarity measures for fuzzy sets, interval-valued intuitionistic fuzzy sets, and picture fuzzy sets and then applied those similarity measures to solve building material recognition problems. Zhang et al. [38] used the technique for order preference by similarity to the ideal solution method to estimate dynamic agents to the positive ideal agent and the negative ideal agent under the intuitionistic fuzzy number conditions. Zhou et al. [39] developed the hesitant fuzzy envelopment analysis model, the deviation-oriented hesitant fuzzy envelopment analysis model, and the score-oriented hesitant fuzzy envelopment analysis model to derive score and deviation values. Hence, the subjective preferences of decision-makers for the attributes can be examined in the evaluation procedure.

In Zhang et al. [4], they only proved that their three similarity measures satisfy three axioms (A1), (A2), and (A3). However, Zhang et al. [4] did not discuss the fourth axiom (A4). Therefore, the first goal of our paper is to provide a patchwork to verify three similarity measures developed by Zhang et al. [4] that satisfy (A4) to complete the proof for well-defined similarity measures.

Based on Liang and Shi [40] and Atanassov [2,41,42], we know that for three $IFSs(X)$ A, B, and C satisfying $A \subseteq B \subseteq C$ if and only if for every x_i in the universe of discourse, $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$ and $v_A(x_i) \leq v_B(x_i) \leq v_C(x_i)$, where μ_A is the membership function and v_A is the non-membership function for the intuitionistic fuzzy set, A.

We present our first theoretic result for the similarity measure proposed by Zhang et al. [4] for fuzzy sets.

Lemma 1. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$, we prove that $S^{FS}(A,C) \leq S^{FS}(A,B)$.*

Proof. We know that

$$S^{FS}(A,C) = \frac{1}{n}\sum_{i=1}^{n}\frac{2\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \tag{14}$$

and

$$S^{FS}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A^2(x_i)+\mu_B^2(x_i)}, \tag{15}$$

under the restriction $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$ for every u_i in $X = \{x_1, x_2, \ldots, x_n\}$.

For $i = 1, 2, \ldots, n$, we compute that

$$\begin{aligned}&\frac{\mu_B(x_i)}{\mu_A^2(x_i)+\mu_B^2(x_i)} - \frac{\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)}\\ &= \frac{[\mu_C(x_i)-\mu_B(x_i)]\left[\mu_B(x_i)\mu_C(x_i)-\mu_A^2(x_i)\right]}{\left[\mu_A^2(x_i)+\mu_B^2(x_i)\right]\left[\mu_A^2(x_i)+\mu_C^2(x_i)\right]}.\end{aligned} \tag{16}$$

Owing to $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, we derive that

$$\frac{\mu_B(x_i)}{\mu_A^2(x_i)+\mu_B^2(x_i)} - \frac{\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \geq 0, \tag{17}$$

and then it yields that

$$\frac{\mu_A(x_i)\mu_B(x_i)}{\mu_A^2(x_i)+\mu_B^2(x_i)} \geq \frac{\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \tag{18}$$

for $i = 1, 2, \ldots, n$, so we verify that $S^{FS}(A,C) \leq S^{FS}(A,B)$. □

Lemma 2. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$, we prove that $S^{FS}(A,C) \leq S^{FS}(B,C)$.*

Proof. We know that

$$S^{FS}(A,C) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)}, \tag{19}$$

and

$$S^{FS}(B,C) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_B(x_i)\mu_C(x_i)}{\mu_B^2(x_i)+\mu_C^2(x_i)}, \tag{20}$$

under the restriction $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$ for every u_i in $X = \{x_1, x_2, \ldots, x_n\}$.

For $i = 1, 2, \ldots, n$, we compute that

$$\begin{aligned} &\frac{\mu_B(x_i)}{\mu_B^2(x_i)+\mu_C^2(x_i)} - \frac{\mu_A(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \\ =\ &\frac{[\mu_B(x_i)-\mu_C(x_i)]\left[\mu_C^2(x_i)-\mu_A(x_i)\mu_B(x_i)\right]}{\left[\mu_A^2(x_i)+\mu_B^2(x_i)\right]\left[\mu_A^2(x_i)+\mu_C^2(x_i)\right]}. \end{aligned} \tag{21}$$

Owing to $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, we derive that

$$\frac{\mu_B(x_i)}{\mu_B^2(x_i)+\mu_C^2(x_i)} - \frac{\mu_A(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \geq 0, \tag{22}$$

and then it yields that

$$\frac{\mu_B(x_i)\mu_C(x_i)}{\mu_B^2(x_i)+\mu_C^2(x_i)} \geq \frac{\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i)+\mu_C^2(x_i)} \tag{23}$$

for $i = 1, 2, \ldots, n$, so we verify that $S^{FS}(A,C) \leq S^{FS}(B,C)$. □

Based on our proven Lemma 1 and Lemma 2, we verify that the first similarity measure proposed by Zhang et al. [4] satisfies the fourth axiom (A4). Hence, we derive our first main result.

Theorem 1. *The first similarity measure proposed by Zhang et al. [4] $S^{FS}(A,B)$ satisfies the fourth axiom (A4).*

To prove that the second and third similarity measures of Zhang et al. [4] satisfy the fourth axiom (A4), we need the following lemma.

Lemma 3. *If $\frac{1}{2} \geq \frac{a}{A} \geq \frac{b}{B}$ and $\frac{1}{2} \geq \frac{c}{C} \geq \frac{d}{D}$, then $\frac{a+c}{A+C} \geq \frac{b+d}{B+D}$, where a, b, c, d and A, B, C, D are positive numbers.*

Proof. From the conditions of Lemma 3, we know that $\frac{a}{A} \geq \frac{b}{B}$ and $\frac{c}{C} \geq \frac{d}{D}$, and then we derive that A is bounded above by $\frac{a}{b}B$ and C is bounded above by $\frac{c}{d}D$.

We observe $\frac{a+c}{A+C}$ to know that

$$\frac{a+c}{A+C} \geq \left.\frac{a+c}{A+C}\right|_{A=\frac{a}{b}B, C=\frac{c}{d}D} = \frac{\alpha\, b+\beta\, d}{\alpha\, B+\beta\, D} \tag{24}$$

where $\alpha = ad$ and $\beta = bc$ are two abbreviations to simplify the expression.

We compute

$$\frac{\alpha\, b+\beta\, d}{\alpha\, B+\beta\, D} - \frac{b+d}{B+D} = \frac{(\alpha-\beta)(bD-dB)}{(\alpha\, B+\beta\, D)(B+D)} \tag{25}$$

to imply that if (a) $\alpha \geq \beta$ and $bD - dB \geq 0$, or (b) $\alpha \leq \beta$ and $bD - dB \leq 0$, then Lemma 3 is valid.

From the conditions of Lemma 3, we know that $\frac{a}{A} \geq \frac{b}{B}$ and $\frac{c}{C} \geq \frac{d}{D}$, and then we derive that a is bounded below by $\frac{A}{B}b$ and c is bounded below by $\frac{C}{D}d$.

We observe $\frac{a+c}{A+C}$ to know that

$$\frac{a+c}{A+C} \geq \left.\frac{a+c}{A+C}\right|_{a=\frac{A}{B}b, c=\frac{C}{D}d} = \frac{\delta\, b+\varphi\, d}{\delta\, B+\varphi\, D}. \tag{26}$$

where $\delta = AD$ and $\varphi = BC$ are two additional abbreviations to simplify the expression.

We compute

$$\frac{\delta\, b+\varphi\, d}{\delta\, B+\varphi\, D} - \frac{b+d}{B+D} = \frac{(\delta-\varphi)(bD-dB)}{(\delta\, B+\varphi D)(B+D)} \tag{27}$$

to imply that if (c) $\varphi \leq \delta$ and $bD - dB \geq 0$, or (d) $\varphi \geq \delta$ and $bD - dB \leq 0$, then Lemma 3 is valid.

There are four cases: (C1) $\alpha \geq \beta$ and $bD - dB \geq 0$, (C2) $\alpha \leq \beta$ and $bD - dB \leq 0$, (C3) $\alpha \leq \beta$ and $bD - dB \geq 0$, and (C4) $\alpha \geq \beta$ and $bD - dB \leq 0$.

We already obtain that Case (C1) is (a) and Case (C2) is (b).

For Case (C3), with the condition $bD - dB \geq 0$, we derive that

$$\begin{aligned} \delta - \varphi &= AD - BC \\ &\geq AD - BC\tfrac{ad}{bc} \\ &\geq AC\tfrac{d}{c} - BC\left(\tfrac{A}{B}b\right)\tfrac{d}{bc} \geq 0, \end{aligned} \tag{28}$$

since $\alpha = ad \leq \beta = bc$ and $D \geq C\frac{d}{c}$.

Hence, we derive that $\varphi \leq \delta$ that is (c) with the condition $bD - dB \geq 0$.

For Case (C4) with the condition $\alpha \geq \beta$ and $bD - dB \leq 0$, we obtain that

$$\begin{aligned} \delta - \varphi &= AD - BC \\ &\leq AD\tfrac{bc}{ad} - BC \\ &\leq \left(B\tfrac{a}{b}\right)D\tfrac{bc}{ad} - B\left(D\tfrac{c}{d}\right) \leq 0, \end{aligned} \tag{29}$$

since $\alpha = ad \leq \beta = bc$, $B\frac{a}{b} \geq A$, and $D\frac{d}{c} \geq C$.

Therefore, we know that $\delta \leq \varphi$ that is (d) with the condition $bD - dB \leq 0$.

Based on the above discussion, we finish the proof of Lemma 3. □

For the second measure of Zhang et al. [4], we begin to verify that it satisfies the fourth axiom (A4).

Lemma 4. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$ we prove that $S^{IFS}(A,C) \leq S^{IFS}(A,B)$.*

Proof. We know that

$$S^{IFS}(A,C) = \frac{1}{n}\sum_{i=1}^{n} \frac{2[\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)]}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)} \tag{30}$$

and

$$S^{IFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{2[\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)]}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)}, \tag{31}$$

under the restriction $\mu_A(x_i) \le \mu_B(x_i) \le \mu_C(x_i)$ and $v_A(x_i) \le v_B(x_i) \le v_C(x_i)$ for every x_i in $X = \{x_1, x_2, \ldots, x_n\}$.

First, we recall Theorem 1 to know that

$$\frac{\mu_A(x_i)\mu_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i)} \ge \frac{\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i)}. \tag{32}$$

We repeated to apply Theorem 1 again to obtain that

$$\frac{v_A(x_i)v_B(x_i)}{v_A^2(x_i) + v_B^2(x_i)} \ge \frac{v_A(x_i)v_C(x_i)}{v_A^2(x_i) + v_C^2(x_i)}. \tag{33}$$

We know that

$$\frac{1}{2} \ge \frac{\mu_A(x_i)\mu_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i)} \tag{34}$$

and

$$\frac{1}{2} \ge \frac{v_A(x_i)v_B(x_i)}{v_A^2(x_i) + v_B^2(x_i)} \tag{35}$$

such that the conditions of Lemma 3 are satisfied.

Next, we use Lemma 3 for Equations (32) and (33) to yield that

$$\frac{\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)} \ge \frac{\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}, \tag{36}$$

for $i = 1, 2, \ldots, n$, so we verify that $S^{IFS}(A,B) \ge S^{IFS}(A,C)$. □

Lemma 5. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$, we prove that $S^{IFS}(A,C) \le S^{IFS}(B,C)$.*

Proof. We know that

$$S^{IFS}(A,C) = \frac{1}{n}\sum_{i=1}^{n} \frac{2[\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)]}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)} \tag{37}$$

and

$$S^{IFS}(B,C) = \frac{1}{n}\sum_{i=1}^{n} \frac{2[\mu_B(x_i)\mu_C(x_i) + v_B(x_i)v_C(x_i)]}{\mu_B^2(x_i) + \mu_C^2(x_i) + v_B^2(x_i) + v_C^2(x_i)} \tag{38}$$

under the restriction $\mu_A(x_i) \le \mu_B(x_i) \le \mu_C(x_i)$ and $v_A(x_i) \le v_B(x_i) \le v_C(x_i)$ for every x_i in $X = \{x_1, x_2, \ldots, x_n\}$.

First, we recall Theorem 1 to know that

$$\frac{\mu_B(x_i)\mu_C(x_i)}{\mu_B^2(x_i) + \mu_C^2(x_i)} \ge \frac{\mu_A(x_i)\mu_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i)}. \tag{39}$$

We repeated to apply Theorem 1 again to obtain that

$$\frac{v_B(x_i)v_C(x_i)}{v_B^2(x_i) + v_C^2(x_i)} \ge \frac{v_A(x_i)v_C(x_i)}{v_A^2(x_i) + v_C^2(x_i)}. \tag{40}$$

We know that

$$\frac{1}{2} \geq \frac{\mu_B(x_i)\mu_C(x_i)}{\mu_B^2(x_i) + \mu_C^2(x_i)}, \tag{41}$$

and

$$\frac{1}{2} \geq \frac{v_B(x_i)v_C(x_i)}{v_B^2(x_i) + v_C^2(x_i)}. \tag{42}$$

such that the conditions of Lemma 3 are satisfied.

Next, we apply Lemma 3 for Equations (39) and (40) to derive that

$$\frac{\mu_B(x_i)\mu_C(x_i) + v_B(x_i)v_C(x_i)}{\mu_B^2(x_i) + \mu_C^2(x_i) + v_B^2(x_i) + v_C^2(x_i)} \geq \frac{\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}, \tag{43}$$

for $i = 1, 2, \ldots, n$, so we verify that $S^{IFS}(B,C) \geq S^{IFS}(A,C)$. □

Based on our Lemmas 4 and 5, we verify that the second similarity measure proposed by Zhang et al. [4] satisfies the fourth axiom (A4). Hence, we derive our second main result.

Theorem 2. *The second similarity measure proposed by Zhang et al. [4] $S^{IFS}(A,B)$ satisfies the fourth axiom (A4).*

For the third measure of Zhang et al. [4], we begin to show that it satisfies the fourth axiom (A4).

Lemma 6. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$, we prove that $WS^{IFS}(A,C) \leq WS^{IFS}(A,B)$.*

Proof. We know that

$$WS^{IFS}(A,B) = \sum_{i=1}^{n} w_i \frac{2[\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)]}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)}, \tag{44}$$

and

$$WS^{IFS}(A,C) = \sum_{i=1}^{n} w_i \frac{2[\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)]}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}, \tag{45}$$

Based on Equation (36), we derived that

$$w_i \frac{\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)} \geq w_i \frac{\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}, \tag{46}$$

with $w_i \geq 0$, for $i = 1, 2, \ldots, n$, so we verify that $WS^{IFS}(A,B) \geq WS^{IFS}(A,C)$. □

Lemma 7. *For three FSs A, B and C satisfying $A \subseteq B \subseteq C$, we prove that $WS^{IFS}(A,C) \leq WS^{IFS}(B,C)$.*

Proof. We know that

$$WS^{IFS}(B,C) = \sum_{i=1}^{n} w_i \frac{2[\mu_B(x_i)\mu_C(x_i) + v_B(x_i)v_C(x_i)]}{\mu_B^2(x_i) + \mu_C^2(x_i) + v_B^2(x_i) + v_C^2(x_i)}, \tag{47}$$

and

$$WS^{IFS}(A,C) = \sum_{i=1}^{n} w_i \frac{2[\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)]}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}. \tag{48}$$

Based on Equation (33), we derived that

$$w_i \frac{\mu_A(x_i)\mu_B(x_i) + v_A(x_i)v_B(x_i)}{\mu_A^2(x_i) + \mu_B^2(x_i) + v_A^2(x_i) + v_B^2(x_i)} \geq w_i \frac{\mu_A(x_i)\mu_C(x_i) + v_A(x_i)v_C(x_i)}{\mu_A^2(x_i) + \mu_C^2(x_i) + v_A^2(x_i) + v_C^2(x_i)}, \tag{49}$$

with $w_i \geq 0$, for $i = 1, 2, \ldots, n$, so we verify that $WS^{IFS}(B, C) \geq WS^{IFS}(A, C)$. □

Based on our proven Lemmas 6 and 7, we verify that the third similarity measure proposed by Zhang et al. [4] satisfies the fourth axiom (A4). Hence, we derive our second main result.

Theorem 3. *The third similarity measure proposed by Zhang et al. [4] $WS^{IFS}(A, B)$ satisfies the fourth axiom (A4).*

Therefore, we provide revisions to prove that the three similarity measures proposed by Zhang et al. [4] all satisfy the fourth axiom (A4) to complete the verification of a well-defined examination for similarity measures proposed by Zhang et al. [4].

5. Numerical Examples

In our first three examples, we reconsider the pattern recognition problem proposed by Zhang et al. [4] with three different settings of weights to illustrate that the weighed similarity measure was proposed by Zhang et al. [4] which will be influenced by weights. We recall the pattern recognition proposed by Zhang et al. [4] with three patterns C_1, C_2, and C_3, and one sample Q, where

$$C_1 = \{\langle x_1, 1, 0\rangle, \langle x_2, 0.8, 0\rangle, \langle x_3, 0.7, 0.1\rangle\}, \tag{50}$$

$$C_2 = \{\langle x_1, 0.8, 0.1\rangle, \langle x_2, 1, 0\rangle, \langle x_3, 0.9, 0\rangle\}, \tag{51}$$

$$C_3 = \{\langle x_1, 0.6, 0.2\rangle, \langle x_2, 0.8, 0\rangle, \langle x_3, 1, 0\rangle\} \tag{52}$$

and

$$Q = \{\langle x_1, 0.5, 0.3\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.8, 0.1\rangle\}. \tag{53}$$

We develop three examples with different settings of w_i, for $i = 1, 2, 3$. For the first example, we follow Zhang et al. [4] to assume that $w_1 = 0.5$, $w_2 = 0.3$ and $w_3 = 0.2$. For the second example, we set that $w_1 = 0.05$, $w_2 = 0.05$ and $w_3 = 0.9$, and then for the third example, we take that $w_1 = 0.09$, $w_2 = 0.01$, and $w_3 = 0.9$. The computation results are listed in the next Table 2.

Table 2. Computation results for Examples 1–3.

Example	$WS^{IFS}(C_1,Q)$	$WS^{IFS}(C_2,Q)$	$WS^{IFS}(C_3,Q)$	Implication
1	0.848318	0.888747	0.957349	$C_3 \succ C_2 \succ C_1$
2	0.975641	0.973963	0.967530	$C_1 \succ C_2 \succ C_3$
3	0.968569	0.974424	0.969626	$C_2 \succ C_3 \succ C_1$

From Table 2, to consider Example 1, we derive that sample Q should have belonged to the pattern C_3. Our derivation is consistent with Zhang et al. [4].

However, for our Example 2, with a different set of w_i, for $i = 1, 2, 3$, then we obtain that the sample Q should have belonged to the pattern C_1. Our result is different from that of Zhang et al. [4].

Moreover, for our Example 3, with another set of w_i, for $i = 1, 2, 3$, then we imply that the sample Q should have belonged to the pattern C_2. Our finding of Example 3 is different from that of Examples 1 and 2. Hence, we can conclude that the weighted similarity measure proposed by Zhang et al. [4] will be significantly influenced by the setting of w_i, for $i = 1, 2, 3$.

For our fourth example, we recall an application of similarity measures under an intuitionistic fuzzy sets environment for the fault diagnosis of turbine generators that was discussed by Li and Wan [43] and Chu et al. [44]. They used the amplitude ratio of vibration signal in six different frequency ranges, less than $0.4f$, $0.5f$, f, $2f$, $3f$ and more than $3f$, as the characteristic values to construct their universe of discourse, where f is the fundamental frequency of the turbine generator. There are three typical failures to be used as the failure patterns: P_1 (oil whip), P_2 (unbalance), and P_3 (misalignment), and two samples B_1 and B_2 to be tested for its pattern. We cite Tables 1 and 2 of Chu et al. [44] for the six different frequency ranges of three patterns and two samples, respectively, in our Table 3 under an intuitionistic fuzzy sets environment.

Table 3. Data for patterns and samples (reproduced from Tables 1 and 2 of Chu et al. [44]).

	Frequency Range					
	<0.4f	***0.5f***	***f***	***2f***	***3f***	***>3f***
Pattern P_1	<0.06,0.84>	<0.84,0.02>	<0.20,0.75>	<0.02,0.89>	<0.20,0.75>	<0.01,0.92>
Pattern P_2	<0.01,0.93>	<0.02,0.90>	<0.90,0.01>	<0.08,0.85>	<0.01,0.89>	<0.02,0.93>
Pattern P_3	<0.01,0.94>	<0.01,0.94>	<0.40,0.42>	<0.40,0.44>	<0.28,0.56>	<0.01,0.61>
Sample B_1	<0.01,0.96>	<0.00,0.97>	<0.37,0.60>	<0.46,0.51>	<0.31,0.66>	<0.21,0.75>
Sample B_2	<0.00,0.98>	<0.05,0.92>	<0.69,0.27>	<0.04,0.93>	<0.03,0.84>	<0.00,0.97>

Based on our previous discussion for the weighted similarity measure WS^{IFS} proposed by Zhang et al. [4], we know that it is influenced by the different settings of w_i for $i = 1, 2, \ldots, 6$ such that we only consider S^{IFS} proposed by Zhang et al. [4] in our fourth example.

To be compatible with Chu et al. [44], Julian et al. [17], Tung et al. [21], Li and Wan [43], Yusoff et al. [45], and Zeng [46], we cite Table 3 of Chu et al. [44] in our Table 4 along with ourfindings afterwe apply the second similarity measure proposed by Zhang et al. [4] of Equation (12).

Table 4. Comparison of seven methods.

	Sample B_1			**Sample B_2**		
	Patterns			**Patterns**		
	P_1	P_2	P_3	P_1	P_2	P_3
[4]	0.772	0.857	0.984	0.768	0.985	0.884
[44]	0.779	0.827	0.918	0.797	0.939	0.822
[17]	0.163	0.393	0.839	0.185	0.795	0.481
[21]	0.582	0.696	0.920	0.593	0.897	0.741
[43]	0.554	0.704	0.926	0.582	0.893	0.721
[45]	0.670	0.745	0.953	0.713	0.933	0.787
[46]	0.582	0.697	0.923	0.593	0.898	0.747
[47]	0.773	0.927	0.980	0.604	0.629	0.606
[48]	0.422	0.431	0.637	0.425	0.555	0.137
[49]	0.366	0.643	0.652	0.401	0.544	0.235
[50]	0.805	0.853	0.904	0.788	0.835	0.713
[51] with (61)	0.797	0.715	0.908	0.633	0.834	0.423
[51] with (62)	0.924	0.920	0.975	0.790	0.902	0.773

In the following, we consider several recent published papers to apply their similarity measures for this pattern recognition problem.

For $\xi = [\xi_1, \xi_2, \xi_3, \xi_4, \omega_1, \omega_2]$ and $\eta = [\eta_1, \eta_2, \eta_3, \eta_4, \kappa_1, \kappa_2]$, two generalized trapezoidal fuzzy numbers, Dutta [47] defined a new similarity measure $D_S(\xi, \eta)$ as

$$D_S(\xi, \eta) = \frac{2(\omega_1\kappa_1 + \omega_2\kappa_2 + |\omega_1 - \kappa_1| + |\omega_2 - \kappa_2| + \sum \xi_i\eta_i)}{\omega_1^2 + \omega_2^2 + \kappa_1^2 + \kappa_2^2 + |\omega_1 - \kappa_1|^2 + |\omega_2 - \kappa_2|^2 + \sum\left(\xi_i^2 + \eta_i^2\right)}, \tag{54}$$

where $[\eta_1, \eta_2, \eta_3, \eta_4]$ is a trapezoidal number with left height ω_1 and right height ω_2.

For an intuitionistic fuzzy set on X, $\mu_A(x) : X \to [0,1]$ is the membership function and $v_A(x) : X \to [0,1]$ is the non-membership function.We can convert the intuitionistic fuzzy into a generalized trapezoidal fuzzy number as follows,

$$\xi = [\mu_A(x), \mu_A(x),\ 1 - v_A(x),\ 1 - v_A(x), 1, 1], \tag{55}$$

and then we can apply the similarity measure proposed by Dutta [47].

For $A = \{[y, \xi_A(y), \eta_A(y), \nu_A(y)] : y \in Y\}$ and $B = \{[y, \xi_B(y), \eta_B(y), \nu_B(y)] : y \in Y\}$, two spherical fuzzy sets, where ξ_A, η_A, and $\nu_{A:}Y \to [0,1]$ are the degree of positive, neutral, and negative membership functions, with $Y = \left\{y_1, \ldots, y_m\right\}$ and $\xi_A(y) + \eta_A(y) + (y) \le 1$, Rafiq et al. [48] developed a cotangent similarity measure, $S_C(A,\ B)$ as follows,

$$S_C(A,\ B) = \frac{1}{m}\sum\nolimits_{i=1}^{m} \cot\left(\frac{\pi}{4} + \Pi_i\right), \tag{56}$$

where Π_i is an abbreviation to simplify the expression, where

$$\Pi_i = \max\left\{\left|\xi_A^2\left(y_i\right) - \xi_{\mathbf{B}}^2\left(y_i\right)\right|, \left|\eta_A^2\left(y_i\right) - \eta_{\mathbf{B}}^2\left(y_i\right)\right|, \left|\nu_A^2\left(y_i\right) - \nu_{\mathbf{B}}^2\left(y_i\right)\right|, \Psi_i\right\}, \tag{57}$$

and Ψ_i is a second abbreviation to simplify the expression, with

$$\Psi_i = \left|\xi_A\left(y_i\right) + \eta_A\left(y_i\right) + \nu_A\left(y_i\right) - \xi_B\left(y_i\right) - \eta_B\left(y_i\right) - \nu_B\left(y_i\right)\right|. \tag{58}$$

We can generalize an intuitionistic fuzzy set $A = \{[y, \xi_A(y), \nu_A(y)] : y \in Y\}$ to a spherical fuzzy set $A = \{[y, \xi_A(y), \eta_A(y), \nu_A(y)] : y \in Y\}$ with $\eta_A\left(y_i\right) = 0$, for $i = 1, 2, \ldots, m$.

For $A = \{[y, \xi_A(y), \eta_A(y), \nu_A(y)] : y \in Y\}$ and $B = \{[y, \xi_B(y), \eta_B(y), \nu_B(y)] : y \in Y\}$, two spherical fuzzy sets, Khan et al. [49] defined a new similarity measure, $S^S(A,\ B)$, as

$$S^S(A,\ B) = \frac{\sum_{i=1}^{m}\left[\xi_A^2\left(y_i\right)\xi_{\mathbf{B}}^2\left(y_i\right) + \eta_A^2\left(y_i\right)\eta_{\mathbf{B}}^2\left(y_i\right) + \nu_A^2\left(y_i\right)\nu_{\mathbf{B}}^2\left(y_i\right)\right]}{\sum_{i=1}^{m}\left\{\max\left\{\xi_A^4\left(y_i\right), \xi_{\mathbf{B}}^4\left(y_i\right)\right\} + \max\left\{\eta_A^4\left(y_i\right), \eta_{\mathbf{B}}^4\left(y_i\right)\right\} + \max\left\{\nu_A^4\left(y_i\right), \nu_{\mathbf{B}}^4\left(y_i\right)\right\}\right\}}. \tag{59}$$

For two intuitionistic fuzzy sets, Muthuraj and Devi [50] constructed a new tangent similarity measure, $T_{IFMS}(A,\ B)$, as follows

$$T_{IFMS}(A,\ B) = 1 - \tfrac{1}{n}\sum_{i=1}^{n} \tan\left[\tfrac{\pi}{12}\left(\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|v_A(x_i) - v_B(x_i)\right| + \left|\pi_A(x_i) - \pi_B(x_i)\right|\right)\right]. \tag{60}$$

For $A = \{[y, s_A(y), i_A(y), d_A(y)] : y \in Y\}$ and $B = \{[y, s_B(y), i_B(y), d_B(y)] : y \in Y\}$, two T-spherical fuzzy sets, where $s_A(y), i_A(y), d_A(y)$ and $r_A(y) : X \to [0,1]$ are the membership, hesitancy, non-membership, and refusal degree, Wu et al. [51] assumed two cosine similarity measures, $TSFCS^1(A, B)$ and $TSFCS^2(A,\ B)$, in the following:

$$TSFCS^1(A,\ B) = \frac{1}{m}\sum\nolimits_{i=1}^{m} \cos\left(\frac{\pi}{2}\max\{\alpha_i, \beta_i, \gamma_i, \delta_i\}\right), \tag{61}$$

where $\left|s_A^4(y_i) - s_B^4(y_i)\right| = \alpha_i$, $\left|i_A^4(y_i) - i_B^4(y_i)\right| = \beta_i$, $\left|d_A^4(y_i) - d_B^4(y_i)\right| = \gamma_i$, and $\left|r_A^4(y_i) - r_B^4(y_i)\right| = \delta_i$ are auxiliary notations to simplify the expression, and

$$\mathrm{TSFCS}^2(A, B) = \frac{1}{m}\sum\nolimits_{i=1}^{m} \cos\left(\frac{\pi}{4}(\alpha_i + \beta_i + \gamma_i + \delta_i)\right). \quad (62)$$

We can generalize an intuitionistic fuzzy set $A = \{[y, \xi_A(y), \nu_A(y)] : y \in Y\}$ to a T-spherical fuzzy set $A = \{[y, s_A(y), i_A(y), d_A(y)] : y \in Y\}$ with $s_A(y_i) = \xi_A(y_i)$, $i_A(y_i) = 1 - \xi_A(y_i) - \nu_A(y_i)$, $d_A(y_i) = \nu_A(y_i)$, and $\eta_A(y_i) = 0$, for $i = 1, 2, \ldots, m$.

Based on similarity measures discussed from Equation (54) to Equation (62), we evaluate the pattern recognition problems of Table 3 and then add them to the following Table 4.

From the fourth column of Table 4, the sample B_1 should have belonged to the pattern P_3 and in the sixth column of Table 4, the sample B_2 should have belonged to the pattern P_2. The results derived by the similarity measure proposed by Zhang et al. [4] are the same as decided by Chu et al. [44], Julian et al. [17], Tung et al. [21], Li and Wan [43], Yusoff et al. [45], Zeng [46], Dutta [47], Rafiq et al. [48], Khan et al. [49], Muthuraj et al. [50], and Wu et al. [51]. Our fourth example illustrates that the similarity measure proposed by Zhang et al. [4] can be applied for a practical application of fault diagnosis of turbine generators.

6. Directions for Future Research

In this paper, we discuss three similarity measures proposed by Zhang et al. [4] that only refer to membership function and non-membership function, without considering the hesitation function. We can predict that to prove the similarity measures based on the inner product including membership, non-membership, and hesitation functions, satisfying the fourth axiom proposed by Li and Cheng [14] will be an interesting research topic. Some other applications require similarity measures. For example, a similarity angle mapper has been widely used as a similarity measure for comparing two vectors in hyperspectral image applications such as Kwan et al. [52] and Qu et al. [53]. Researchers applying similarity metrics in hyperspectral images will be an interesting topic for future practitioners.

7. Conclusions

In this paper, we first provide a patchwork to prove that three similarity measures proposed by Zhang et al. [4] satisfy the fourth axiom (A4) proposed by Li and Cheng [14]. Next, we examine the same example proposed by Zhang et al. [4] for a pattern recognition problem to point out that their third similarity measure, the weighted similarity measure, is dependent on weights such that how to derive a proper setting for weights becomes a critical issue. Finally, we provide a practical application for the second similarity measure of Zhang et al. [4] to demonstrate the usefulness of their second similarity measure.

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References

1. Zadeh, L.A. Fuzzy sets. *Info. Control* **1965**, *8*, 338–353. [CrossRef]
2. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Ye, J. Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Math. Comput. Model.* **2011**, *53*, 91–97. [CrossRef]

4. Zhang, L.; Xu, X.; Chen, X. A New Similarity Measure for Intuitionistic Fuzzy Sets and Its Applications. *Int. J. Info. Manag. Sci.* **2012**, *23*, 229–239.
5. Deng, P.S.; Yen, C.; Tung, C.; Yu, Y.; Chu, P. A technical note for the deteriorating inventory model with exponential time-varying demand and partial backlogging. *Int. J. Info. Manag. Sci.* **2006**, *17*, 101–108.
6. Tang, D.W.; Chao, H.C.; Chuang, J.P. A note on the inventory model for deteriorating items with exponential declining demand and partial backlogging. *Int. J. Info. Manag. Sci.* **2013**, *24*, 167–173.
7. Lan, C.; Yu, Y.; Lin, R.H.; Tung, C.; Yen, C.; Deng, P.S. A note on the improved algebraic method for the EPQ model with stochastic lead time. *Int. J. Info. Manag. Sci.* **2007**, *18*, 91–96.
8. Yang, G.K.; Hung, K.C.; Julian, P. AdoptingLanchester model to the Ardennes Campaign with deadlock situation in the shift time between defense and attack. *Int. J. Info. Manag. Sci.* **2013**, *24*, 349–362.
9. Deng, P.S. Improved inventory models with ramp type demand and Weibull deterioration. *Int. J. Info. Manag. Sci.* **2005**, *16*, 79–86.
10. Chang, S.K.J.; Lei, H.L.; Jung, S.T.; Lin, R.H.J.; Lin, J.S.J.; Lan, C.H.; Yu, Y.C.; Chuang, J.P.C. Note on Deriving Weights from Pairwise Comparison Matrices in AHP. *Int. J. Info. Manag. Sci.* **2008**, *19*, 507–517.
11. Jung, S.; Lin, J.S.; Chuang, J.P.C. A note on "an EOQ model for items with Weibull distributed deterioration, shortages and power demand pattern". *Int. J. Inf. Manag. Sci.* **2008**, *19*, 667–672.
12. Deng, P.S.; Yang, G.K.; Chen, H.; Chu, P.; Huang, D. The criterion for the optimal solution of inventory model with stock-dependent consumption rate. *Int. J. Inf. Manag. Sci.* **2005**, *16*, 97–109.
13. Gerstenkorn, T.; Manko, J. Correlation of intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1991**, *44*, 39–43. [CrossRef]
14. Li, D.; Cheng, C. New similarity measures of intuitionistic fuzzy sets and application to pattern recognition. *Pattern Recognit. Lett.* **2002**, *23*, 221–225.
15. Hung, K.; Wang, P. An integrated intuitionistic fuzzy similarity measures for medical problems. *Int. J. Comput. Intell. Syst.* **2014**, *7*, 327–343. [CrossRef]
16. Hung, K.; Lin, J.; Chu, P. An extended algorithm of similarity measures and its application to radar target recognition based on intuitionistic fuzzy sets. *J. Test. Eval.* **2015**, *43*, 878–887. [CrossRef]
17. Julian, P.; Hung, K.C.; Lin, S.J. On the Mitchell similarity measure and its application to pattern recognition. *Pattern Recognit. Lett.* **2012**, *33*, 1219–1223. [CrossRef]
18. Chu, C.; Guo, Y. Developing similarity based IPA under intuitionistic fuzzy sets to assess leisure bikeways. *Tour. Manag.* **2015**, *47*, 47–57. [CrossRef]
19. Yen, P.C.P.; Fan, K.; Chao, H.C.J. A new method for similarity measures for pattern recognition. *Appl. Math. Model.* **2013**, *37*, 5335–5342. [CrossRef]
20. Hung, K.; Lin, K. A new intuitionistic fuzzy cosine similarity measures and its application. In Proceedings of the IEEE International Conference on Industrial Engineering and Engineering Management, Hong Kong, China, 10–13 December 2012; pp. 2194–2198.
21. Tung, C.; Liu, S.; Wang, B.S. A comment on "on the Mitchell similarity measure and its application to pattern recognition". *Pattern Recognit. Lett.* **2013**, *34*, 453–455. [CrossRef]
22. Hung, K.; Lin, K. Long-term business cycle forecasting through a potential intuitionistic fuzzy least-squares support vector regression approach. *Inf. Sci.* **2013**, *224*, 37–48. [CrossRef]
23. Tung, C.; Hopscotch, C. Discussion on similarity measure of its complement. *J. Discret. Math. Sci. Cryptogr.* **2015**, *18*, 417–432. [CrossRef]
24. Aggarwal, A.; Mehra, A.; Chandra, S.; Khan, I. Solving Atanassov's I-fuzzy linear programming problems using Hurwicz's criterion. *Fuzzy Inform. Eng.* **2018**, *10*, 339–361. [CrossRef]
25. Farhadinia, B.; Xu, Z. Hesitant fuzzy information measures derived from T-norms and S-norms. *Iran. J. Fuzzy Syst.* **2018**, *15*, 157–175.
26. Fei, L.; Wang, H.; Chen, L.; Deng, Y. A new vector valued similarity measure for intuitionistic fuzzy sets based on OWA operators. *Iran. J. Fuzzy Syst.* **2019**, *16*, 113–126.
27. Joshi, R.; Kumar, S. A new approach in multiple attribute decision making using exponential hesitant fuzzy entropy. *Int. J. Info. Manag. Sci.* **2019**, *30*, 305–322.
28. Khanmohammadi, E.; Malmir, B.; Safari, H.; Zandieh, M. A new approach to strategic objectives ranking based on fuzzy logarithmic least squares method and fuzzy similarity technique. *Oper. Res. Perspect.* **2019**, *6*, 100122. [CrossRef]
29. Li, J.; Liu, Y. Property analysis of triple implication method for approximate reasoning on Atanassov's intuitionistic fuzzy sets. *Iran. J. Fuzzy Syst.* **2018**, *15*, 95–116.

30. Lin, K. A new distance measure for MCDM problem using TOPSIS method. In Proceedings of the Proceedings—International Conference on Machine Learning and Data Engineering, iCMLDE 2019, Taipei, Taiwan, 2–4 December 2019; pp. 19–24.
31. Mishra, A.R.; Rani, P. Interval-valued intuitionistic fuzzy WASPAS method: Application in reservoir flood control management policy. *Group Decis. Negot.* **2018**, *27*, 1047–1078. [CrossRef]
32. Rani, P.; Jain, D.; Hooda, D.S. Shapley function based interval-valued intuitionistic fuzzy VIKOR technique for correlative multi-criteria decision making problems. *Iran. J. Fuzzy Syst.* **2018**, *15*, 25–54.
33. Rouyendegh, B.D. The intuitionistic fuzzy ELECTRE model. *Int. J. Manag. Sci. Eng.Manag.* **2018**, *13*, 139–145. [CrossRef]
34. Shen, F.; Ma, X.; Li, Z.; Xu, Z.; Cai, D. An extended intuitionistic fuzzy TOPSIS method based on a new distance measure with an application to credit risk evaluation. *Inf. Sci.* **2018**, *428*, 105–119. [CrossRef]
35. Shokeen, J.; Rana, C. Fuzzy sets, advanced fuzzy sets and hybrids. In Proceedings of the 2017 International Conference on Energy, Communication, Data Analytics and Soft Computing, ICECDS, Chennai, Tamil Nadu, India, 1 August 2017; pp. 2538–2542.
36. Wang, Y.; Wang, L.; Sangaiah, A.K. Generalized Pythagorean fuzzy information aggregation operators for multi-criteria decision making. In Proceedings of the ICNC-FSKD 2017—13th International Conference on Natural Computation, Fuzzy Systems, and Knowledge Discovery, Guilin, China, 29–31 July 2017; pp. 1410–1415.
37. Wei, G. Some similarity measures for picture fuzzy sets and their applications. *Iran. J. Fuzzy Syst.* **2018**, *15*, 77–89.
38. Zhang, L.; Liu, J.; Huang, B.; Li, H.; Zhou, X. Dynamic agent evaluation using intuitionistic fuzzy TOPSIS. In Proceedings of the 2018 IEEE 22nd International Conference on Computer Supported Cooperative Work in Design, CSCWD 2018, Nanjing, China, 9–11 May 2018; pp. 407–413.
39. Zhou, W.; Chen, J.; Xu, Z.; Meng, S. Hesitant fuzzy preference envelopment analysis and alternative improvement. *Inf. Sci.* **2018**, *465*, 105–117. [CrossRef]
40. Liang, Z.; Shi, P. Similarity Measures on Intuitionistic Fuzzy Sets. *Pattern Recognit. Lett.* **2003**, *24*, 2687–2693. [CrossRef]
41. Atanassov, K. More on intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *33*, 37–46. [CrossRef]
42. Atanassov, K. *Intuitionistic Fuzzy Sets, Theory and Applications*; Physica: Heidelberg, NY, USA, 1999.
43. Li, H.; Wan, S. Improved BP method in vibration failure diagnosis of steam turbine generator set. *J. Chongqing Univ.* **1999**, *22*, 47–52.
44. Chu, C.; Hung, K.; Julian, P. A complete pattern recognition approach under Atanassov's intuitionistic fuzzy sets. *Knowl. Based Syst.* **2014**, *66*, 36–45. [CrossRef]
45. Yusoff, B.; Taib, I.; Abdullah, L.; Wahab, A.F. A new similarity measure on intuitionistic fuzzy sets. *Int. J. Comput. Math. Sci.* **2011**, *5*, 70–74.
46. Zeng, S. Some intuitionistic fuzzy weighted distance measures and their application to group decision making. *Group Decis. Negot.* **2013**, *22*, 281–298. [CrossRef]
47. Dutta, P. An advanced dice similarity measure of generalized fuzzy numbers and its application in multicriteria decision making. *Arab J. Basic Appl. Sci.* **2020**, *27*, 75–92. [CrossRef]
48. Rafiq, M.; Ashraf, S.; Abdullah, S.; Mahmood, T.; Muhammad, S. The cosine similarity measures of sphericalfuzzy sets and their applications in decision making. *J. Intell. Fuzzy Syst.* **2019**, *36*, 6059–6073. [CrossRef]
49. Khan, M.J.; Kumam, P.; Deebani, W.; Kumam, W.; Shah, Z. Distance and similarity measures for spherical fuzzy sets and their applications in selecting mega projects. *Mathematics* **2020**, *8*, 519. [CrossRef]
50. Muthuraj, R.; Devi, S. New similarity measure between intuitionistic fuzzy multisets based on tangent function and its application in medical diagnosis. *Int. J. Recent Tech. Eng.* **2019**, *8*, 161–165.
51. Wu, M.; Chen, T.; Fan, J. Similarity measures of T-spherical fuzzy sets based on the cosine function and their applications in pattern recognition. *IEEE Access* **2020**, *8*, 98181–98192. [CrossRef]

52. Kwan, C.; Budavari, B.; Gao, F.; Zhu, X. A hybrid color mapping approach to fusing MODIS and landsat images for forward prediction. *Remote Sens.* **2018**, *10*, 520. [CrossRef]
53. Qu, Y.; Qi, H.; Ayhan, B.; Kwan, C.; Kidd, R. Does multispectral/hyperspectralpansharpening improve the performance of anomaly detection? In Proceedings of the International Geoscience and Remote Sensing Symposium (IGARSS), Fort Worth, TX, USA, 23–28 July 2017; pp. 6130–6133.