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A Fast Non-Linear Symmetry Approach for Guaranteed Consensus in Network of Multi-Agent Systems

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Abstract: There has been tremendous work on multi-agent systems (MAS) in recent years. MAS consist of multiple autonomous agents that interact with each order to solve a complex problem. Several applications of MAS can be found in computer networks, smart grids, and the modeling of complex systems. Despite numerous benefits, a significant challenge for MAS is achieving a consensus among agents in a shared target task, which is difficult without applying certain mathematical equations. Non-linear models offer better possibility of resolving consensus for MAS; however, existing non-linear models are considerably complicated and present no guarantees for achieving consensus. This paper proposes a non-linear mathematical model of semi symmetry quadratic operator (SSQO) in order to resolve the issue of consensus in networks of MAS. The model is based on stochastic quadratic operator theory, with added new notations. An important feature for the proposed model is low complexity, fast consensus, and a guaranteed capability to reach a consensus. We present an evaluation of the proposed SSQO model and comparison with other existing models. We demonstrate that an average consensus can be achieved with our model in addition to the emulation effects for the MAS consensus.

Keywords: consensus problem; multi-agent systems; semi symmetry quadratic operator

1. Introduction

In the last decade, increased attention was directed towards the coordination and control of MAS, which appeared in various situations, such as wireless sensor networks and team cooperation. New studies have recently emerged that focus on the problems related to consensus in MAS [1]. The issue of consensus is considered to be a central problem in multi-output multi-input systems (MIMOs) that have many applications in technology [2]. In this context, the consensus is defined as the way in which individual group members agree to cooperate and generate consensus among themselves in order to achieve one objective at the same time [3].

MAS are defined as a dynamic network topology consisting of several smart agents that communicate with each other to locally exchange information in order to implement a task. Consensus or agreement is one of the most important behaviors required for every group in our natural lives. In recent decades, artificial intelligence has been introduced in various fields that benefit the world in many complex and welfare services. MAS have a great role in applying these services, including applications of physics, engineering, biology, mathematics, and social sciences [4]. The consensus in MAS is the most important factor that focuses on the performance of the functions that are assigned to it.

These agents communicate with each other wirelessly (local) to exchange information dynamically, where consensus is achieved through a specific protocol. However, achieving consensus in a dynamic network topology is extremely complex. There are several types of research that have developed mathematical protocols in which agents rely on consensus to form a network. These mathematical protocols are either linear or non-linear. Non-linear protocols has been proved to be more effective than linear protocols to achieve consensus in the wireless communication topology of the dynamic agent system network. However, researchers are faced with the challenge of creating a non-linear protocol to a dynamic nodes network consensus that is easy to calculate and achieve consensus in a short period of time. This paper aims to build a new non-linear protocol for a wireless network topology of dynamic multi-agent for the purpose of achieving consensus with a less complicated and less computational time.

The issue of consensus has been first discussed in relation to DeGroot's model [5], whereby each group member interacts with its neighboring states using a linear stochastic matrix until all of them reach the same limit. Thereupon, Berger [6] has proved that the consensus of DeGroot's model depends on the transition matrix as well as the vector column of initial states. Other researchers have reported in [7,8] offered an agreement that has been based on a network of graphs. In [9], a distributed algorithm has been generalized for the consensus in fixed topology. In [10], an arranged motion of particulars in a group has been controlled by a specific model in order to update the information from the closest neighbor. A convergence test has been done for acceleration and flocking behavior cases in [11]. Both approaches of linear and non-linear consensus protocols have been tested for distributed systems in [12] based on undirected graph theory. A comprehensive survey of several consensus models is found in [1,13].

The work has been presented by [14] discusses the approach to solve the averaging problem through the application of suitable assumptions that are based on linear iterative form. A similar study has been reported in [15] explores the cases of time to reach the consensus in time-varying networks. Meanwhile, a sensor model estimator has been developed in [16] in order to solve the problem of estimating the state of the system by exchanging information between them, where it has been discovered that in time-invariant, the cases will converge. In [17], a consensus model of a leader-follow has been designed in an event-trigger technique in consideration of a completely dynamic leader to control the tracking problem for second-order MAS.

A considerable challenge for this research involves investigating non-linear dynamic models for team agents [18]. New different approaches of non-linear models and algorithms have already been offered to resolve the issue of consensus in many types of node networks [19–24]. A proposed stochastic matrix in a non-linear model has been generated in order to control the consensus for MAS by [25,26]. An optimistic optimization approach with simple black box has been devised in a form of a non-linear structure for controlling the agents' behavior in [27]. A non-linear class of fuzzy logic systems [28] has been discussed for the adaptive problem, where it has been concluded that the uncertain statuses can be approximated using the fuzzy logic method. A good result has been produced in a proposed quantization system in [29] for the feedback of the $H - infinity$ problem in the control systems network. Furthermore, a basic theory that is based on random consensus dynamics has been established for certain application areas [30], while a distributed dual averaging algorithm is proposed in [31] for optimizing the cooperative consensus in delay cooperation in the computational multi-agent network. The balanced statuses of the individual members have been tested in [32] under local communications networks. Additionally, a consensus in a strongly connected graph has been applied while using a novel distributed model in discrete-time situation in [33]. The work detailed in [34] involves a proposed stochastic matrix with a positive diagonal for the interactions among the network agents. In addition, symmetric communications have been studied in [35] for the consensus of multi-agents networks, while [36] has been used a strategic method for infracted networks using malignant agents to reach a consensus. The attributes of the linear consensus model have been projected to resolve the problem of consensus in slight communication networks that are based on the Markov process [37].

The study and analysis of the behaviors of a network of agents is a complex process and a source of interest that has attracted many researchers in recent years, especially with the increase in the use of artificial intelligence technology in our lives. There are many problems that have risen in this area, and the problem of consensus is one of the most important of these problems. This problem occurs when the agents' group wants to perform a task; it requires dialogues among them and coordination to approve the ideas and agreement to carry out this task with one consensus. This consensus requires elaborate rules that can bring dispersal ideas closer together among agents and make them agree over the course of the task without interruption or delay. These rules have been studied in many previous pieces of research through which the proxies reach a consensus by adopting protocols and algorithms for mathematical equations. In this case, two types of mathematical protocols have been adopted in order to achieve consensus, the consensus in discrete-time, and consensus in continuous-time. Many researches presented in [38–41] are concerned with consensus in discrete-time, while there are also researches, such as presented in [42–46], which are concerned with consensus in continuous-time. However, most researches focused on the former (consensus in discrete-time), because it is considered the most challenging, as there are unstable subjective values in the matrix for each agent, which indicates that the problem of consensus for MAS in the discrete-time is more complicated compared to the case of continuous-time [47,48]. Therefore, attention is given to consensus in discrete-time for MAS in this work.

It has been shown that cubic stochastic matrices constitute powerful tools for consensus model that can significantly simplify the analysis for convergence. Quadratic stochastic consensus for MAS has been investigated in [49–55]. From this point, it is worth investigating a novel, less complicated consensus of non-linear control independently. This work aims to study the properties of semi symmetry quadratic operator (SSQO) convergence. We have obtained an essential condition for ensuring that the underlying cubic stochastic matrix has simple yet specified values. It has been shown that the notion of structural balance for the cubic stochastic matrix plays a crucial role in investigating the respective condition. We apply the condition to the consensus problems on MAS in order to demonstrate the importance of the obtained condition. A general notion of consensus, called 'non-linear consensus', is introduced, which means that the initial values of all individual members are limited equally. This has allowed us to obtain certain necessary and sufficient conditions for SSQO. These SSQO results extend and complement some existing results on the convergence of classes of quadratic stochastic operators (QSO) [56–61].

A major research contribution of this paper is the establishment of new conditions of the non-linear protocol of SSQO for consensus results, which are useful in the study of distributed coordination of MAS. In previous works dealing with the agreement coordination problems (or problems of consensus), is to identify decentralized strategies in order to ensure convergence to a common value.

2. Background and Methods

In this section, preliminary notions and notations for SSQO and based theories will be provided. The proposed model of SSQO is based on QSO [62]. New rules for the transition matrix of QSO are to be included in order to produce a less complicated model to achieve the consensus for MAS. In this case, the statuses of individual members $(s_1, s_2, s_3, \ldots, s_m)$ will be duplicated as row s_i and column s_j and produced with the transition matrix $A_{ij,k}$ that is based to the following protocol:

$$S_k^{(t+1)} = \sum_{ij,k=1}^{m} s_i^{(t)} A_{ij,k}\, s_j^{(t)} \tag{1}$$

which means

$$S_k^{(t+1)} = \sum_{ij,k=1}^{m} s_i^{(t)} A_{ij,k}\, s_j^{(t)} = \left(\begin{array}{c} \begin{pmatrix} s_1^{(t)} & s_2^{(t)} & \cdots & s_m^{(t)} \end{pmatrix} \begin{pmatrix} a_{11,1} & a_{12,1} & \cdots & a_{1m,1} \\ a_{21,1} & a_{22,1} & \cdots & a_{2m,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,1} & a_{m2,1} & \cdots & a_{mm,1} \end{pmatrix} \begin{pmatrix} s_1^{(t)} \\ s_2^{(t)} \\ \vdots \\ s_m^{(t)} \end{pmatrix} \\ \begin{pmatrix} s_1^{(t)} & s_2^{(t)} & \cdots & s_m^{(t)} \end{pmatrix} \begin{pmatrix} a_{11,2} & a_{12,2} & \cdots & a_{1m,2} \\ a_{21,2} & a_{22,2} & \cdots & a_{2m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,2} & a_{m2,2} & \cdots & a_{mm,2} \end{pmatrix} \begin{pmatrix} s_1^{(t)} \\ s_2^{(t)} \\ \vdots \\ s_m^{(t)} \end{pmatrix} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \begin{pmatrix} s_1^{(t)} & s_2^{(t)} & \cdots & s_m^{(t)} \end{pmatrix} \begin{pmatrix} a_{11,m} & a_{12,m} & \cdots & a_{1m,m} \\ a_{21,m} & a_{22,m} & \cdots & a_{2m,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,m} & a_{m2,m} & \cdots & a_{mm,m} \end{pmatrix} \begin{pmatrix} s_1^{(t)} \\ s_2^{(t)} \\ \vdots \\ s_m^{(t)} \end{pmatrix} \end{array} \right) \tag{2}$$

where s refers to agent, m the total number of agents, i and j the indexed number for each agent in row and column forms, and $A_{ij,k}$ is the transition matrix of the communicated agents.

$S_k^{(t+1)}$ is the general operator for QSO to determine the limit behavior for each member of $s_i^{(t)}$, where i is in row form and j in column form; however, $A_{ij,k}$ signifies the distributed matrices $(A_{ij,1}, A_{ij,2}, \ldots, A_{ij,k})$ as each matrix, $A_{ij,k}$ consists of the communications of each agent $s_i^{(t)}$ with the other agents.

Subsequently, the process will be updated using the new outputs as new inputs for the next iteration $(t+1)$. The iterations are continued until the fixed values are reached (consensus), in the form of $s^{(t+1)} = s^{(t)}$.

The transition matrix $A_{ij,k}$ in QSO is distributed to m matrices containing random values, where their sum is a matrix consisting of elements $a_{ij,k}$ equal to one, as following:

$$a_{ij,1} + a_{ij,2} + \cdots + a_{ij,k} = 1 \tag{3}$$

However, the values of the initial statuses are random and the sum of them equals one. Because the evolutionary matrices are split into m matrices and have random elements, the structure of the equations system of QSO will consist of long polynomials equations. Hence, it is necessary to establish a new class from QSO that has low computations, such as DeGroot's linear model, as detailed in [63]. Furthermore, some studies on some classes of QSO such as doubly stochastic quadratic operators DSQO [2], modified doubly stochastic quadratic operators MDSQO [51], extreme doubly stochastic quadratic operators EDSQO [54,64] and cubic quadratic stochastic operators CQSO [18] already have a consensus MAS with some restricted conditions and drawbacks in the consensus.

Now, let us present the proposed model of SSQO for the consensus in MAS. The SSQO has the new notations in the transition matrix $A_{ij,k}$ in Equation (1). The transition matrix of stochastic matrix (SM) has entries elements ($a_{ii.k} = a_{ji,k}$), which are either 0 or $\frac{1}{2}$, where the value $\frac{1}{2}$ means that the individual member interacts with itself and 0 means idle state. Furthermore, the sum of each distributed matrices $(A_{ij,1}, A_{ij,2}, \ldots, A_{ij,k})$ should have the same condition focusing on the matrix in QSO, which stipulates that the sum of its elements equals to m, as follows

$$A_{ij,k} = \left\{ a_{ii.k} = a_{ji.k} = (\tfrac{1}{2} \mid 0),\ a_{ji.k}\ ,\ \sum_{ij=1}^{m} a_{ij,k} = m,\ \sum_{k=1}^{m} a_{ij,k} = 1 \right\} \tag{4}$$

where i and j signify the number of rows and columns, while k is the number of matrices.

3. Theoretical Result

Theorem 1. *Suppose that a team of individual agents in MAS receive information and should perform analysis. The node members will communicate with each other locally via SSQO to exchange the information for analysis and produce a decision. The team will reach to an average agreement to send the decision if at least one node has a positive initial status.*

Proof. Consider a group of agents $\left(S_1^0, S_2^0, \ldots, S_m^0\right)$ that have at least one positive initial state where $0 \leq s_i^0 \leq 1$, $s_k^0 \in S^{m-1}$, while the others could be nonnegative. These states are $MAX(s_i^0)$ or $MIN(s_i^0)$. □

Allow for the idea to participate amongst agents themselves via rules of the transition matrix $A_{ij,k}$ of SSQO.

The $A_{ij,k}$ represents sharing ideas in a stochastic matrix for the status of agents $s_i^{(t)}$ and $s_j^{(t)}$. Consequently, each agent in the group shares and updates its status idea via protocol SSQO (refer to Equations (1) and (2)), as follows

$$S_i^{(t+1)} = \left(s_i^{(t)} * A_{ij,k}\right) s_j^{(t)} = A_{ij,k}\, s_{i,j}^{(t)} \tag{5}$$

Here, the process for each agent $s_i^{(t+1)}$ starts by multiplying the transition matrix $A_{ij,k}$ by the current statuses of all agents in row form $s_i^{(t)}$, then this product $\left(s_i^{(t)} * A_{ij,k}\right)$ is multiplied again by the current statuses of agents, but, this time, in column form $s_j^{(t)}$ to produce a new status for the agent. In other words, to produce a new status for the agent, the transition matrix $A_{ij,k}$ is multiplied by the current statuses of all agents in both row and column $s_{i,j}^{(t)}$ forms.

We want to show that, by protocol SSQO, each agent of MAS has communications of two agents $\left(s_i^{(t)} * s_j^{(t)}\right)$ with a coefficient equal to one or of $\left(s_i^{(t)} * s_i^{(t)}\right)$ with coefficient equal to $\left(\frac{1}{2}\right)$.

It means that each group has the probable communications of

$$\left(MIN(s_i^{(t)}\right) * MAX(s_j^{(t)})) \text{ or } \left(MAX(s_i^{(t)}\right) * MAX(s_i^{(t)})) \text{ or } \left(MIN(s_i^{(t)}\right) * MIN(s_i^{(t)})) \tag{6}$$

where $s_i^{(t)}$ and $s_i^{(t)}$ are either $MIN(s_i^{(t)})$ or $MAX(s_i^{(t)})$.

Because the $s_i^{(0)}$ is $0 \leq s_i^0 \leq 1$ and stochastic, as well as the coefficients for the transition matrix $A_{ij,k}$ in SSQO, then we get that

$$MIN(s_i^{(t)}) \leq MIN(s_i^{(t+1)}) \tag{7}$$

$$MAX(s_i^{(t)}) \geq MAX(s_i^{(t+1)}) \tag{8}$$

where $MIN(s_i^{(t)})$ increases and $MAX(s_i^{(t)})$ decreases.

We obtain

$$MIN(s_i^{(0)}) \ll MIN(s_i^{(1)}) \ll MIN(s_i^{(2)}) \ll \ldots \ll MIN(s_i^{(t)}) \tag{9}$$

$$MAX(s_i^{(0)}) \gg MAX(s_i^{(1)}) \gg MAX(s_i^{(2)}) \gg \ldots \gg MAX(s_i^{(t)}) \tag{10}$$

where that means $MIN(s_i^{(t)})$ gets much less " $\ll$ " than $MIN(s_i^{(t+1)})$, and $MAX(s_i^{(t)})$ gets much greater " $\gg$ " than $MAX(s_i^{(t+1)})$.

Hence, we get that

$$MIN(s_i^{(t)}) \leq \ldots \leq MIN(s_i^{(t+1)}) \leq \ldots \leq MAX(s_i^{(t+1)}) \leq \ldots \leq MAX(s_i^{(t)}) \tag{11}$$

Subsequently, we can see that in the middle of the inequality

$$\ldots \leq MIN(s_i^{(t+1)}) \leq \ldots \leq MAX(s_i^{(t+1)}) \leq \tag{12}$$

the $MIN(s_i^{(t+1)})$ approximates to $MAX(s_i^{(t+1)})$ and because $A_{ij,k}$ is cubic stochastic, the $MIN(s_i^{(t+1)})$ and $MAX(s_i^{(t+1)})$ will be bounded and

$$MIN(s_i^{(t+1)}) = MAX(s_i^{(t+1)}) \tag{13}$$

In other words, at time t, the evolutionary stochastic matrix transfers the statuses of $\left[MIN(s_i^{(0)}\right), MAX(s_i^{(0)})]$ to $MAX(s_i^{(0)}) \geq s_i^t \geq MIN(s_i^{(0)})$.

Because $MIN(s_i^{(0)})$ increases piecemeal in the Equation (9) and, at the same time, $MAX(s_i^{(0)})$ decreases piecemeal in the Equation (10); subsequently, after t repetitions, $MIN(s_i^{(0)})$ and $MAX(s_i^{(0)})$ reach the same value equally without a loss of generality, which means that consensus is achieved and reached to the average $1/m$.

The implication now is that the $s_i^{(t+1)}$ has a (common) factor $\left(\frac{1}{m}\right)$ of $s_i^{(t)}$ as

$$s_i^{(t+1)} = \frac{1}{m} * \left(s_1^{(0)} + s_2^{(0)} + \ldots + s_m^{(0)}\right), \sum_{i=1}^{m} s_i = 1, \tag{14}$$

in any t, and $k = [1, m]$.

Meanwhile, because $\left(s_1^{(0)} + s_2^{(0)} + \ldots + s_m^{(0)}\right) = 1$, we get

$$s_i^{(t+1)} = \frac{1}{m} \tag{15}$$

which means $s_i^{(t+1)}$ is bounded and fixed to $\left(\frac{1}{m}\right)$.

This also means also that $MAX(s_i^{(0)})$ *and* $MIN(s_i^{(0)})$ are bounded on $(1/m)$.

Because the $\lim_{t\to\infty} s_i^{(t)}$ are equal then

$$\lim_{t\to\infty} d(s_i^{(t)}) = \lim_{t\to\infty} MAS(s_i^{(t)}) - \lim_{t\to\infty} MIN(s_i^{(t)}) = 0, \tag{16}$$

Consequently, $s_k^{(t+1)}$ converges.

Remark 1. *We should indicate that our model of protocol SSQO has been investigated for the required efficiency to avoid the drawbacks in the models of DeGroot's linear model [63] and non-linear models of CQSO [18], DSQO [2], and EDSQO [54], where the SSQO protocol*

- *is a non-linear model as well as DeGroot's model, which means that it has fast convergence;*
- *is distinguished by a less complicated computation as well as the non-linear stochastic operators of CQSO, DSQO, and EDSQO; and,*
- *avoids the problem of periodic and non-changing initial values as in the case of EDSQO.*

In the next section (Section 4: Discussion and Numerical Solution), we provide some practical examples in order to support the theoretic results for Theorem 1 and, for Remark 1, we provide comparison results in Section 5.

4. Discussion and Numerical Solution

In this section, we consider several cases of the SSQO protocol with a view to illustrate the related outcomes, where we define the consensus of three agents (S_1, S_2, S_3), four agents (S_1, S_2, S_3, S_4), and five agents $(S_1, S_2, S_3, S_4, S_5)$ utilizing the SSQO rules in cases of a) $A_{ij,k}$ in Equation (1) is SM and (b)) $A_{ij,k}$ in Equation (1) is doubly stochastic matrix (DSM). The evolutionary matrices are presented as:

$$\left(A_{ij,1} \quad , \quad A_{ij,2} \quad , \quad \dots \quad , \quad A_{ij,m}\right)$$
$$\begin{pmatrix} a_{11,1} & a_{12,1} & \cdots & a_{1m,1} \\ a_{21,1} & a_{22,1} & \cdots & a_{2m,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,1} & a_{m2,1} & \cdots & a_{mm,1} \end{pmatrix}, \begin{pmatrix} a_{11,2} & a_{12,2} & \cdots & a_{1m,2} \\ a_{21,2} & a_{22,2} & \cdots & a_{2m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,2} & a_{m2,2} & \cdots & a_{mm,2} \end{pmatrix},$$
$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \begin{pmatrix} a_{11,m} & a_{12,m} & \cdots & a_{1m,m} \\ a_{21,m} & a_{22,m} & \cdots & a_{2m,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1,m} & a_{m2,m} & \cdots & a_{mm,m} \end{pmatrix}$$

Here, the rules of SM are that the elements $a_{ij,k}$ are nonnegative and

$$a_{ij,1} + a_{ij,2} + \cdots + a_{ij,m} = 1. \tag{17}$$

where i and j are row and column, respectively, and k is distributed evaluation matrix for each agent.

Although the rules of DSM are that elements $a_{ij,k}$ are nonnegative and

$$a_{i1,k} + a_{i1,k} + \cdots + a_{im,k} = 1.a_{1j,k} + a_{2j,k} + \cdots + a_{mj,k} = 1.a_{ij,1} + a_{ij,2} + \cdots + a_{ij,m} = 1. \tag{18}$$

Now, it is possible to present examples for all cases.

4.1. Three Agents

In the case of SM:

$$A_{ij,1} = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, A_{ij,3} = \begin{pmatrix} 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$$

Using Equation (1), we obtain

$$\begin{aligned} S_1^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_1^{(t)} s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} \\ S_2^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_2^{(t)} s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \\ S_3^{(t+1)} &= s_1^{(t)} s_3^{(t)} + s_2^{(t)} s_3^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \end{aligned} \tag{19}$$

In the case of DSM:

$$A_{ij,1} = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}, A_{ij,3} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

Using Equation (1), we obtain

$$\begin{aligned} S_1^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_2^{(t)} s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \\ S_2^{(t+1)} &= s_1^{(t)} s_3^{(t)} + s_2^{(t)} s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} \\ S_3^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_1^{(t)} s_3^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \end{aligned} \tag{20}$$

Figure 1 shows the consensus for extreme values of the initial statuses for three agents $\left(s_1^{(0)} = 1,\ s_2^{(0)} = 0,\ \ s_3^{(0)} = 0\right)$.

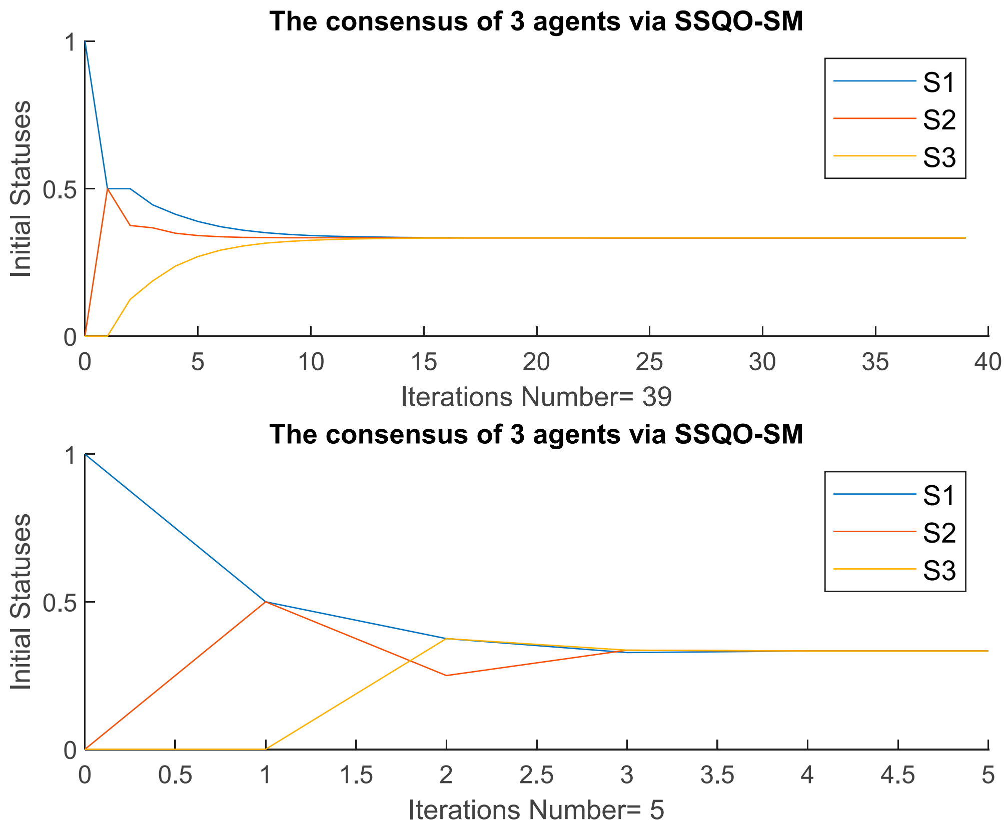

Figure 1. The consensus of three agents by semi symmetry quadratic operator (SSQO) using stochastic matrix (SM) and doubly stochastic matrix (DSM) with initial statuses of (0, 1, 0).

Figure 2 shows the consensus for random values of the initial statuses for three agents $(s_1^{(0)} = 0.162,\ s_2^{(0)} = 0.222,\ \ s_3^{(0)} = 0.616)$.

From the simulation results presented in Figures 1 and 2, we can see that SSQO achieves the consensus for three agents with initial values, with one agent having a full value while the others have zero value, as shown in Figure 1, and with positive random initial values, as shown in Figure 2. Moreover, the consensus is reached faster using DSM is than using SM. The second figure of Figures 1 and 2 shows this.

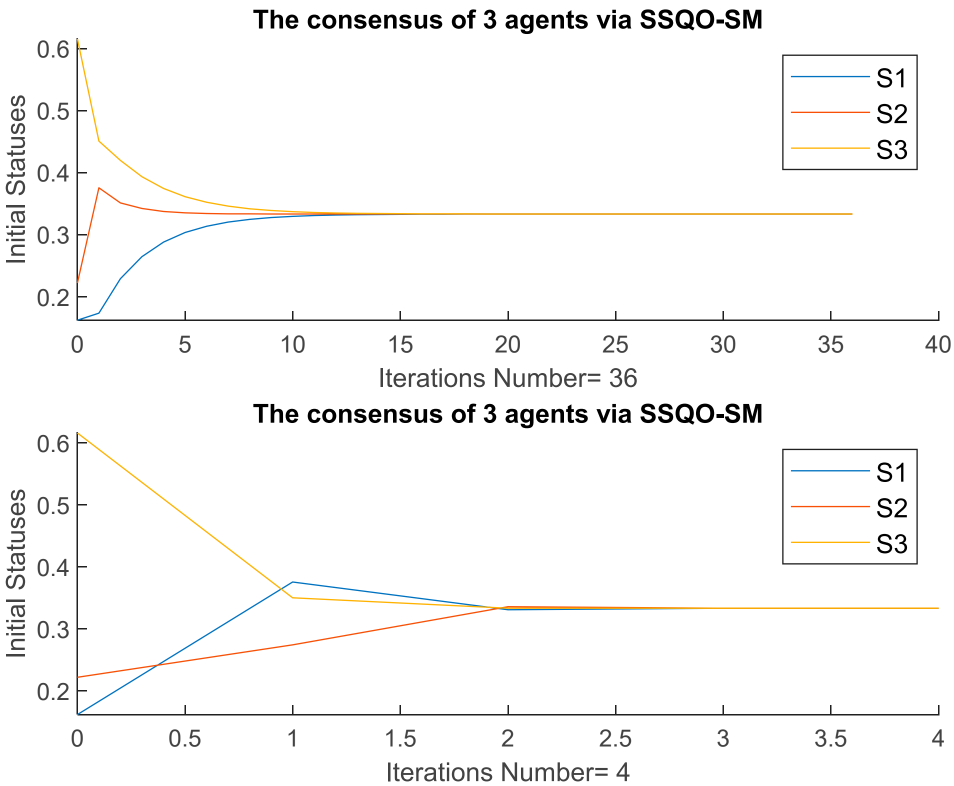

Figure 2. The consensus of three agents by SSQO using SM and DSM with initial statuses of (0.162, 0.222, 0.616) (random).

4.2. Four Agents

In the case of SM:

$$A_{ij,1} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 & 0.5 \end{pmatrix}$$

$$A_{ij,3} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}, A_{ij,4} = \begin{pmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

Using Equation (1), we obtain

$$\begin{aligned} S_1^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_1^{(t)} s_3^{(t)} + s_2^{(t)} s_4^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \\ S_2^{(t+1)} &= s_1^{(t)} s_3^{(t)} + s_1^{(t)} s_4^{(t)} + s_3^{(t)} s_4^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} \\ S_3^{(t+1)} &= s_1^{(t)} s_2^{(t)} + s_1^{(t)} s_4^{(t)} + s_2^{(t)} s_3^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_4{}^2}{2}\right)^{(t)} \\ S_4^{(t+1)} &= s_2^{(t)} s_3^{(t)} + s_2^{(t)} s_4^{(t)} + s_3^{(t)} s_4^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_4{}^2}{2}\right)^{(t)} \end{aligned} \tag{21}$$

In the case of DSM:

$$A_{ij,1} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix},$$

$$A_{ij,3} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}, A_{ij,4} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}.$$

Using Equation (1), we obtain

$$\begin{aligned} S_1^{(t+1)} &= s_1^{(t)}s_2^{(t)} + s_1^{(t)}s_3^{(t)} + s_2^{(t)}s_4^{(t)} + s_3^{(t)}s_4^{(t)} \\ S_2^{(t+1)} &= s_1^{(t)}s_2^{(t)} + s_1^{(t)}s_3^{(t)} + s_2^{(t)}s_4^{(t)} + s_3^{(t)}s_4^{(t)} \\ S_3^{(t+1)} &= s_1^{(t)}s_4^{(t)} + s_2^{(t)}s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} + \left(\frac{s_4{}^2}{2}\right)^{(t)} \\ S_4^{(t+1)} &= s_1^{(t)}s_4^{(t)} + s_2^{(t)}s_3^{(t)} + \left(\frac{s_1{}^2}{2}\right)^{(t)} + \left(\frac{s_2{}^2}{2}\right)^{(t)} + \left(\frac{s_3{}^2}{2}\right)^{(t)} + \left(\frac{s_4{}^2}{2}\right)^{(t)} \end{aligned} \tag{22}$$

Figure 3 shows the consensus for extreme values of the initial statuses for 4 agents $\left(s_1^{(0)} = 0,\ s_2^{(0)} = 0,\ s_3^{(0)} = 1,\ s_4^{(0)} = 0\right)$.

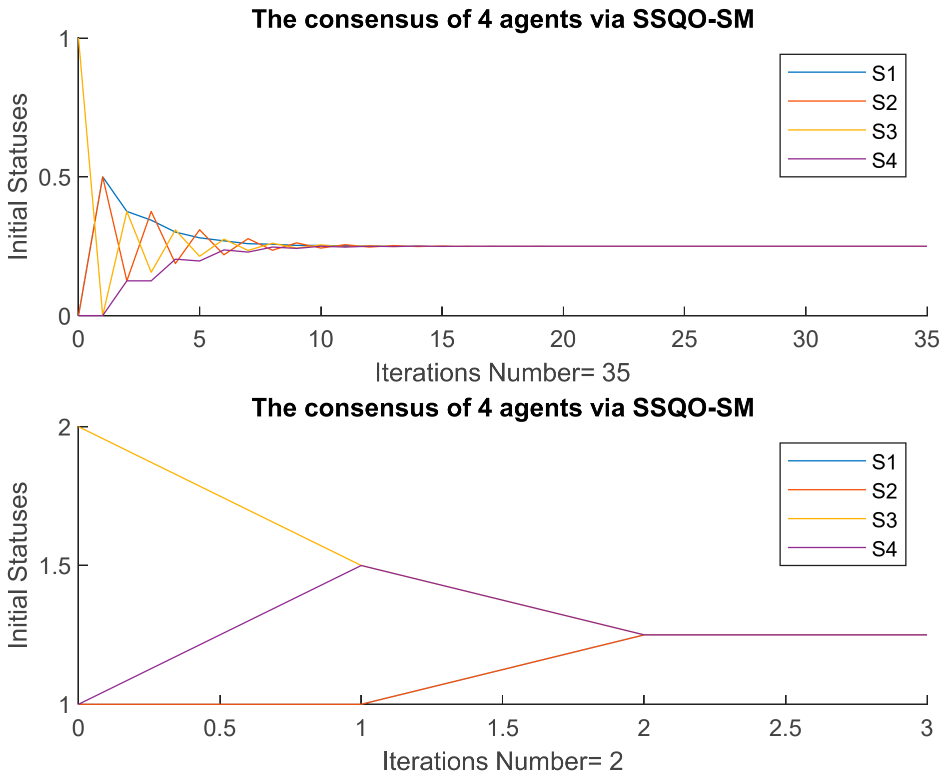

Figure 3. The consensus of four agents SSQO using SM and DSM with initial statuses of (0, 0, 1, 0).

Figure 4 shows the consensus for random values of the initial statuses for four agents $\left(s_1^{(0)} = 0.12,\ s_2^{(0)} = 0.18,\ s_3^{(0)} = 0.60,\ s_4^{(0)} = 0.10\right)$.

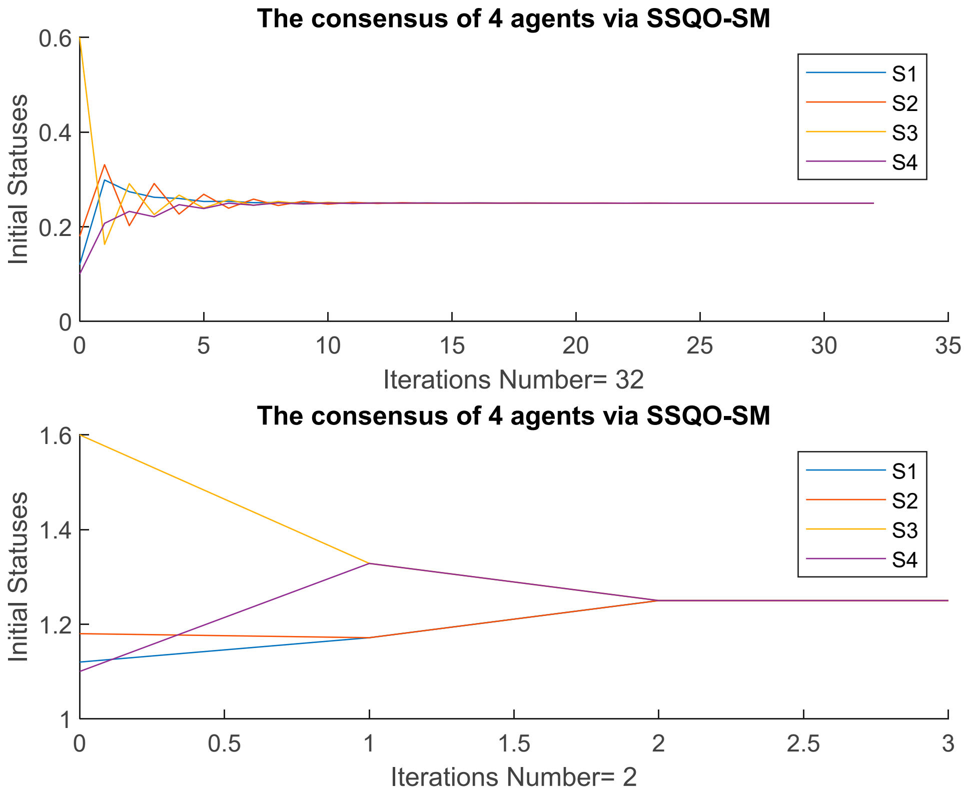

Figure 4. The consensus of four agents by SSQO by SM and DSM with initial statuses of (0.12, 0.18, 0.60, 0.10) (random).

The simulation result, as shown in in Figures 3 and 4, highlights that SSQO achieves the consensus for four agents with initial values. One agent has full value, while the others have zero value, as shown in Figure 3, while, in another case, the initial values are positive random values, as shown in Figure 4. Moreover, the consensus is reached faster using DSM than using SM. We can see this in Figures 3 and 4.

4.3. Five Agents

In the case of SM:

$$A_{ij,1} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{ij,3} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \end{pmatrix}, A_{ij,4} = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix},$$

$$A_{ij,5} = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$$

Using Equation (1), we obtain

$$\begin{aligned}
S_1^{(t+1)} &= s_1^{(t)}s_2^{(t)} + s_1^{(t)}s_3^{(t)} + s_1^{(t)}s_4^{(t)} + s_1^{(t)}s_5^{(t)} + \left(\frac{s_1^2}{2}\right)^{(t)} + \left(\frac{s_2^2}{2}\right)^{(t)} \\
S_2^{(t+1)} &= s_2^{(t)}s_1^{(t)} + s_2^{(t)}s_3^{(t)} + s_2^{(t)}s_4^{(t)} + s_3^{(t)}s_5^{(t)} + \left(\frac{s_2^2}{2}\right)^{(t)} + \left(\frac{s_3^2}{2}\right)^{(t)} \\
S_3^{(t+1)} &= s_3^{(t)}s_1^{(t)} + s_3^{(t)}s_2^{(t)} + s_3^{(t)}s_4^{(t)} + s_3^{(t)}s_5^{(t)} + \left(\frac{s_3^2}{2}\right)^{(t)} + \left(\frac{s_4^2}{2}\right)^{(t)} \\
S_4^{(t+1)} &= s_4^{(t)}s_1^{(t)} + s_4^{(t)}s_2^{(t)} + s_4^{(t)}s_3^{(t)} + s_4^{(t)}s_5^{(t)} + \left(\frac{s_4^2}{2}\right)^{(t)} + \left(\frac{s_5^2}{2}\right)^{(t)} \\
S_5^{(t+1)} &= s_5^{(t)}s_1^{(t)} + s_5^{(t)}s_2^{(t)} + s_5^{(t)}s_3^{(t)} + s_5^{(t)}s_4^{(t)} + \left(\frac{s_5^2}{2}\right)^{(t)} + \left(\frac{s_1^2}{2}\right)^{(t)}
\end{aligned} \tag{23}$$

In the case of DSM:

$$A_{ij,1} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}, A_{ij,2} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

$$A_{ij,3} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \end{pmatrix} A_{ij,4} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{ij,5} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{pmatrix}$$

Using Equation (1), we obtain

$$\begin{aligned}
S_1^{(t+1)} &= s_1^{(t)}s_2^{(t)} + s_1^{(t)}s_4^{(t)} + s_3^{(t)}s_5^{(t)} + s_4^{(t)}s_5^{(t)} + \left(\frac{s_2^2}{2}\right)^{(t)} + \left(\frac{s_3^2}{2}\right)^{(t)} \\
S_2^{(t+1)} &= s_1^{(t)}s_2^{(t)} + s_2^{(t)}s_3^{(t)} + s_4^{(t)}s_5^{(t)} + \left(\frac{s_1^2}{2}\right)^{(t)} + \left(\frac{s_3^2}{2}\right)^{(t)} + \left(\frac{s_4^2}{2}\right)^{(t)} + \left(\frac{s_5^2}{2}\right)^{(t)} \\
S_3^{(t+1)} &= s_1^{(t)}s_3^{(t)} + s_1^{(t)}s_5^{(t)} + s_2^{(t)}s_4^{(t)} + s_2^{(t)}s_5^{(t)} + s_3^{(t)}s_4^{(t)} \\
S_4^{(t+1)} &= s_1^{(t)}s_3^{(t)} + s_1^{(t)}s_5^{(t)} + s_2^{(t)}s_4^{(t)} + s_2^{(t)}s_5^{(t)} + s_3^{(t)}s_4^{(t)} \\
S_5^{(t+1)} &= s_1^{(t)}s_4^{(t)} + s_2^{(t)}s_3^{(t)} + s_3^{(t)}s_5^{(t)} + \left(\frac{s_1^2}{2}\right)^{(t)} + \left(\frac{s_2^2}{2}\right)^{(t)} + \left(\frac{s_4^2}{2}\right)^{(t)} + \left(\frac{s_5^2}{2}\right)^{(t)}
\end{aligned} \tag{24}$$

Figure 5 shows the consensus for extreme values of the initial statuses for five agents $\left(s_1^{(0)} = 0,\ s_2^{(0)} = 1,\ s_3^{(0)} = 0,\ s_4^{(0)} = 0,\ s_5^{(0)} = 0\right)$.

Figure 6 shows the consensus for random values of the initial statuses for five agents $\left(s_1^{(0)} = 0.02, s_2^{(0)} = 0.09, s_3^{(0)} = 0.15, s_4^{(0)} = 0.20, s_5^{(0)} = 0.54\right)$.

Figures 5 and 6 present the consensus for five agents, where one agent has full value while the others have zero value, as shown in Figure 5, while, in another case, the initial values are positive and random, as shown in Figure 6. Additionally, the consensus is reached faster using DSM than using SM. We can see this in the second figure of Figures 5 and 6.

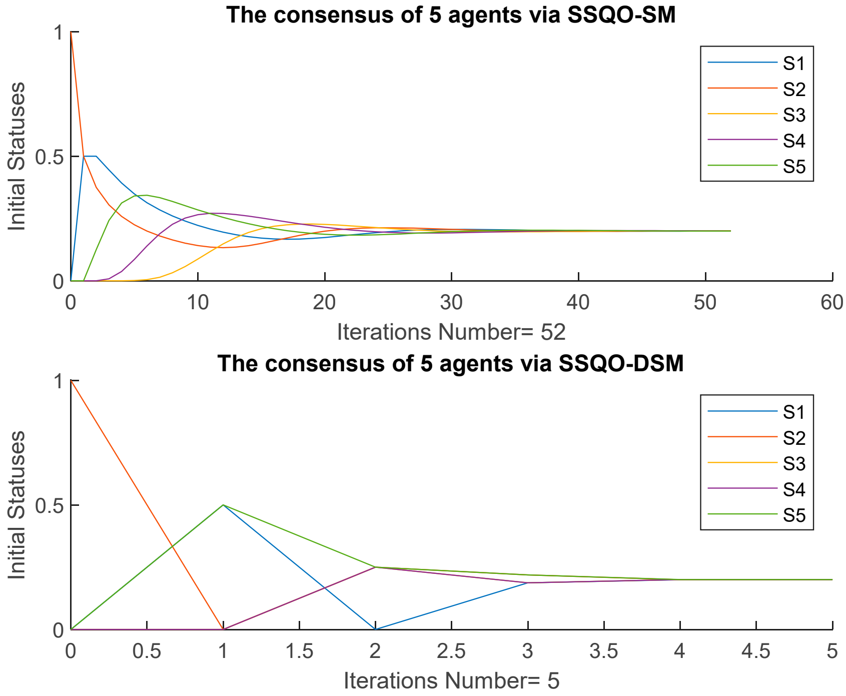

Figure 5. The consensus of five agents by SSQO using SM and DSM with initial statuses of (0, 1, 0, 0, 0).

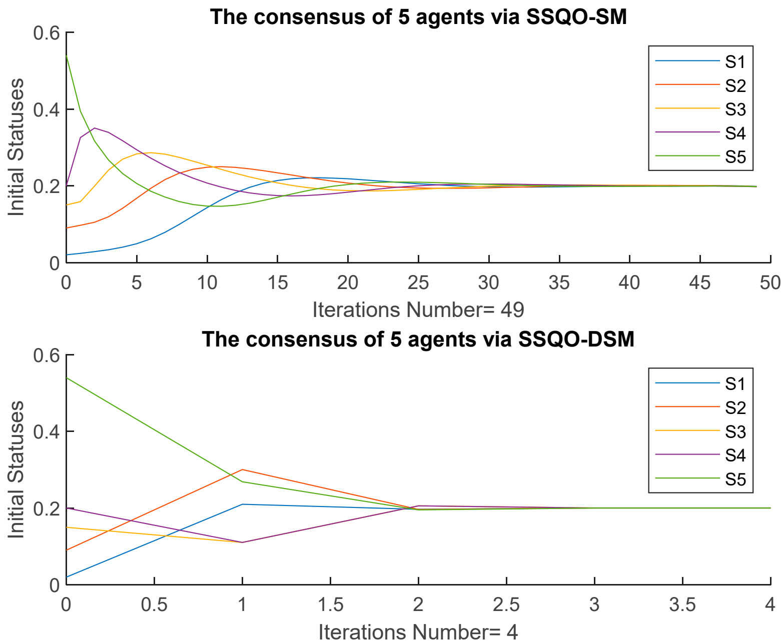

Figure 6. The consensus of five agents by SSQO by SM and DSM for initial statuses of (0.02, 0.09, 0.15, 0.20, 0.54) (random).

The consensus achieved in the initial statuses utilizing SSQO rules with SM and DSM is outlined in Figures 1–6. It can be observed that the consensus is reached faster by DSM than by SM. We can see in Figures 1–6 that the number of iterations in DSM required to reach a consensus is fewer than by SM, with 2–5 iterations as compared to 35–52 iterations, respectively.

Here, we show some examples of three, four, and five agents (although the simulation can also be applied for consensus of finite agents). Figure 7 shows the consensus of 10 agents and Figure 8 for 1000 agents.

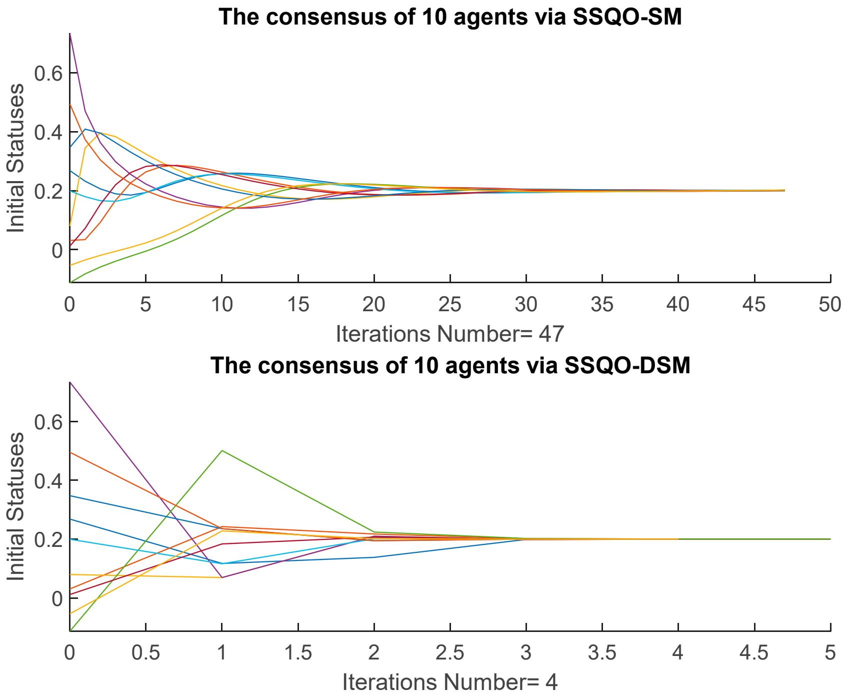

Figure 7. The consensus of 10 agents by SSQO by SM and DSM with random initial statuses.

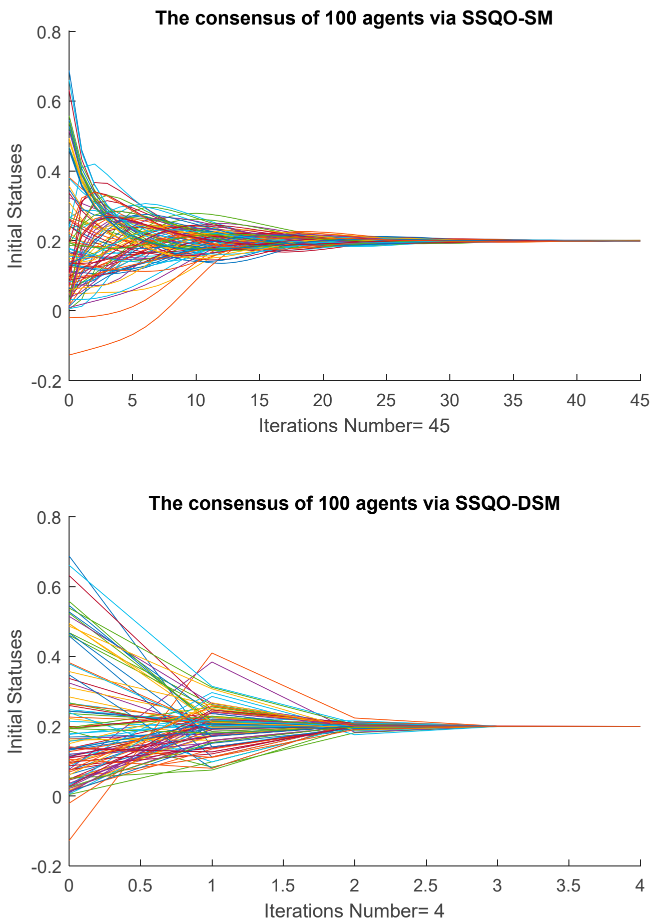

Figure 8. The consensus of 100 agents by SSQO by SM and DSM with random initial statuses.

To conclude, in this section, we present some examples for three, four, and five agents using SSQO with the methods of calculation in order to prove that the SSQO reach a consensus. Figures 1–6 emphasize the result for Theorem 1, where Figures 1 and 2 for three agents, Figures 3 and 4 for four agents, and Figures 5 and 6 for five agents. Finally, the simulation results presented in Figures 7 and 8 show the results for 10 and 100 agents, respectively.

5. Comparison of the Consensus SSQO Model with Other Consensus Models

It is worthwhile to evaluate the efficiency of the proposed SSQO model in comparison to other models of the same structure such as DeGroot [63], CQSO [18], DSQO [2], and EDSQO [54]. The abstracted Table 1 illustrates the advantages and disadvantages of each model. It also shows the evolutionary operator process and the transition matrix rules for each model; whereas Table 2 indicates the special cases of drawbacks.

Figures 9 and 10 clearly identify the weaknesses of the existing models: Figures 9 and 10 show that no consensus is reached by three agents for one and 10 times with different initial statuses. No consensus is reached due to the transition matrix being periodic (DeGroot and CSQO), and selfish communication in the transition matrix (DSQO and EDSQO). As mentioned in [63], the DeGroot model reaches a consensus if and only if the transition matrix is not periodic. As mentioned in [18], the CSQO model is based on the DeGroot model and, thus, also cannot reach a consensus in the periodic case of in the transition matrix. As mentioned in [2], the models DSQO and EDSQO cannot reach a consensus in the case of the selfish communication in the transition matrix. Moreover, in terms of the limit behavior of the agent statuses, we can observe that DSQO displays smoother and long-period oscillations, while DeGroot, CSQO, and EDSQO show a period one cycle with iterations, due to the fact that DSQO has no vertices (extreme) values in the transition matrix, while DeGroot, CSQO, and EDSQO have such values in the transition matrix. After 50 iterations in DSQO, the agent statuses may diverge; the elements of each agent obtain a share factor from the product of the statuses in the transition matrix; in other words, the communication among all agents only with one agent.

Thus, in this paper, we propose a new non-linear consensus model that converges to a consensus and involves less computation. In fact, Figures 9 and 10 show the weakness for the existing models of DeGroot, CSQO, DSQO, and EDSQO in one and 10 times for different random initial statuses for three agents $(S1, S2, S3)$, respectively.

In general, we have noted that the transition matrix for SSQO could be any value for nondiagonal elements that have an interaction, because the sum of $a_{ji.k}$ and $a_{ij,k}$ under the conditions of SSQO is stochastic and equals one in any way for those agents who interact with others. Because the $a_{ji.k}$ and $a_{ij,k}$ are stochastic, it makes the transition matrix fixable. It means that the interactions among agents are not under restricted conditions.

The contribution of this work is the development of a new consensus model that combines four distinct advantages. The first advantage is that the model is non-linear, the second is that the consensus is achieved fast, the third is that it requires less computation, and the fourth is that the consensus is achieved in all cases following the condition rules of the transition matrix.

Table 1. Evaluation of SSQO with DeGroot, CSQO, DSQO, and EDSQO consensus.

	Name	Advantages	Disadvantages	Evolutionary Operators	Transition Matrix
1	DeGroot [63]	• Less computation	• Linear • Slow consensus • Not convergence in some cases	$s_i^{((t)+1)} = \sum_{i=1}^{m} A_{ij} s_i^{(t)},$	$A_{ij} = \left\{ a_{ij} \geq 0, \ , \ \sum_{i=1}^{m} P_{ij} = 1 \right\}$
2	CSQO [18]	• Non-linear • Fast consensus	• More computation (Long polynomial) • Restricted conditions	$s_i^{((t)+1)} = \sum_{i,j=1}^{m} s_i^{(t)} A_{ij,k} s_j^{(t)},$	$A_{ij,k} = \left\{ \begin{array}{c} a_{ij,k} \geq 0, \ \sum_{i=1}^{m} a_{j,k} = 1, \ \sum_{j=1}^{m} a_{i,k} = 1, \\ \sum_{k=1}^{m} P_{ij,k} = 1, \ \forall i,j,k = 1, \ldots, m \end{array} \right\}$
3	DSQO [2]	• Non-linear • Fast consensus	• More computation (Long polynomial) • Not convergence in some cases • Restricted conditions	$s_i^{((t)+1)} = \sum_{i,j=1}^{m} s_i^{(t)} A_{ij,k} s_j^{(t)},$	$A_{ij,k} = \left\{ \begin{array}{c} a_{ij,k} \geq 0, a_{ij.k} = a_{ji,k} \ , \\ \sum_{ij=1}^{m} a_{ij,k} = m, \ \sum_{k=1}^{m} a_{ij,k} = 1, \\ \sum_{i,j \in \alpha} a_{ij} \leq \lvert \alpha \rvert, \\ \alpha \subset \{1, \ 2, \ 3, \ \ldots, \ m\}, \\ \forall i,j,k = 1, \ \ldots, m \end{array} \right\}$
4	EDSQO [54]	• Non-linear • Less computation • Fast consensus	• Not convergence in some cases • Restricted conditions	$s_i^{((t)+1)} = \sum_{i,j=1}^{m} s_i^{(t)} A_{ij,k} s_j^{(t)},$	$A_{ij,k} = \left\{ \begin{array}{c} a_{ii,k} = (0 \vert 1), a_{ij,k} = \left(0 \middle\vert \frac{1}{2} \middle\vert 1\right), \\ a_{ji.k} = a_{ij,k} \ , \ \sum_{ij=1}^{m} a_{ij,k} = m, \\ \sum_{k=1}^{m} a_{ij,k} = 1, \ \sum_{i,j \in \alpha} a_{ij} \leq \lvert \alpha \rvert, \\ \alpha \subset \{1, \ 2, \ 3, \ \ldots, \ m\}, \ \forall i,j,k = 1, \ \ldots, m \end{array} \right\}$
5	SSQO proposed model	• Non-linear • Less computation • Fast consensus • Convergence in all cases	• Some restricted conditions	$s_i^{((t)+1)} = \sum_{i,j=1}^{m} s_i^{(t)} A_{ij,k} s_j^{(t)},$	$A_{ij,k} = \left\{ \begin{array}{c} a_{ii.k} = a_{ji.k} = \left(\frac{1}{2} \middle\vert 0\right), \\ \sum_{ij=1}^{m} a_{ij,k} = m, \sum_{k=1}^{m} a_{ij,k} = 1 \end{array} \right\}.$

Table 2. Description of the cases of the transition matrix for DeGroot, CSQO, DSQO, and EDSQO that cannot reach to consensus with advantaged proposed model SSQO.

	Name	Advantages	Disadvantages
1	DeGroot [63]	$A_{ij,1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	This is a transition matrix under the rules of the DeGroot linear consensus model, which is the drawback that cannot reach to consensus.
2	CSQO [18]	$A_{ij,1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_{ij,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $A_{ij,3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.	This is a transition matrix under the rules of the CSQO non-linear consensus model, which is the drawback that cannot reach to consensus.
3	DSQO [2]	$A_{ij,1} = \begin{pmatrix} 1 & 0.7 & 0.3 \\ 0.7 & 0 & 0 \\ 0.3 & 0 & 0 \end{pmatrix}$, $A_{ij,2} = \begin{pmatrix} 0 & 0.2 & 0 \\ 0.2 & 1 & 0.8 \\ 0 & 0.8 & 0 \end{pmatrix}$, $A_{ij,3} = \begin{pmatrix} 0 & 0.1 & 0.7 \\ 0.1 & 0 & 0.2 \\ 0.7 & 0.2 & 1 \end{pmatrix}$	This is a transition matrix under the rules of the DSQO non-linear consensus model, which is the drawback that cannot reach to consensus.
4	EDSQO [54]	$A_{ij,1} = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}$, $A_{ij,2} = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix}$, $A_{ij,3} = \begin{pmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$.	This is a transition matrix under the rules of the EDSQO non-linear consensus model, which is the drawback that cannot reach to consensus.
5	SSQO proposed model	$A_{ij,1} = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}$, $A_{ij,2} = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$, $A_{ij,3} = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}$.	This is a transition matrix under the rules of the proposed SSQO non-linear consensus model, which avoids the drawback that cannot reach to consensus.

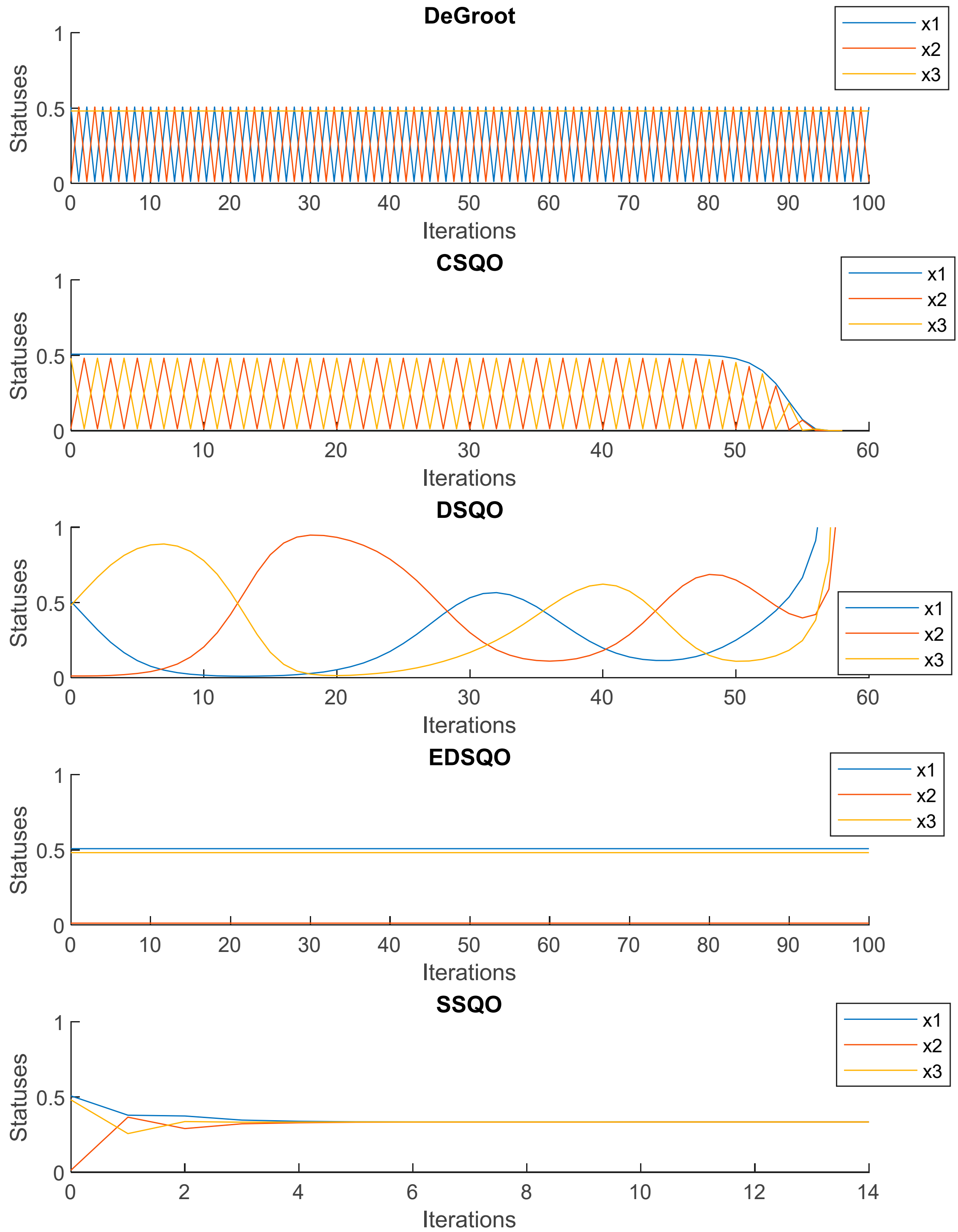

Figure 9. The Disagreement Cases of DeGroot, CSQO, DSQO, and EDSQO as compared to SSQO for three agents in one time for three agents in different random initial statuses.

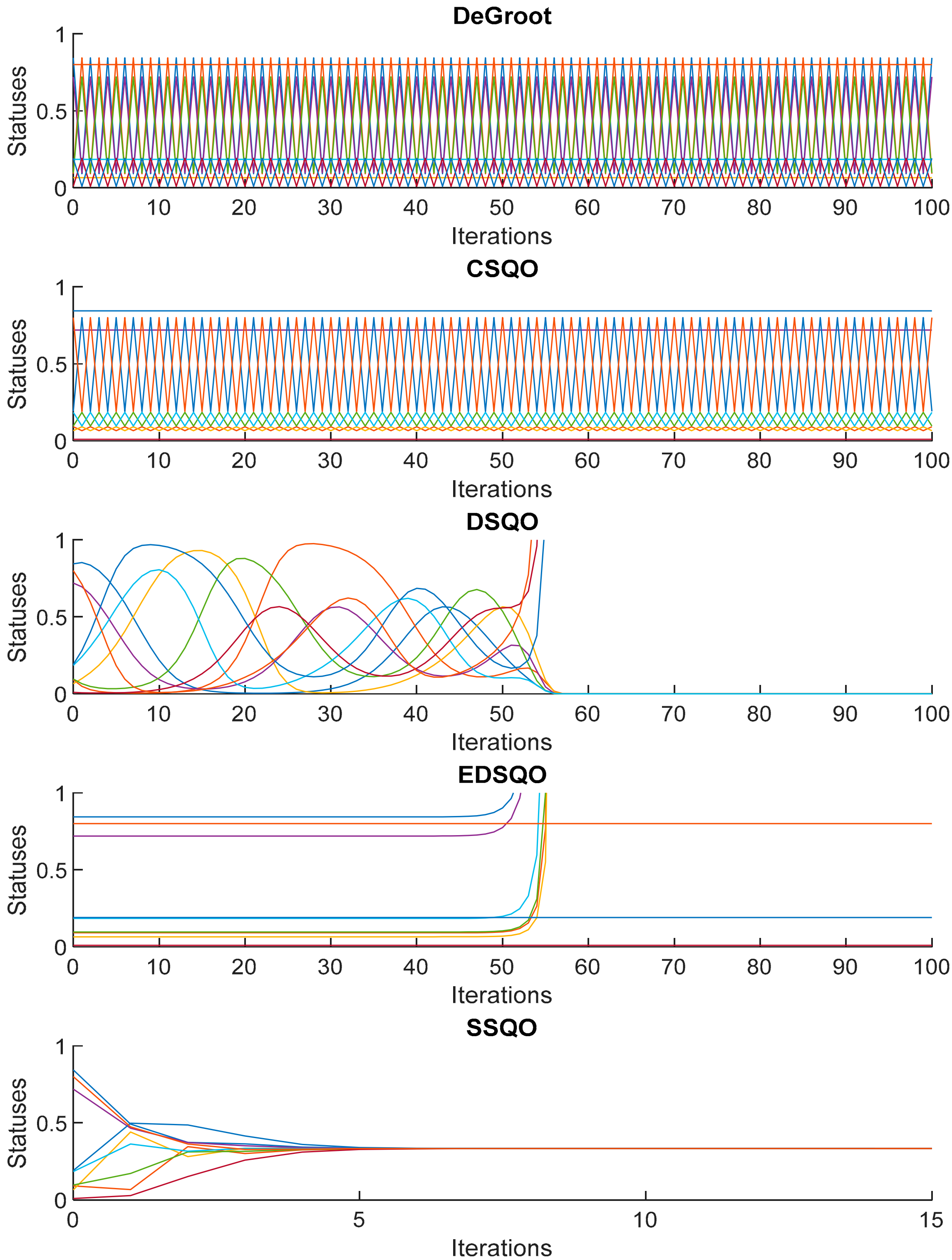

Figure 10. The Disagreement Cases of DeGroot, CSQO, DSQO, and EDSQO as compared to SSQO for three agents in 10 times for three agents in different random initial statuses.

6. Conclusions

In this paper, we have established a new less complicated non-linear convergence protocol of SSQO to resolve the consensus problem in MAS. This protocol generalizes both the linear model of DeGroot's and non-linear model of QSO. We have considered several new concepts for the SSQO non-linear model that allows for dynamic agents to agree in the distributed environment through teamwork in networks under local updated interconnections. The work has shown that two important limitations must be considered when addressing the consensus for MAS, which are initial statuses of agents should be at least one positive and the evolutionary matrices should be carried out while using a cubic stochastic non-linear matrix. The investigation into the SSQO protocol has demonstrated its required properties, meaning that the model should be non-linear and enable fast convergence to consensus and less complicated computation.

Author Contributions: Conceptualization, R.A.; methodology, R.A.; software, R.A.; validation, S.A.; formal analysis, R.A.; investigation, S.A.; resources, S.A.; writing—original draft preparation, R.A.; writing—review and editing, S.A., visualization, S.A.; supervision, R.A.; project administration, S.A.; funding acquisition, S.A. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest.

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