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On Extendability of the Principle of Equivalent Utility

Małgorzata Chudziak * and Marek Żołdak

Institute of Mathematics, University of Rzeszów, Pigonia 1, 35-959 Rzeszów, Poland; marek_z2@ur.edu.pl
* Correspondence: mchudziak@ur.edu.pl

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Abstract: An insurance premium principle is a way of assigning to every risk a real number, interpreted as a premium for insuring risk. There are several methods of defining the principle. In this paper, we deal with the principle of equivalent utility under the rank-dependent utility model. The principle, generated by utility function and probability distortion function, is based on the assumption of the symmetry between the decisions of accepting and rejecting risk. It is known that the principle of equivalent utility can be uniquely extended from the family of ternary risks. However, the extension from the family of binary risks need not be unique. Therefore, the following problem arises: characterizing those principles that coincide on the family of all binary risks. We reduce the problem thus to the multiplicative Pexider functional equation on a region. Applying the form of continuous solutions of the equation, we solve the problem completely.

Keywords: insurance premium; extension; utility function; probability distortion function; Choquet integral; Pexider functional equation

MSC: 39B22; 91B30

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1. Introduction

Assume that $(\Omega, \mathcal{F}, P)$ is a nonatomic probability space and that $\mathcal{X}$ is a family of all bounded random variables on $(\Omega, \mathcal{F}, P)$. Furthermore, let

$$\mathcal{X}_+ := \{X \in \mathcal{X} : X \geq 0 \ \ P - a.e.\}.$$

Elements of $\mathcal{X}_+$ represent the risk to be insured by an insurance company. An insurance contract pricing consists of assigning to any $X \in \mathcal{X}_+$ a nonnegative real number, interpreted as a premium for insuring X. One of the methods of insurance contract pricing is *the principle of equivalent utility* introduced by Bühlmann [1]. To define the principle, assume that the insurance company possesses a preference relation $\preceq$ over the elements of $\mathcal{X}_+$. Such a relation induces the indifference relation $\sim$ on $\mathcal{X}_+$ in the following natural way: for every $X, Y \in \mathcal{X}_+$

$$X \sim Y \quad \text{if and only if} \quad X \preceq Y \ \text{ and } \ Y \preceq X. \tag{1}$$

Suppose that the company, having an initial wealth level $w \in [0, \infty)$, is going to decide whether to accept or reject the application for a risk $X \in \mathcal{X}_+$. If the application is accepted, the initial wealth level will increase by the insurance premium, say $H(X)$, but the company will bear the risk X. Therefore, this decision is represented by the random variable $w + H(X) - X$. If, however, the application is rejected, the company will remain at the initial wealth level. The principle of equivalent utility is based on the assumption of the symmetry between these decisions. More precisely, it postulates that the premium $H(X)$ should be determined in such a way that the company remains indifferent between accepting the risk and rejecting it, that is

$$w + H(X) - X \sim w. \tag{2}$$

Obviously, in general, one cannot expect the existence of $H(X)$ or its uniqueness. However, it is known that, if the preference relation $\preceq$ satisfies the axioms of expected utility, then for every $X \in \mathcal{X}$ the number $H(X)$ is uniquely determined by Equation (2). Some results concerning the properties of the principle under expected utility can be found, e.g., in References [1–4].

In this paper, we deal with the principle of equivalent utility under the rank-dependent utility model. This behaviorally motivated model, proposed by Quiggin [5], is based on the observation that, making decisions under risks, people usually set a reference point and they perceive the results of risky decisions above this point as profits and the results below it as losses. Furthermore, decision makers distort probabilities. Thus, the rank-dependent utility model combines a value function with a probability distortion function, that is a nondecreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$. More precisely, a preference relation under this model is represented by the Choquet integral. Recall that the Choquet integral with respect to the probability distortion function g is defined as follows:

$$E_g[X] = \int_{-\infty}^{0} (g(P(X > t)) - 1)\, dt + \int_{0}^{\infty} g(P(X > t))\, dt \quad \text{for} \quad X \in \mathcal{X}. \tag{3}$$

More details concerning rank-dependent utility can be found, e.g., in Reference [6].

The principle of equivalent utility under the rank-dependent utility model has been introduced by Heilpern [7]. It has been shown in Reference [7] that, in this setting, Equation (2) becomes

$$E_g[u(w + H_{(w,u,g)}(X) - X)] = u(w) \tag{4}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. It turns out (cf. Reference [8], Remark 4) that, if g is continuous, then, for every $X \in \mathcal{X}_+$, Equation (4) determines the number $H_{(w,u,g)}(X)$ uniquely. Several properties of the premium defined by Equation (4) have been studied in Reference [7] under the assumption that u is concave and g is convex. Tsanakas and Desli [9] investigated the properties of this premium regarding sensitivity to portfolio size and to risk aggregation. For more details concerning broad classes of risk measures generated by the principle of equivalent utility, we refer to Reference [10].

Note that, in general, Equation (4) has no explicit solution. However, in some exceptional cases, $H_{(w,u,g)}(X)$ can be expressed in an explicit way for every $X \in \mathcal{X}_+$. In particular, if u is linear, then

$$H_{(w,u,g)}(X) = E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+,$$

where $\bar{g} : [0,1] \to [0,1]$, given by

$$\bar{g} = 1 - g(1 - p) \quad \text{for} \quad p \in [0,1], \tag{5}$$

is the probability distortion function *conjugated to* g. Furthermore, if $u(x) = a(1 - e^{-cx})$ for $x \in \mathbb{R}$ (with some $a, c > 0$), then

$$H_{(w,u,g)}(X) = \frac{1}{c} \ln E_{\bar{g}}[e^{cX}] \quad \text{for} \quad X \in \mathcal{X}_+.$$

2. Problem Formulation

It follows from Equation (4) that the premium for a risk depends only on its probability distribution. Therefore, in the sequel, we identify the risks with their probability distributions. For every $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $p \in (0,1)$, by $\langle x_1, x_2; 1-p, p \rangle$, we denote any random variable $X \in \mathcal{X}$

such that $P(X = x_1) = 1 - p$ and $P(X = x_2) = p$. Moreover, for every $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 < x_2 < x_3$ and every $p, q \in (0,1)$ with $p + q < 1$, $\langle x_1, x_2, x_3; 1-p-q, p, q\rangle$ denotes any random variable $X \in \mathcal{X}$ such that $P(X = x_1) = 1 - p - q$, $P(X = x_2) = p$, and $P(X = x_3) = q$. Note that, as the space $(\Omega, \mathcal{F}, P)$ is nonatomic, such random variables exist. Let

$$\mathcal{X}^{(2)} := \{\langle x_1, x_2; 1-p, p\rangle : x_1, x_2 \in \mathbb{R},\ x_1 < x_2,\ p \in (0,1)\}$$

and

$$\mathcal{X}^{(3)} := \{\langle x_1, x_2, x_3; 1-p-q, p, q\rangle : x_1, x_2, x_3 \in \mathbb{R},\ x_1 < x_2 < x_3,\ p, q \in (0,1),\ p+q < 1\}.$$

Furthermore, we set

$$\mathcal{X}^{(2)}_+ := \{\langle x_1, x_2; 1-p, p\rangle \in \mathcal{X}^{(2)} : x_1 \geq 0\}$$

and

$$\mathcal{X}^{(3)}_0 := \{\langle x_1, x_2, x_3; 1-p-q, p, q\rangle \in \mathcal{X}^{(3)} : x_1 = 0\}.$$

Recently, Chudziak [11] has considered the extension problem for the principle of equivalent utility under Cumulative Prospect Theory. In this setting, the premium $H_{(w,u,g,h)}(X)$ for a risk $X \in \mathcal{X}_+$ is defined as a unique solution of the following equation:

$$E_{gh}[u(w + H_{(w,u,g,h)}(X) - X)] = u(w) \tag{6}$$

where

$$E_{gh}[X] = E_g[\max\{X, 0\}] - E_h[\max\{-X, 0\}] \quad \text{for} \quad X \in \mathcal{X}$$

is the generalized Choquet integral related to the probability distortion functions g (for gains) and h (for losses). The principle of equivalent utility under Cumulative Prospect Theory has been introduced by Kałuszka and Krzeszowiec [12]. The existence and uniqueness of the principle defined by Equation (6) have been characterized in Reference [8]. Several properties of the premium have been considered in References [12,13].

It has been proved in Reference [11] that the principle determined by Equation (6) can be uniquely extended from the family $\mathcal{X}^{(3)}_0$ onto $\mathcal{X}_+$. More precisely, if

$$H_{(w,u_1,g_1,h_1)}(X) = H_{(w,u_2,g_2,h_2)}(X) \quad \text{for} \quad X \in \mathcal{X}^{(3)}_0,$$

where $w \in [0, \infty)$ and, for $i \in \{1,2\}$, g_i and h_i are continuous probability distortion functions and $u_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u_i(0) = 0$, then

$$H_{(w,u_1,g_1,h_1)}(X) = H_{(w,u_2,g_2,h_2)}(X) \quad \text{for} \quad X \in \mathcal{X}_+.$$

Since $E_{g\bar{g}}[X] = E_g[X]$ for $X \in \mathcal{X}$ (cf. Reference [12]), Equation (4) is a particular case of Reference (6). Thus, this result applies also to the principle of equivalent utility under rank-dependent utility. That is, if $w \in [0, \infty)$ and, for $i \in \{1,2\}$, g_i is a continuous probability distortion function and $u_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u_i(0) = 0$, then

$$H_{(w,u_1,g_1)}(X) = H_{(w,u_2,g_2)}(X) \quad \text{for} \quad X \in \mathcal{X}^{(3)}_0$$

implies

$$H_{(w,u_1,g_1)}(X) = H_{(w,u_2,g_2)}(X) \quad \text{for} \quad X \in \mathcal{X}_+.$$

However, the above result fails to hold with $\mathcal{X}_0^{(3)}$ replaced by $\mathcal{X}_+^{(2)}$ (cf. Example 1). Thus, the following problem arises naturally: for a given $w \in [0, \infty)$, characterizing those pairs (u_1, g_1) and (u_2, g_2) for which

$$H_{(w,u_1,g_1)}(X) = H_{(w,u_2,g_2)}(X) \quad \text{for} \quad X \in \mathcal{X}_+^{(2)}. \tag{7}$$

The aim of this paper is to present a complete solution to this problem. A crucial role in our considerations is played by the continuous solutions of the multiplicative Pexider equation on a region.

3. Preliminary Results

We begin this section with three remarks which will be useful in our further considerations.

Remark 1. *Let g be a probability distortion function. It follows from Equation (3) that, if $X = \langle x_1, x_2; 1 - p, p\rangle \in \mathcal{X}^{(2)}$, then*

$$E_g[X] = (1 - g(p))x_1 + g(p)x_2. \tag{8}$$

Furthermore, if $X = \langle x_1, x_2, x_3; 1 - p - q, p, q\rangle \in \mathcal{X}^{(3)}$, then

$$E_g[X] = (1 - g(p + q))x_1 + (g(p + q) - g(q))x_2 + g(q)x_3. \tag{9}$$

Remark 2. *Assume that $w \in [0, \infty)$, g is a probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. Then, for every $X = \langle x_1, x_2; 1 - p, p\rangle \in \mathcal{X}_+^{(2)}$, we have*

$$u(w + H_{(w,u,g)}(X) - X) = \langle u(w + H_{(w,u,g)}(X) - x_2), u(w + H_{(w,u,g)}(X) - x_1); p, 1 - p\rangle.$$

Therefore, in view of Equation (8), Equation (4) becomes

$$(1 - g(1 - p))u(w + H_{(w,u,g)}(X) - x_2) + g(1 - p)u(w + H_{(w,u,g)}(X) - x_1) = u(w). \tag{10}$$

Similarly, taking into account Equation (9), we conclude that, for every $X = \langle x_1, x_2, x_3; 1 - p - q, p, q\rangle \in \mathcal{X}_+^{(3)}$, Equation (4) takes the following form:

$$\begin{gathered}(1 - g(1 - q))u(w + H_{(w,u,g)}(X) - x_3) + (g(1 - q) - g(1 - p - q))u(w + H_{(w,u,g)}(X) - x_2) \\ +g(1 - p - q)u(w + H_{(w,u,g)}(X) - x_1) = u(w).\end{gathered} \tag{11}$$

Remark 3. *Assume that $w \in [0, \infty)$, g is a probability distortion function, and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function such that $u(0) = 0$. Let $X = \langle x_1, x_2; 1 - p, p\rangle \in \mathcal{X}_+^{(2)}$. Note that, if $H_{(w,u,g)}(X)$ were not greater than x_1, then the left-hand side of Equation (10) would be smaller than $u(w)$. On the other hand, if $H_{(w,u,g)}(X)$ were not smaller that x_2, then the left-hand side of Equation (10) would be greater than $u(w)$. Therefore, we have*

$$x_1 < H_{(w,u,g)}(\langle x_1, x_2; 1 - p, p\rangle) < x_2 \quad \textit{for} \quad \langle x_1, x_2; 1 - p, p\rangle \in \mathcal{X}_+^{(2)}. \tag{12}$$

The following example shows that, under the rank-dependent utility model, the extension of the principle of equivalent utility from the family of binary risks need not be unique.

Example 1. *Let $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ be given by*

$$u_1(x) = x \quad \textit{for} \quad x \in \mathbb{R}$$

and

$$u_2(x) = \begin{cases} 2x & \textit{for} \quad x \in (-\infty, w), \\ x + w & \textit{for} \quad x \in [w, \infty), \end{cases} \tag{13}$$

respectively. Then, obviously, u_1 and u_2 are strictly increasing and continuous and $u_1(0) = u_2(0) = 0$. Furthermore, let $g_1, g_2 : [0,1] \to [0,1]$ be of the following form:

$$g_1(p) = p \quad \text{for} \quad p \in [0,1]$$

and

$$g_2(p) = \frac{2p}{p+1} \quad \text{for} \quad p \in [0,1], \tag{14}$$

respectively. Then, g_1 and g_2 are continuous probability distortion functions. Since $E_{g_1}[X] = E[X]$ for $X \in \mathcal{X}_+$ (cf. Reference [12]), in view of Equation (4), for every $w \in [0,\infty)$ and $X \in \mathcal{X}_+$, $H_{(w,u_1,g_1)}(X)$ is a solution of the following equation:

$$E[w + H_{(w,u_1,g_1)}(X) - X] = w.$$

Hence,

$$H_{(w,u_1,g_1)}(X) = E[X] \quad \text{for} \quad X \in \mathcal{X}_+. \tag{15}$$

Note that we have also

$$H_{(w,u_2,g_2)}(X) = E[X] \quad \text{for} \quad X \in \mathcal{X}_+^{(2)}. \tag{16}$$

In fact, if $X = \langle x_1, x_2; 1-p, p\rangle \in \mathcal{X}_+^{(2)}$, then $E[X] = (1-p)x_1 + px_2$, and so, in view of Equations (13) and (14), we get

$$\begin{aligned}
&(1 - g(1-p))u_2(w + E[X] - x_2) + g(1-p)u_2(w + E[X] - x_1) \\
&= \left(1 - \frac{2-2p}{2-p}\right) u_2(w - (1-p)(x_2 - x_1)) + \frac{2-2p}{2-p} u_2(w + p(x_2 - x_1)) \\
&= \frac{2p}{2-p}(w - (1-p)(x_2 - x_1)) + \frac{2-2p}{2-p}(2w + p(x_2 - x_1)) = 2w = u_2(w).
\end{aligned}$$

Thus, taking into account Equation (10), we obtain Equation (16). From Equations (15) and (16), we derive Equation (7).

On the other hand, taking $X = \langle 0, 1, 2; 1/3, 1/3, 1/3\rangle \in \mathcal{X}_0^{(3)}$, we get $E[X] = 1$ and so

$$\begin{aligned}
&\left(1 - g_2\left(\frac{2}{3}\right)\right) u_2(w + E[X] - 2) + \left(g_2\left(\frac{2}{3}\right) - g_2\left(\frac{1}{3}\right)\right) u_2(w + E[X] - 1) + g_2\left(\frac{1}{3}\right) u_2(w + E[X]) \\
&= \frac{2}{5}(w-1) + \frac{3}{5}w + \frac{1}{2}(2w+1) = 2w + \frac{1}{10} > 2w = u_2(w).
\end{aligned}$$

Hence, in view of Equation (11), we obtain $H_{(w,u_2,g_2)}(X) \neq E[X]$ which, together with Equation (15), gives $H_{(w,u_1,g_1)}(X) \neq H_{(w,u_2,g_2)}(X)$.

The following result concerning continuous solutions of the multiplicative Pexider equation on a region will play an important role in our considerations.

Lemma 1. *Let $D \subseteq (0,\infty)^2$ be a non-empty, open, and connected set such that*

$$D_1 := \{t \in (0,\infty) : (t,y) \in D \text{ for some } y \in (0,\infty)\} = (0,\infty).$$

Furthermore, let

$$D_2 := \{y \in (0,\infty) : (t,y) \in D \text{ for some } t \in (0,\infty)\}$$

and

$$D_+ := \{ty : (t,y) \in D\}.$$

Assume that $K : D_+ \to \mathbb{R}$, $L : D_2 \to (0, \infty)$ *and* $\phi : (0, \infty) \to (0, \infty)$. *If* ϕ *is a strictly increasing continuous function and the triple* (K, L, ϕ) *satisfies following equation*

$$K(ty) = \phi(t)L(y) \quad \textit{for} \quad (t, y) \in D \tag{17}$$

then there exist $a, b, r \in (0, \infty)$ *such that*

$$K(x) = abx^r \quad \textit{for} \quad x \in D_+, \tag{18}$$

$$\phi(t) = at^r \quad \textit{for} \quad t \in (0, \infty) \tag{19}$$

and

$$L(y) = by^r \quad \textit{for} \quad y \in D_2. \tag{20}$$

Proof. Assume that ϕ is a strictly increasing continuous function and that the triple (K, L, ϕ) satisfies Equation (17). It follows from Equation (17) that $K(x) > 0$ for $x \in D_+$. Thus, applying Reference [14], Corollary 3, we obtain that there exist $a, b \in \mathbb{R} \setminus \{0\}$ and a function $m : (0, \infty) \to (0, \infty)$ such that

$$K(x) = abm(x) \quad \text{for} \quad x \in D_+, \tag{21}$$

$$\phi(t) = am(t) \quad \text{for} \quad t \in (0, \infty), \tag{22}$$

$$L(y) = bm(y) \quad \text{for} \quad y \in D_2 \tag{23}$$

and

$$m(xy) = m(x)m(y) \quad \text{for} \quad x, y \in (0, \infty).$$

Since ϕ is continuous, in view of Equation (22), so is m. Thus, applying Reference [15], Theorem 13.1.6, we conclude that there exists $r \in \mathbb{R} \setminus \{0\}$ such that

$$m(x) = x^r \quad \text{for} \quad x \in (0, \infty).$$

Hence, from Equations (21)–(23) we derive Equations (18)–(20), respectively. Furthermore, since ϕ is strictly increasing, $\phi(t) > 0$ for $t \in (0, \infty)$, and $L(y) > 0$ for $y \in D_2$, in view of Equations (19) and (20), we get $a, b, r \in (0, \infty)$. □

In the proof of our main result, we will also need the following lemma.

Lemma 2. *Assume that* $w \in [0, \infty)$, g *is a continuous probability distortion functions and* $u : \mathbb{R} \to \mathbb{R}$ *is a continuous strictly increasing function with* $u(0) = 0$. *Let* $f : (0, \infty) \times (0, 1) \to \mathbb{R}$ *be given by*

$$f(x, p) = H_{(w,u,g)}(\langle 0, x; 1-p, p \rangle) \quad \textit{for} \quad x \in (0, \infty), \ p \in (0, 1). \tag{24}$$

Then, we obtain the following:

(i)

$$(1 - g(1-p))u(w + f(x, p) - x) + g(1-p)u(w + f(x, p)) = u(w) \quad \textit{for} \quad x \in (0, \infty), \ p \in (0, 1); \tag{25}$$

(ii)

$$f(x, p) \in (0, x) \quad \textit{for} \quad x \in (0, \infty), \ p \in (0, 1); \tag{26}$$

(iii) For every $x \in (0, \infty)$, *the function* $f(x, \cdot)$ *is continuous, where*

$$\lim_{p \to 0^+} f(x, p) = 0 \quad \textit{and} \quad \lim_{p \to 1^-} f(x, p) = x;$$

(iv) *For every $p \in (0,1)$, the function $f(\cdot, p)$ is continuous and $\lim_{x\to 0^+} f(x,p) = 0$.*

Proof. Note that Equation (25) follows directly from Equations (10) and (24). Furthermore, Equations (12) and (24) imply Equation (26).

In order to prove (iii), fix $x \in (0,\infty)$. Suppose that $f(x,\cdot)$ is not continuous at the point $p_0 \in (0,1)$. Then, there exists a sequence $(p_n : n \in \mathbb{N})$ of elements of $(0,1)$ such that $\lim_{n\to\infty} p_n = p_0$ but $(f(x,p_n) : n \in \mathbb{N})$ does not tend to $f(x,p_0)$. It follows from Equation (26) that the sequence $(f(x,p_n) : n \in \mathbb{N})$ is bounded. Thus, there exists a subsequence $(p_{n_k} : k \in \mathbb{N})$ of the sequence $(p_n : n \in \mathbb{N})$ such that $\lim_{k\to\infty} f(x,p_{n_k}) =: y \neq f(x,p_0)$. Furthermore, in view of Equation (25), we get

$$(1-g(1-p_0))u(w+f(x,p_0)-x) + g(1-p_0)u(w+f(x,p_0)) = u(w) \tag{27}$$

and

$$(1-g(1-p_{n_k}))u(w+f(x,p_{n_k})-x) + g(1-p_{n_k})u(w+f(x,p_{n_k})) = u(w) \quad \text{for} \quad k \in \mathbb{N}. \tag{28}$$

Since u and g are continuous, letting in Equation (28) $k \to \infty$ and subtracting from Equation (27) the equality obtained in this way, we obtain

$$(1-g(1-p_0))(u(w+f(x,p_0)-x) - u(w+y-x)) + g(1-p_0)(u(w+f(x,p_0)) - u(w+y)) = 0.$$

This yields a contradiction, as u is strictly increasing and $y \neq f(x,p_0)$. Thus, we have proved that $f(x,\cdot)$ is continuous.

Now, we show that $\lim_{p\to 0^+} f(x,p) = 0$. Suppose that this is not true. Then, arguing as previously, we conclude that there exists a sequence $(p_n : n \in \mathbb{N})$ of elements of $(0,1)$ such that $\lim_{n\to\infty} p_n = 0$ but $\lim_{n\to\infty} f(x,p_n) =: y \neq 0$. Moreover, by Equation (25), we get

$$(1-g(1-p_n))u(w+f(x,p_n)-x) + g(1-p_n)u(w+f(x,p_n)) = u(w) \quad \text{for} \quad n \in \mathbb{N}.$$

Hence, as g is a continuous probability distortion function and u is continuous, letting $n \to \infty$, we obtain $u(w+y) = u(w)$. Since u is strictly increasing, this gives a contradiction and proves that $\lim_{p\to 0^+} f(x,p) = 0$. Using the same arguments, one can show that $\lim_{p\to 1^-} f(x,p) = x$. Therefore, (iii) is proved.

The proof of (iv) is similar. □

4. Main Result

Now, we are going to formulate and prove the main result of the paper.

Theorem 3. *Let $w \in [0,\infty)$. Assume that, for $i \in \{1,2\}$, g_i is a continuous probability distortion function and $u_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u_i(0) = 0$. Then, Equation (7) holds if and only if there exist $a, b, r \in (0,\infty)$ such that*

$$u_2(x) = \begin{cases} bu_1(w)^r - b(u_1(w) - u_1(x))^r & \textit{for} \quad x \in (-\infty, w), \\ bu_1(w)^r + ab(u_1(x) - u_1(w))^r & \textit{for} \quad x \in [w,\infty) \end{cases} \tag{29}$$

and

$$g_2(p) = \frac{g_1(p)^r}{g_1(p)^r + a(1-g_1(p)^r)} \quad \textit{for} \quad p \in [0,1]. \tag{30}$$

Proof. Assume that Equation (7) is valid. Let $f : (0,\infty) \times (0,1) \to \mathbb{R}$ be given by

$$f(x,p) = H_{(w,u_1,g_1)}(\langle 0, x; 1-p, p\rangle) \quad \text{for} \quad x \in (0,\infty), \ p \in (0,1).$$

Then, taking into account Equation (7) and applying Lemma 8(i), for every $x \in (0, \infty)$, $p \in (0, 1)$, and $i \in \{1, 2\}$, we obtain

$$(1 - g_i(1-p))u_i(w + f(x,p) - x) + g_i(1-p)u_i(w + f(x,p)) = u_i(w). \tag{31}$$

Hence,

$$(1 - g_i(1-p))v_i(x - f(x,p)) + g_i(1-p)v_i(-f(x,p)) = 0 \ \text{ for } \ x \in (0,\infty),\ p \in (0,1),\ i \in \{1,2\} \tag{32}$$

where $v_i : \mathbb{R} \to \mathbb{R}$ for $i \in \{1, 2\}$ is given by

$$v_i(x) = u_i(w) - u_i(w - x) \quad \text{for} \quad x \in \mathbb{R}. \tag{33}$$

Obviously, for $i \in \{1, 2\}$, v_i is strictly increasing and continuous with $v_i(0) = 0$. Moreover, it follows from Equation (32) that

$$v_i(-f(x,p)) = -\phi_i(p)v_i(x - f(x,p)) \ \text{ for } \ x \in (0,\infty),\ p \in (0,1),\ i \in \{1,2\} \tag{34}$$

where $\phi_i : (0, 1) \to (0, \infty)$ for $i \in \{1, 2\}$ is given by

$$\phi_i(p) = \frac{1 - g_i(1-p)}{g_i(1-p)} \quad \text{for} \quad p \in (0,1). \tag{35}$$

Note that, for $i \in \{1, 2\}$, ϕ_i is a continuous strictly increasing function with

$$\lim_{p \to 0^+} \phi_i(p) = 0, \qquad \lim_{p \to 1^-} \phi_i(p) = \infty. \tag{36}$$

Hence, for $i \in \{1, 2\}$, ϕ_i is an increasing homeomorphism of $(0, 1)$ onto $(0, \infty)$. Furthermore, in view of Equation (34), we get

$$-\phi_i(p)v_i(x - f(x,p)) \in v_i(\mathbb{R}) \ \text{ for } \ x \in (0,\infty),\ p \in (0,1),\ i \in \{1,2\}$$

and

$$v_1^{-1}(-\phi_i(p)v_1(x - f(x,p))) = v_2^{-1}(-\phi_2(p)v_2(x - f(x,p))) \ \text{ for } \ x \in (0,\infty),\ p \in (0,1).$$

Since ϕ_1 is a homeomorphism of $(0, 1)$ onto $(0, \infty)$, replacing in the last equality p by $\phi_1^{-1}(t)$, we obtain

$$v_1^{-1}(-tv_1(x - f(x, \phi_1^{-1}(t)))) = v_2^{-1}(-\phi(t)v_2(x - f(x, \phi_1^{-1}(t)))) \ \text{ for } \ x, t \in (0,\infty) \tag{37}$$

where

$$\phi := \phi_2 \circ \phi_1^{-1}. \tag{38}$$

Note that ϕ is an increasing homeomorphism on $(0, \infty)$. Moreover, taking

$$S(t) := \{x - f(x, \phi_1^{-1}(t)) : x \in (0,\infty)\} \quad \text{for} \quad t \in (0,\infty),$$

in view of Equation (37), we get

$$v_1^{-1}(-tv_1(z)) = v_2^{-1}(-\phi(t)v_2(z)) \quad \text{for} \quad t \in (0,\infty),\ z \in S(t).$$

Therefore,

$$v_1^{-1}(-ty) = v_2^{-1}(-\phi(t)v_2(v_1^{-1}(y))) \quad \text{for} \quad t \in (0,\infty),\ y \in v_1(S(t)). \tag{39}$$

Since v_1 and ϕ_1 are continuous, applying Lemma 2(iii), we obtain that, for every $x \in (0,\infty)$, the function

$$(0,\infty) \ni t \to v_1(x - f(x, \phi_1^{-1}(t))) \tag{40}$$

is continuous. Moreover, as v_1 is strictly increasing with $v_1(0) = 0$, taking into account Lemma 2(ii), we get

$$v_1(x - f(x, \phi_1^{-1}(t))) > 0 \quad \text{for} \quad x, t \in (0,\infty).$$

Thus, for every $x \in (0,\infty)$, the set

$$D^{(x)} := \{(t,y) \in (0,\infty)^2 : y < v_1(x - f(x, \phi_1^{-1}(t)))\}$$

is non-empty and open. Furthermore, we have

$$D^{(x)} = T^{(x)}((0,\infty) \times (0, v_1(x))) \quad \text{for} \quad x \in (0,\infty)$$

where $T^{(x)} : (0,\infty)^2 \to \mathbb{R}^2$ is of the following form:

$$T^{(x)}(x_1, x_2) = \left(x_1, v_1(x - f(x, \phi_1^{-1}(x_1)))\frac{x_2}{v_1(x)}\right) \quad \text{for} \quad (x_1, x_2) \in (0,\infty) \times (0, v_1(x)).$$

Since, for every $x \in (0,\infty)$, the function given by Equation (40) is continuous, so is $T^{(x)}$. Hence, $D^{(x)}$ is connected for $x \in (0,\infty)$. Note also that, for every $x_1, x_2 \in (0,\infty)$, we have

$$\left(1, \frac{1}{2}\min\{v_1(x_i - f(x_i, \phi_1^{-1}(1))) : i \in \{1,2\}\}\right) \in D^{(x_1)} \cap D^{(x_2)} \neq \emptyset.$$

Therefore, the set

$$D := \bigcup_{x \in (0,\infty)} D^{(x)}$$

is non-empty, open, and connected.

Let $(t,y) \in D$. Then, $(t,y) \in D^{(x)}$ for some $x \in (0,\infty)$, that is $0 < y < v_1(x - f(x, \phi_1^{-1}(t)))$. Moreover, according to Lemma 2(iv), the function

$$(0,\infty) \ni x \to v_1(x - f(x, \phi_1^{-1}(t)))$$

is continuous and, as v_1 is continuous with $v_1(0) = 0$, we get

$$\lim_{x \to 0} v_1(x - f(x, \phi_1^{-1}(t))) = 0.$$

Hence, $y = v_1(x_0 - f(x_0, \phi_1^{-1}(t)))$ for some $x_0 \in (0,\infty)$ and so $y \in v_1(S(t))$. Therefore, taking into account Equation (39), we conclude that

$$(v_2 \circ v_1^{-1})(-ty) = -\phi(t)(v_2 \circ v_1^{-1})(y) \quad \text{for} \quad (t,y) \in D. \tag{41}$$

Obviously, we have

$$D_1 = (0,\infty). \tag{42}$$

Moreover,

$$D_2 = v_1((0,\infty)). \tag{43}$$

In fact, if $y \in D_2$, then $(t,y) \in D$ for some $t \in (0,\infty)$. Thus, there exists $x \in (0,\infty)$ such that $(t,y) \in D^{(x)}$ and so

$$v_1(0) = 0 < y < v_1(x - f(x, \phi_1^{-1}(t))).$$

Since v_1 is strictly increasing and continuous, this implies that $y \in v_1((0,\infty))$. Conversely, if $y \in v_1((0,\infty))$, then $y = v_1(z)$ for some $z \in (0,\infty)$. Moreover, as ϕ is an increasing homeomorphism on $(0,\infty)$, we have $\lim_{t\to 0^+} \phi_1^{-1}(t) = 0$. Thus, taking $x \in (z,\infty)$, applying Lemma 2(*iii*), and using the continuity of v_1, we obtain $\lim_{t\to 0^+} v_1(x - f(x, \phi_1^{-1}(t))) = v_1(x)$. Therefore, as $y = v_1(z) < v_1(x)$, for sufficiently small $t \in (0,\infty)$, we have $y < v_1(x - f(x, \phi_1^{-1}(t)))$ and so $(t, y) \in D^{(x)} \subset D$. Hence, $y \in D_2$. In this way, we have proved Equation (43).

Let

$$L := (v_2 \circ v_1^{-1})_{|v_1((0,\infty))}. \tag{44}$$

Since, for $i \in \{1,2\}$, v_i is strictly increasing with $v_i(0) = 0$, in view of Equation (43), we get $L : D_2 \to (0,\infty)$. Moreover, defining $K : D_+ \to \mathbb{R}$ by

$$K(z) = -(v_2 \circ v_1^{-1})(-z) \quad \text{for} \quad z \in D_+,$$

from Equation (41) we derive that the triple (K, L, ϕ) satisfies Equation (17). Thus, using again the fact that ϕ is an increasing homeomorphism on $(0,\infty)$ and applying Lemma 1, we obtain that there exist $a, b, r \in (0,\infty)$ such that ϕ and L are of the forms in Equations (19) and (20), respectively. It follows from Equations (19) and (38) that

$$\phi_2(p) = a\phi_1(p)^r \quad \text{for} \quad p \in (0,1).$$

Hence, as $g_1(0) = g_2(0) = 0$ and $g_1(1) = g_2(1) = 1$, taking into account Equation (35), we obtain Equation (30).

From Equations (20), (33), and (44), we deduce that

$$u_2(x) = bu_1(w)^r - b(u_1(w) - u_1(x))^r \quad \text{for} \quad x \in (-\infty, w). \tag{45}$$

We are going to show that

$$u_2(x) = bu_1(w)^r + ab(u_1(x) - u_1(w))^r \quad \text{for} \quad x \in [w, \infty). \tag{46}$$

Since u_i is continuous for $i \in \{1,2\}$, it follows from Equation (45) that

$$u_2(w) = bu_1(w)^r. \tag{47}$$

Thus, Equation (46) holds for $x = w$. Fix $x \in (w,\infty)$. First, consider the case where $w > 0$. Then, $0 < x - w < x$, and so applying Lemma 2(*iii*), we get that $f(x, p) = x - w$ for some $p \in (0,1)$. Hence, in view of Equation (31), we obtain $g_i(1-p)u_i(x) = u_i(w)$ for $i \in \{1,2\}$. Therefore, taking into account Equations (30) and (47), we obtain

$$u_2(x) = \frac{u_2(w)}{g_2(1-p)} = bu_1(w)^r \frac{g_1(1-p)^r + a(1 - g_1(1-p))^r}{g_1(1-p)^r} = bu_1(w)^r \left(1 + a\frac{(1 - g_1(1-p))^r}{g_1(1-p)^r}\right)$$

$$= bu_1(w)^r + abu_1(w)^r \left(1 - \frac{u_1(w)}{u_1(x)}\right)^r \left(\frac{u_1(x)}{u_1(w)}\right)^r = bu_1(w)^r + ab(u_1(x) - u_1(w))^r.$$

If $w = 0$ then, in view of Equation (45), we get $u_2(x) = -b(-u_1(x))^r$. Thus, taking into account Equations (30) and (31) (with $w = 0$) and in view of Lemma 2(*ii*), for every $p \in (0,1)$, we obtain

$$u_2(f(x,p)) = -\frac{1 - g_2(1-p)}{g_2(1-p)} u_2(f(x,p) - x) = -\frac{a(1 - g_1(1-p))^r}{g_1(1-p)^r}(-b(-u_1(f(x,p) - x))^r)$$

$$= ab\left(-\frac{(1 - g_1(1-p))u_1(f(x,p) - x)}{g_1(1-p)}\right)^r = abu_1(f(x,p))^r.$$

Hence,

$$u_2(f(x,p)) = abu_1(f(x,p))^r \quad \text{for} \quad p \in (0,1).$$

Since u_i is continuous for $i \in \{1,2\}$, letting in this equality $p \to 1^-$ and applying Lemma 2(iii), we get $u_2(x) = abu_1(x)^r$. This proves Equation (46) in the case $w = 0$. Obviously, Equations (45) and (46) imply Equation (29).

In order to prove the converse statement, assume that there exist $a, b, r \in (0, \infty)$ such that Equations (29) and (30) hold. Let $X = \langle x_1, x_2; 1-p, p\rangle \in \mathcal{X}_+^{(2)}$. Then, in view of Equation (10), we obtain

$$(1 - g_1(1-p))u_1(w + H_{(w,u_1,g_1)}(X) - x_2) + g_1(1-p)u_1(w + H_{(w,u_1,g_1)}(X) - x_1) = u_1(w).$$

Hence,

$$(1-g_1(1-p))(u_1(w) - u_1(w + H_{(w,u_1,g_1)}(X) - x_2)) = g_1(1-p)(u_1(w + H_{(w,u_1,g_1)}(X) - x_1) - u_1(w)).$$

Moreover, it follows from Equation (12) that

$$w + H_{(w,u_1,g_1)}(X) - x_2 < w < w + H_{(w,u_1,g_1)}(X) - x_1.$$

Therefore, taking into account Equations (29) and (30) and applying Equation (10) again, we get

$$\begin{aligned}
&(1 - g_2(1-p))u_2(w + H_{(w,u_1,g_1)}(X) - x_2) + g_2(1-p)u_2(w + H_{(w,u_1,g_1)}(X) - x_1) \\
&= \frac{a(1-g_1(1-p))^r}{g_1(1-p)^r + a(1-g_1(1-p)^r)}(bu_1(w)^r - b(u_1(w) - u_1(w + H_{(w,u_1,g_1)}(X) - x_2))^r) \\
&+ \frac{g_1(1-p)^r}{g_1(1-p)^r + a(1-g_1(1-p)^r)}(bu_1(w)^r + ab(u_1(w + H_{(w,u_1,g_1)}(X) - x_1) - u_1(w))^r) \\
&= bu_1(w)^r = u_2(w).
\end{aligned}$$

Hence, $H_{(w,u_1,g_1)}(X) = H_{(w,u_2,g_2)}(X)$. $\square$

5. Conclusions

The principle of equivalent utility is a method of insurance contract pricing. It is based on the assumption of symmetry between the decisions of accepting and rejecting risk. It is known that under the rank-dependent utility model, the principle possesses a unique extension from the family of ternary risks. However, the extension from the family of binary risk need not be unique. In this paper, we establish a characterization of the principles that coincides on the family of binary risks. It is given in terms of the relations between the pairs of utility and probability distortion functions generating the principles. This result can play important roles in the study of the principle of equivalent utility. In fact, having a premium with known generators and applying our results, one can describe a family of all premiums which coincides with a given premium on the family of binary risks.

Recently, the principle of equivalent utility under Cumulative Prospect Theory has been intensively investigated. It seems to be interesting to establish an analogous characterization in this setting. Some partial results in this direction can be found in Reference [16].

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