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CQ-Type Algorithm for Reckoning Best Proximity Points of EP-Operators

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Abstract: We introduce a new class of non-self mappings by means of a condition which is called the (EP)-condition. This class includes proximal generalized nonexpansive mappings. It is shown that the existence of best proximity points for (EP)-mappings is equivalent to the existence of an approximate best proximity point sequence generated by a three-step iterative process. We also construct a CQ-type algorithm which generates a strongly convergent sequence to the best proximity point for a given (EP)-mapping.

Keywords: non-self mapping; best proximity point; CQ algorithm

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1. Introduction

The Banach contraction principle, which is the central result of the metric fixed point theory, has for decades been a source of inspiration for many authors. It states that any contraction mapping acting on a complete metric space has a unique fixed point, which is the limit of a sequence obtained by successive iterations of the given mapping. The attempts to extend this fundamental result have generated an impressive amount of scientific papers, as well as new areas of research. For instance, the theory of nonexpansive mappings, which naturally generalizes contraction mappings, has been a central topic during the last five decades. Fundamental existence results for nonexpansive mappings have been obtained by Kirk [1], Browder [2], and Göhde [3]. Later on, even wider classes of mappings were proposed and studied (see for instance Suzuki [4], García-Falset et al. [5]). At the same time, besides Picard's iteration used for contractions, some authors have introduced other iteration schemes (such as Mann [6] and Ishikawa [7]). This was in part due to the fact that Picard's iterative sequence for nonexpansive mappings does not necessarily converge. For more recently introduced iterative schemes, one can see Noor [8], Agrawal et al. [9], Abbas and Nazir, [10], Sintunavarat and Pitea [11], Thakur et al. [12–14], etc.

Another natural extension is to consider non-self mappings between two disjoint sets instead of mappings of a set into itself. In this setting, however, there is no point asking for fixed points, but instead one looks for best proximity points. More precisely, let $T\colon X \to Y$ be a mapping between two subsets X and Y of a metric space E. A *best proximity point* $x \in X$ is a point such that $d(x, Tx)$ is minimal. The interest for this type of problem was ignited by Fan [15]. Later on, authors such as Reich [16], Seghal and Singh [17], Naraghirad [18], and others have picked up on this subject and extended Fan's result in multiple ways.

The results presented in this paper relate to the above mentioned context as follows. Firstly, we consider the iterative process introduced by Thakur et al. [12] (which we shall call henceforth

TTP16), but for mappings satisfying the condition (E), introduced by García-Falset et al. [5], extending Lemma 3.1 and, respectively, Theorem 3.2 from [12].

Secondly, we adapt the iterative process TTP16 to the setting of non-self mappings and define a new class of operators which are required to have the (EP)-property (see below). This class includes proximal generalized nonexpansive mappings, introduced by Gabeleh [19]. It is shown that the (EP)-mappings have best proximity points if and only if the iterative sequence generated by the adapted TTP16 process is an approximate best proximity point sequence.

In the last section, we construct an algorithm which is a hybrid between the CQ algorithm of Nakajo and Takahashi [20] (see also Takahashi [21] and Jakob [22]) and the adapted TTP16 iterative process. The motivation in this case being the strong convergence of the sequences generated by the algorithm to best proximity points for (EP)-mappings.

2. Preliminaries

Let X and Y be two nonempty subsets of a Banach space $(E, \|\cdot\|)$. Throughout this paper the following notations will be used:

$$\begin{aligned} d(X,Y) &= \inf\{\|x-y\| : x \in X, y \in Y\}; \\ d(x,Y) &= \inf\{\|x-y\| : y \in Y\}; \\ P_X(y) &= \{x \in X : \|x-y\| = d(y,Y)\}; \\ X_0 &= \{x \in X : \|x-y'\| = d(X,Y),\ \text{for some } y' \in Y\}; \\ Y_0 &= \{y \in Y : \|x'-y\| = d(X,Y),\ \text{for some } x' \in X\}. \end{aligned}$$

Definition 1 ([23])**.** *A pair (X,Y) of nonempty subsets of a normed vector space with $X_0 \neq \emptyset$ is said to have the P-property if and and only if for any $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$,*

$$\begin{cases} \|x_1 - y_1\| = d(X,Y) \\ \|x_2 - y_2\| = d(X,Y) \end{cases} \Longrightarrow \|x_1 - x_2\| = \|y_1 - y_2\|.$$

Lemma 1. *Let X and Y be two nonempty closed bounded and convex subsets of a Banach space. If the pair (X,Y) has the P-property, then both X_0 and Y_0 are closed bounded and convex sets.*

Proof. To prove that X_0 is a closed set, take a sequence $\{x_n\} \subset X_0$, converging in the norm to some point $p \in X$. As $\{x_n\} \subset X_0$, one can associate a sequence $\{y_n\} \subset Y_0$, such that $\|x_n - y_n\| = d(X,Y)$ for all n. On the other hand, the P-property implies that $\|x_n - x_m\| = \|y_n - y_m\|$, for all n and m. Thus, $\{y_n\} \subset Y_0$ is a Cauchy sequence, which converges to some $q \in Y$, since Y is a closed set. Using now the inequality

$$\|p-q\| \leq \|p - x_n\| + \|x_n - y_n\| + \|y_n - q\|,$$

we conclude that $\|p-q\| = d(X,Y)$, meaning that $p \in X_0$. Thus X_0 is a closed set.

The set X_0 is bounded since X is bounded.

To prove the convexity of the set X_0, take $x_1, x_2 \in X_0$ and $\alpha \in [0,1]$. There exist $y_1, y_2 \in Y_0$ such that

$$\|x_1 - y_1\| = d(X,Y) = \|x_2 - y_2\|.$$

From the convexity of the set Y we get

$$\begin{aligned} \|\alpha x_1 + (1-\alpha)x_2 - (\alpha y_1 + (1-\alpha)y_2)\| &\leq \alpha\|x_1 - y_1\| + (1-\alpha)\|x_2 - y_2\| \\ &= d(X,Y). \end{aligned}$$

Thus X_0 is convex. The proof for Y_0 is similar. □

A Banach space E is called *uniformly convex* (see for instance [24]) if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for $x, y \in E$,

$$\left.\begin{array}{l} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x-y\| > \varepsilon \end{array}\right\} \implies \left\|\frac{x+y}{2}\right\| \leq \delta.$$

Let C be a nonempty closed convex subset of a Banach space E. Given a bounded sequence $\{x_n\} \subset E$, setting, for a given $x \in E$,

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|,$$

one defines the *asymptotic radius*

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and, respectively, the *asymptotic center*

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\},$$

of the sequence $\{x_n\}$ with respect to C.

In a uniformly convex Banach space the asymptotic center of a bounded sequence consists of a single element [25]. In a paper published in 2011, García-Falset et al. introduced a new class of mappings satisfying the so-called condition (E) defined as follows.

Definition 2 ([5])**.** *Let C be a nonempty subset of a Banach space $(E, \|\cdot\|)$. We say that a mapping $T\colon C \to E$ satisfies the condition* (E_μ) *if there exists $\mu \geq 1$ such that for all $x, y \in C$,*

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|.$$

A mapping T is said to satisfy the condition (E) *whenever it satisfies* (E_μ) *for some $\mu \geq 1$.*

This condition is weaker than Suzuki's condition (C) for generalized nonexpansive mappings, a fact which follows from [4] Lemma 7. Recently Thakur et al. [12] have introduced a new iterative process, whose convergence to best proximity points of maps which satisfy the condition (E) we shall study. The iterative process, for a mapping satisfying the condition (E), is as follows.

$$\left.\begin{array}{rl} x_1 & \in C \\ x_{n+1} & = Ty_n \\ y_n & = T\left((1-\alpha_n)\, x_n + \alpha_n z_n\right) \\ z_n & = (1-\beta_n)\, x_n + \beta_n T x_n \end{array}\right\} \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

The following lemma is the counterpart of Lemma 3.1 from [12], but for mappings satisfying the condition (E). We shall denote the set of fixed points of a mapping T by $F(T)$.

Lemma 2. *Let C be a nonempty closed convex subset of a Banach space $(E, \|\cdot\|)$, and let $T\colon C \to C$ be a mapping satisfying the condition* (E) *such that $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by the iterative process Equation (1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$. As the mapping T satisfies condition (E), we have

$$\|Tx - p\| = \|p - Tx\| \leq \mu\|p - Tp\| + \|x - p\| = \|x - p\|, \tag{2}$$

for any $x \in C$.

Applying Equation (2) and using the triangle axiom, one has

$$\begin{aligned} \|z_n - p\| &= \| (1-\beta_n) x_n + \beta_n T x_n - p\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n \|Tx_n - p\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3}$$

Similarly, using Equation (3), we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n) x_n + \alpha_n z_n) - p\| \\ &\leq \| (1-\alpha_n) x_n + \alpha_n z_n - p\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|z_n - p\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{4}$$

Now Equations (2) and (4) yield

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\|, \end{aligned} \tag{5}$$

which means that the sequence $\{\|x_n - p\|\}$ is bounded and nonincreasing for any $p \in F(T)$. Thus, the limit $\lim\limits_{n\to\infty} \|x_n - p\|$ exists. □

The following theorem is an extension of Theorem 3.2 from [12] to the class of mappings satisfying condition (E). It is worth to compare it with Theorems 2 and 3 from [5]. We shall need the following technical lemma.

Lemma 3 ([26])**.** *Suppose $(E, \|\cdot\|)$ is a uniformly convex Banach space and $\{t_n\}$ is a sequence bounded away from 0 and 1, i.e., $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $\limsup\limits_{n\to\infty} \|x_n\| \leq a$, $\limsup\limits_{n\to\infty} \|y_n\| \leq a$ and $\limsup\limits_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = a$ hold for some $a \geq 0$. Then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.*

Theorem 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T: C \to C$ be a mapping satisfying condition (E). Given a point $x_1 \in C$, let the sequence $\{x_n\}$, $n \geq 1$, be generated by the iterative process Equation (1) with $\{\alpha_n\}$ and $\{\beta_n\}$ bounded away from 0 and 1. Then $F(T) \neq \emptyset$ if and only if the sequence $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ (i.e., $\{x_n\}$ is an approximate fixed point sequence).*

Proof. Let $p \in F(T) \neq \emptyset$. According to Lemma 2 the limit

$$a := \lim_{n\to\infty} \|x_n - p\|$$

exists and $\{x_n\}$ is a bounded sequence. Using Equations (2) and (3) respectively, we have

$$\lim_{n\to\infty} \|z_n - p\| \leq \lim_{n\to\infty} \|x_n - p\| = a, \tag{6}$$

$$\lim_{n\to\infty} \|Tx_n - p\| \leq \lim_{n\to\infty} \|x_n - p\| = a. \tag{7}$$

On the other hand, using Equations (2) and (5), together with the properties of the norm, we get

$$\begin{aligned}\|x_{n+1}-p\| &\leq \|y_n-p\| \\ &= \|T\left(\left(1-\alpha_n\right)x_n+\alpha_n z_n\right)-p\| \\ &\leq \|\left(1-\alpha_n\right)x_n+\alpha_n z_n-p\| \\ &\leq \left(1-\alpha_n\right)\|x_n-p\|+\alpha_n\|z_n-p\| \\ &= \|x_n-p\|-\alpha_n\|x_n-p\|+\alpha_n\|z_n-p\|,\end{aligned}$$

or, equivalently,

$$\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\alpha_n} \leq \|z_n-p\|-\|x_n-p\|.$$

Thus,

$$\|x_{n+1}-p\|-\|x_n-p\| \leq \|z_n-p\|-\|x_n-p\|$$

implying

$$\|x_{n+1}-p\| \leq \|z_n-p\|.$$

Whereas from Equation (2) we have that $\|z_n-p\| \leq \|x_n-p\|$ and thus

$$a = \lim_{n\to\infty}\|z_n-p\|. \tag{8}$$

It follows

$$\lim_{n\to\infty}\|\beta_n\left(Tx_n-p\right)+\left(1-\beta_n\right)\left(x_n-p\right)\| = \lim_{n\to\infty}\|z_n-p\| = a. \tag{9}$$

Thus, the conditions of Lemma 3 are satisfied yielding $\lim\limits_{n\to\infty}\|Tx_n-x_n\| = 0$.

Conversely, assume that $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty}\|Tx_n-x_n\| = 0$. Take a point $p \in A\left(C,\{X_n\}\right)$. Using the fact that the mapping T satisfies the condition (E), we have

$$\begin{aligned}r\left(Tp,\{x_n\}\right) &= \limsup_{n\to\infty}\|x_n-Tp\| \\ &\leq \limsup_{n\to\infty}\left(\mu\|Tx_n-x_n\|+\|x_n-p\|\right) \\ &= \limsup_{n\to\infty}\|x_n-p\| \\ &= r\left(p,\{x_n\}\right),\end{aligned}$$

which means that Tp lies in $A\left(C,\{X_n\}\right)$. On the other hand, since E is uniformly convex, $A\left(C,\{X_n\}\right)$ is a singleton and hence $Tp = p$. □

Corollary 1. *Let C be a nonempty compact convex subset of a uniformly convex Banach space and let $\{x_n\}$ and T be as in Theorem 1. If $F\left(T\right) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. If $F\left(T\right) \neq \emptyset$, then, according to Theorem 1 $\lim\limits_{n\to\infty}\|x_n-Tx_n\| = 0$. As C is assumed to be compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ to some point $p \in C$. Since the mapping T satisfies the condition (E), for all $n \geq 1$ and some $\mu \geq 1$, we have

$$\|x_{n_k}-Tp\| \leq \mu\|x_{n_k}-Tx_{n_k}\|+\|x_{n_k}-p\|.$$

The uniqueness of the limit implies that $\{x_{n_k}\}$ converges strongly to Tp, meaning that $Tp \in F\left(T\right)$. On the other hand, according to Lemma 2, the limit $\lim\limits_{n\to\infty}\|x_n-p\|$ exists which completes the proof. □

3. Best Proximity Point Problem for (EP)-Mappings

Let X and Y be two convex subsets in a Banach space. A non-self mapping $T\colon X \to Y$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in X.$$

Gabeleh [19] introduced a condition on mappings which is weaker than nonexpansiveness and which resembles Suzuki's condition (C), but in the context of non-self mappings.

Definition 3 ([19])**.** *Let (X, Y) be a pair of of nonempty subsets of a Banach space. A mapping $T\colon X \to Y$ is said to be proximal generalized nonexpansive if and only if for all $x, y, u, v \in X$ such that $\|u - Tx\| = d(X, Y) = \|v - Ty\|$,*

$$\frac{1}{2}\left(\|x - Tx\| - d(X, Y)\right) \leq \|x - y\| \implies \|u - v\| \leq \|x - y\|.$$

The above definition can be widened by taking some $\lambda \in (0, 1)$ instead of $1/2$.

Next we introduce a new condition on non-self mappings which can be seen as the analogue of the condition (E) introduced by García-Falset et al. [5] and which involves the metric projection.

Definition 4. *Let (X, Y) be a pair of of nonempty subsets of a Banach space $(E, \|\cdot\|)$ such that $X_0 \neq \emptyset$ and denote by $P_{X_0}\colon E \to X_0$ the metric projection operator onto X_0. A mapping $T\colon X \to Y$ is said to satisfy the condition* (EP) *if and only if*

$$\|x - P_{X_0}Ty\| \leq \mu\|x - P_{X_0}Tx\| + \|x - y\|, \text{ for all } x, y \in X. \tag{10}$$

Proposition 1. *Any proximal generalized nonexpansive mapping satisfies the condition* (EP).

Proof. From Definition 3 it is clear that $u, v \in X_0$ (and hence $X_0 \neq \emptyset$) and that $Tx \in Y_0$. Also, from the definition of the metric projection we have $u = P_{X_0}Tx$ and $v = P_{X_0}Ty$. For any $\lambda \in (0, 1)$ we have

$$\begin{aligned} \lambda\left(\|x - Tx\| - d(X, Y)\right) &= \lambda\left(\|x - Tx\| - \|P_{X_0}Tx - Tx\|\right) \\ &\leq \lambda\|x - P_{X_0}Tx\| \\ &\leq \|x - P_{X_0}Tx\|. \end{aligned}$$

Since the mapping T is proximal generalized nonexpansive, it follows that

$$\|P_{X_0}Tx - P_{X_0}Ty\| \leq \|x - P_{X_0}Tx\|. \tag{11}$$

On the other hand, the triangle inequality together the inequality Equation (11), yield

$$\begin{aligned} \|x - P_{X_0}Ty\| &\leq \|x - P_{X_0}Tx\| + \|P_{X_0}Tx - P_{X_0}Ty\| \\ &\leq 2\|x - P_{X_0}Tx\| + \|x - y\|, \end{aligned}$$

which means that the condition (EP) is satisfied for $\mu = 2$. □

Next, we adapt the iterative process Equation (1) for the case of non-self mappings using the metric projection as follows.

$$\left.\begin{aligned} x_1 &\in X_0 \\ x_{n+1} &= P_{X_0}Ty_n \\ y_n &= P_{X_0}T\left((1 - \alpha_n)x_n + \alpha_n z_n\right) \\ z_n &= (1 - \beta_n)x_n + \beta_n P_{X_0}Tx_n \end{aligned}\right\} \tag{12}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences bounded away from 0 and 1.

It is clear from Lemma 1 that the set X_0 is convex. Also, since the iterative process Equation (12) involves the metric projection onto X_0 and convex combinations of elements from X_0, it is clear that $\{x_n\} \subset X_0$.

The notion of approximate fixed point sequence has a natural extension in the context of best proximity point problem.

Definition 5 ([19])**.** *Let (X, Y) be a pair of nonempty sets of a Banach space and $T : X \to Y$ be a non-self mapping. A sequence $\{x_n\} \subset X$ is said to be an approximate best proximity point sequence for T if and only if*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = d(X, Y).$$

Theorem 2. *Let (X, Y) be a pair of nonempty subsets of a Banach space E, where the pair has the P-property, X is convex, Y is closed and convex, and $X_0 \neq \emptyset$. Suppose the mapping $T\colon X \to Y$ satisfies the condition* (EP) *with $T(X_0) \subseteq Y_0$ and let $\{x_n\}$ be the sequence generated by the iterative process (12). Then, the mapping T has a best proximity point if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = d(X, Y)$.*

Proof. According to Lemma 1 the set X_0 is closed and convex. If p is a best proximity point for the mapping T, then p is a fixed point for the mapping $P_{X_0}T\colon X_0 \to X_0$, i.e., $F\left(P_{X_0}T\right) \neq \emptyset$. Thus, according to Theorem 1, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - P_{X_0}Tx_n\| = 0$. Also, since $T(X_0) \subseteq Y_0$, we have that $\|P_{X_0}Tx_n - Tx_n\| = d(X, Y)$. Taking $n \to \infty$ in the inequality

$$\|x_n - Tx_n\| \leq \|x_n - P_{X_0}Tx_n\| + \|P_{X_0}Tx_n - Tx_n\|, \;\; n \geq 1, \tag{13}$$

yields $\lim_{n\to\infty} \|x_n - Tx_n\| = d(X, Y)$.

Conversely, suppose that $\lim_{n\to\infty} \|x_n - Tx_n\| = d(X, Y)$. Using this fact while passing to the limit in Equation (13) gives $\lim_{n\to\infty} \|x_n - P_{X_0}Tx_n\| = 0$. Since by assumption the sequence $\{x_n\}$ is bounded, according to Theorem 1, there exists $p \in X_0$ such that $P_{X_0}Tp = p$, which means that $\|p - Tp\| = d(X, Y)$. □

Corollary 2. *Let (X, Y), T, and $\{x_n\}$ be as in Theorem 2 and suppose additionally that X is compact. If $F(P_{X_0}T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by the iterative process* (12) *converges strongly to a best proximity point of T.*

Proof. Since X is compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging strongly to some point $z \in X$. Also, since $F(P_{X_0}T) \neq \emptyset$, we have that $\lim_{n\to\infty} \|x_n - P_{X_0}Tx_n\| = 0$. Letting $k \to \infty$ in the relation

$$\|x_{n_k} - P_{X_0}Tz\| \leq \mu\|x_{n_k} - P_{X_0}Tx_{n_k}\| + \|x_{n_k} - z\|,$$

we obtain that x_{n_k} converges strongly to $P_{X_0}Tz$ and by the uniqueness of the limit we have $z = P_{X_0}Tz$, i.e., $z \in F(P_{X_0}T)$. Applying now Lemma 2 yields the conclusion. □

4. Strong Convergence via a CQ-Type Algorithm

In this section we introduce an algorithm which is a hybrid between the iterative process (12) and the CQ algorithm introduced by Nakajo and Takahashi [20]. The main outcome is the strong convergence of the resulting sequence. Before dealing with the main result, let us establish the following preliminaries.

Let H be a real Hilbert and denote the inner product by $\langle \cdot, \cdot \rangle$ and, respectively, the norm by $\| \cdot \|$. Let X and Y be nonempty closed and convex subsets of H. Given a mapping $T\colon X \to Y$, we denote the set of its best proximity points by X_T, i.e.,

$$X_T = \{x \in X\colon d(x, Tx) = d(X, Y)\}.$$

Clearly $X_T \subseteq X_0$ (for details, one can see [27]).

For a sequence $\{x_n\} \subset X$ let

$$w_\omega(\{x_n\}) = \{x \colon \exists \{x_{n_k}\} \subset \{x_n\}, x_{n_k} \rightharpoonup x\},$$

where $\rightharpoonup$ denotes the weak convergence, be the weak ω-limit set.

Lemma 4 ([28])**.** *Let K be a closed and convex subset of a real Hilbert space H and let P_K be the metric projection from H onto K. Then, given $x \in H$ and $z \in K$,*

$$z = P_K x \ \text{ if and only if } \ \langle x - z, y - z \rangle \leq 0,$$

for all $y \in K$.

Lemma 5 ([28])**.** *Let K be a closed and convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and let $x \in H$. Let $q = P_K x$. If $\{x_n\}$ is such that $w_\omega(\{x_n\}) \subset K$ and satisfies the condition*

$$\|x_n - x\| \leq \|x - q\|, \ \text{ for all } n \in \mathbb{N},$$

then $x_n \to q$.

A Banach space $(E, \|\cdot\|)$ is said to have the Opial property if, for every sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup z$, the inequality

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds whenever $y \neq z$. It is worth mentioning that any Hilbert space has the Opial property (for a proof, please see [29]).

Lemma 6 (Theorem 1, [5])**.** *Let C be a nonempty subset of a Banach space E and let $T \colon C \to E$ be a given mapping. If*

a) *there exists a sequence $\{x_n\} \subset C$ such that $\|x_n - Tx_n\| \to 0$ and $z_n \rightharpoonup z$,*

b) *T satisfies the condition* (E) *on C,*

c) *$(E, \|\cdot\|)$ has the Opial property,*

then $Tz = z$.

Consider now the following algorithm:

$$\begin{aligned}
&x_0 \in X_0 \ \text{ arbitrary}, \\
&z_n = (1 - \beta_n) x_n + \beta_n P_{X_0} T x_n, \\
&y_n = P_{X_0} T ((1 - \alpha_n) x_n + \alpha_n z_n), \\
&w_n = P_{X_0} T y_n, \\
&Q_n = \{u \in X_0 \colon \langle x_n - u, x_n - x_0 \rangle \leq 0\}; \\
&C_n = \{u \in X_0 \colon \max\{\|w_n - u\|, \|y_n - u\|, \|z_n - u\|\} \leq \|x_n - u\|\}, \\
&x_{n+1} = P_{(C_n \cap Q_n)} x_0,
\end{aligned} \tag{14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences bounded away from 0 and 1.

Clearly the projection is well defined since the set X_0 is closed and convex, according to Lemma 1.

Theorem 3. *Let (X, Y) be a pair of nonempty closed and convex subsets of a real Hilbert space, and suppose the pair has the P-property. Let $T \colon X \to Y$ be a mapping which satisfies the condition* (EP) *such that X_T is a nonempty convex subset of X_0. Then, the sequence $\{x_n\}$, generated by the algorithm (12), converges to a best* *proximity point. In particular, it converges to p, where $p = P_{X_T}(x_0)$. Moreover, the same holds true for the sequences $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$.*

Proof. Let $x_0 \in X_0$. Clearly the sets Q_n and C_n respectively, are closed and convex subsets of X. Let us prove that $X_T \subset C_n \cap Q_n$.

Let $z \in X_T$. Clearly, $d(z, Tz) = d(X, Y)$, i.e., $d\left(z, P_{X_0}Tz\right) = 0$. Keeping in mind that the mapping T satisfies the condition (EP), we have

$$\begin{aligned} \|z_n - z\| &= \|(1-\beta_n)x_n + \beta_n P_{X_0}Tx_n - z\| \\ &\le (1-\beta_n)\|x_n - z\| + \beta_n\|P_{X_0}Tx_n - z\| \\ &\le (1-\beta_n)\|x_n - z\| + \beta_n\left(\mu\|P_{X_0}Tz - z\| + \|x_n - z\|\right) \\ &\le \|x_n - z\|. \end{aligned} \tag{15}$$

Similarly, we get the inequality

$$\begin{aligned} \|y_n - z\| &= \|P_{X_0}T\left((1-\alpha_n)x_n + \alpha_n z_n\right) - z\| \\ &\le \mu\|P_{X_0}Tz - z\| + \|(1-\alpha_n)x_n + \alpha_n z_n - z\| \\ &\le (1-\alpha_n)\|z_n - z\| + \alpha_n\|z_n - z\| \\ &= \|z_n - z\|, \end{aligned} \tag{16}$$

and, respectively,

$$\begin{aligned} \|w_n - z\| &= \|P_{X_0}T\left(P_{X_0}T\left((1-\alpha_n)x_n + \alpha_n z_n\right)\right) - z\| \\ &\le \mu\|P_{X_0}Tz - z\| + \|P_{X_0}T\left((1-\alpha_n)x_n + \alpha_n z_n\right) - z\| \\ &\le \mu\|P_{X_0}Tz - z\| + \|(1-\alpha_n)x_n + \alpha_n z_n - z\| \\ &\le (1-\alpha_n)\|z_n - z\| + \alpha_n\|z_n - z\| \\ &= \|z_n - z\|. \end{aligned} \tag{17}$$

Hence, $z \in C_n$, i.e., $X_T \subset C_n$.

The inclusion $X_T \subset Q_n$ follows by induction. Indeed, it is clear from the definition that $Q_0 = X_0$ and that $X_T \subset X_0$, respectively. Assume $X_T \subset Q_n$. As C_n and Q_n are closed and convex sets, for $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, according to Lemma 4, one has $\langle x_{n+1} - z, x_{n+1} - x_0\rangle \le 0$ for all $z \in C_n \cap Q_n$. Using again the definition of the set Q_n and noticing that $X_T \subset C_n \cap Q_n$ yields $X_T \subset Q_{n+1}$, which completes the induction.

Let $p = P_{X_T}(x_0)$. Since $X_T \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \le \|p - x_0\|, \tag{18}$$

which also means that the sequence $\{x_n\}$ is bounded.

Since $x_{n+1} \in Q_n$, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \langle x_{n+1} - x_n, x_{n+1} - x_n\rangle \\ &= \langle x_{n+1} - x_0 + x_0 - x_n, x_{n+1} - x_0 + x_0 - x_n\rangle \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \end{aligned}$$

implying $\|x_{n+1} - x_n\| \to 0$ for $n \to \infty$.

On the other hand, the triangle axiom and the definition of C_n yield

$$\begin{aligned} \|z_n - x_n\| &\le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \end{aligned}$$

and thus $\|z_n - x_n\| \to 0$ for $n \to \infty$.

Noticing that $\|z_n - x_n\| = \beta_n \|P_{X_0} T x_n - x_n\|$, it follows that $\|P_{X_0} T x_n - x_n\| \to 0$ as $n \to \infty$, since the sequence $\{\beta_n\}$ is bounded away from 0 and 1.

Consider now the mapping $P_{X_0} T \colon X_0 \to X_0$, which clearly satisfies the condition (E). The set of its fixed points is the set X_T. Recalling that any Hilbert space has the Opial property, while Applying Lemma 6, yields the inclusion $w_\omega(\{x_n\}) \subset X_T$. This fact, together with inequality Equation (18), according to Lemma 5, provides the strong convergence of the sequence $\{x_n\}$ to the point $p = P_{X_T}(x_0)$.

Turning now to the strong convergence of the other sequences, we have

$$\begin{aligned} \|w_n - x_n\| &\leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \end{aligned}$$

and thus $\|w_n - x_n\| \to 0$. Similarly, one obtains $\|y_n - x_n\| \to 0$.

Lastly, the strong convergence of the sequences $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ follow by taking $n \to \infty$ in the inequalities

$$\begin{aligned} \|w_n - p\| &\leq \|w_n - x_n\| + \|x_n - p\|, \\ \|y_n - p\| &\leq \|y_n - x_n\| + \|x_n - p\|, \\ \|z_n - p\| &\leq \|z_n - x_n\| + \|x_n - p\|. \end{aligned}$$

□

5. Conclusions

The starting point of our study in this paper has two main ingredients. One of them is the iterative process introduced by Thakur et al. [12], for Suzuki generalized nonexpansive mappings. The other is a class of mappings satisfying the condition (E), introduced by García-Falset et al. and which is even larger. We firstly extended the main results from [12] to the case of mappings satisfying condition (E). Afterwards, we have progressed to the setting of best proximity point problem, which is a generalization of the fixed point problem, by introducing a new class of non-self mappings. These generalize the class of proximal generalized nonexpansive mappings introduced by Gabeleh [19]. We have also adapted the iterative process from [12] to the setting of non-self mappings, using the metric projection, and have studied the convergence of the resulting iterative sequence. In the last part, we have constructed a CQ-type algorithm [20] for the iterative process under consideration and have proved the strong convergence of the resulting sequence to a best proximity point for mappings satisfying the condition (EP).

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