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Viscosity Approximation Methods for a General Variational Inequality System and Fixed Point Problems in Banach Spaces

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Abstract: In Banach spaces, we study the problem of solving a more general variational inequality system for an asymptotically non-expansive mapping. We give a new viscosity approximation scheme to find a common element. Some strong convergence theorems of the proposed iterative method are obtained. A numerical experiment is given to show the implementation and efficiency of our main theorem. Our results presented in this paper generalize and complement many recent ones.

Keywords: strong convergence; fixed point; general variational inequality system; asymptotically non-expansive mapping; Banach space

MSC: 47H10; 47H09; 47J25

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1. Introduction

Throughout this paper, let X be a real Banach space and $E \subset X$ be a nonempty subset. Let $T : E \to E$ be a mapping, the set of fixed points of T is denoted by $F(T)$. If there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n\xi - T^n\eta\| \leq k_n\|\xi - \eta\|, \forall \xi, \eta \in E,$$

then T is said to be asymptotically nonexpansive. T is uniformly asymptotically regular, if $\lim_{n\to\infty} \|T^{n+1}\xi - T^n\xi\| = 0, \forall \xi \in E$. If $k_n \equiv 1$, then T is said to be nonexpansive. Recall that T is known as a contractive mapping on E if there exists a constant $\rho \in (0, 1)$ such that $\|T\xi - T\eta\| \leq \rho\|\xi - \eta\|, \forall \xi, \eta \in E$.

An operator $A : E \to X$ is called accretive if there exists $j(\xi - \eta) \in J(\xi - \eta)$ such that

$$\langle A\xi - A\eta, j(\xi - \eta)\rangle \geq 0, \forall \xi, \eta \in E,$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping on X. An operator $A : E \to X$ is called α-inverse strongly accretive if for $\alpha > 0$ and $j(\xi - \eta) \in J(\xi - \eta)$, we have

$$\langle A\xi - A\eta, j(\xi - \eta)\rangle \geq \alpha\|A\xi - A\eta\|^2, \forall \xi, \eta \in E.$$

For any $\epsilon \in (0, 2]$, we denote the the modulus of convexity $\delta_X(\epsilon) > 0$ of X as follows:

$$\delta_X(\epsilon) = inf\{1 - \frac{1}{2}\|\xi + \eta\| : \|\xi\|, \|\eta\| \leq 1, \|\xi - \eta\| \geq \epsilon\}.$$

X is said to be uniformly convex if $\delta_X(0) = 0$. Let ρ_X be the modulus of smoothness of X defined by:

$$\rho_X(s) = \sup\{\frac{\|\xi + s\eta\| + \|\xi - s\eta\|}{2} - 1 : \|\xi\| = 1, \|\eta\| = 1\}.$$

A Banach space X is said to be uniformly smooth if $\lim_{n\to\infty} \rho_X(s)/s = 0$. Moreover, X is uniformly smooth if and only if the norm of X is uniformly Fréchet differentiable.

A mapping $Q : X \to E$ is called sunny if Q has the following property:

$$Q(s\xi + (1-s)Q\xi) = Q\xi, \xi \in X, s \geq 0,$$

whenever $s\xi + (1-s)Q\xi \in X$. A mapping $Q : X \to E$ is said to be a retraction if $Q\xi = \xi$ for all $\xi \in E$. It is well known that a sunny nonexpansive retraction is also sunny and nonexpansive.

Variational inequality theory has played a significant role in nonlinear analysis and the optimization problem. Many iterative methods have been used to solve variational inequality problems due to the applications in some branches of applied science, convex optimization, mathematical physics, and operator studies, see [1–9] and the references therein. In fact, the classical variational inequality problem in Banach spaces is to find $q \in E$ such that

$$\langle Aq, j(x-q)\rangle \geq 0, \forall x \in E.$$

In 2010, Yao et al. [10] proposed a system to find $(u, v) \in E \times E$ such that:

$$\begin{cases} \langle Av + u - v, j(x-u)\rangle \geq 0, \forall x \in E, \\ \langle Bu + v - u, j(x-v)\rangle \geq 0, \forall x \in E, \end{cases}$$

which is called the general variational inequality system in Banach spaces. They proved a strong convergence result of the following sequence to a solution of the variational inequality system:

$$\begin{cases} y_n = Q_E(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_E(y_n - Ay_n), \end{cases}$$

where Q_E is the sunny nonexpansive retract from X onto E.

Recently, many authors have focused their efforts on studying generalized variational inequality systems with variational inequality constraints, see [11–17] and the references therein. Especially, in 2019, Ceng et al. [11] studied a general system of variational inequalities in Banach spaces:

$$\begin{cases} \langle \lambda Av + u - v, j(x-u)\rangle \geq 0, \forall x \in E, \\ \langle \mu Bu + v - u, j(x-v)\rangle \geq 0, \forall x \in E, \end{cases} \tag{1}$$

and they considered an implicit composite extra-gradient-like iterative algorithm for countable family Lipschitzian pseudo-contractive mappings and proved strong convergence results in Banach spaces. Cai et al. [14] showed the following viscosity iteration method for the strict pseudo-contraction and non-expansive mapping:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ u_n = Q_E(y_n - \mu By_n), \\ z_n = Q_E(u_n - \lambda Au_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T_\alpha z_n, \end{cases}$$

where $T_\alpha x = \alpha x + (1-\alpha)Tx$. They proved that x_n converges strongly to a common element of the fixed point set and the set of solutions of the problem (1).

Inspired and motivated by the work of researchers, we consider the following problem about the general variational inequality system in Banach spaces:

$$\begin{cases} \langle (I-\lambda A)(tx^{\dagger}+(1-t)y^{\dagger})-x^{\dagger}, j(x-x^{\dagger})\rangle \leq 0, \forall x \in E, \\ \langle (I-\mu B)x^{\dagger}-y^{\dagger}, j(x-y^{\dagger})\rangle \leq 0, \forall x \in E. \end{cases} \tag{2}$$

When $t=0$, this is the general system of variational inequalities (1). We present a viscosity iterative algorithm for the general variational inequality system (2) and an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ and:

$$\begin{cases} w_n = Q_E(I-\mu B)x_n, \\ z_n = Q_E(I-\lambda A)(tx_n+(1-t)w_n), \\ u_n = \delta_n x_n + (1-\delta_n) z_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n u_n. \end{cases} \tag{3}$$

Then, the strong convergence theorem of this iterative scheme in Banach spaces is proven. Finally, we give the numerical experiments to show the implementation and efficiency of our main theorem. We study this viscosity approximation method to find a common element of the fixed point set of an asymptotically nonexpansive mapping and the set of solutions of the general variational inequality system in Banach spaces. Our results presented in this paper generalize and complement many recent ones [3,5,6,9,10,12–14,17].

2. Preliminaries

In this section, we recall some lemmas which are needed in the proof of our main results.

Lemma 1 ([18]). *Let X be a smooth Banach space. Assume that $Q: X \to E$ is a retract and J is the normalized duality mapping on X. Then the following statements are equivalent:*
(a) Q is sunny and nonexpansive;
(b) $\|Q\xi - Q\eta\|^2 \leq \langle \xi-\eta, J(Q\xi-Q\eta)\rangle, \forall \xi,\eta \in X$;
(c) $\langle \xi - Q\xi, J(\eta - Q\xi)\rangle \leq 0, \forall \xi \in X, \eta \in E$.

Lemma 2 ([19]). *Suppose that $\{v_n\}$ is a sequence of nonnegative real numbers satisfying:*

$$v_{n+1} \leq (1-b_n)v_n + b_n\sigma_n, \forall n \geq 0,$$

where $\{b_n\} \subset (0,1)$ and $\sigma_n \subset R$ satisfying the following conditions:
(i) $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} b_n = +\infty$;
(ii) either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |b_n\sigma_n| < +\infty$.
Then $\lim_{n\to\infty} v_n = 0$.

Lemma 3 ([14]). *Let X be a real Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset. If the operator $A: E \to X$ is α-inverse strongly accretive, then we have:*

$$\|(I-\lambda A)\xi - (I-\lambda A)\eta\|^2 \leq \|\xi-\eta\|^2 - \lambda(2\alpha - c\lambda)\|A\xi - A\eta\|^2,$$

where $\lambda > 0$. If $0 < \lambda < \frac{2\alpha}{c}$, then $I-\lambda A$ is nonexpansive.

Lemma 4 ([20]). *Let X be a real Banach space and $\{x_n\}$, $\{y_n\}$ be two bounded sequences of X. Let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. If $x_{n+1} = (1-\beta_n)x_n + \beta_n p_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|p_{n+1}-p_n\| - \|x_{n+1}-x_n\|) \leq 0$, then $\lim_{n\to\infty} \|p_n - x_n\| = 0$.*

Lemma 5 ([21]). *Let X be a real Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset. And let $T : E \to E$ be an asymptotically nonexpansive mapping with a fixed point. Suppose that X admits a weakly sequentially continuous duality mapping. Then the mapping $I - T$ is demiclosed at zero, i.e., where I is the identity mapping, i.e., if $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, then $x = Tx$.*

Lemma 6 ([22]). *Let X be a real Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset and $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : E \to E$ be a contractive mapping. Then the sequence x_s defined by $x_s = sf(x_s) + (1-s)Tx_s, s \in (0,1)$ converges strongly to a point in $F(T)$. Suppose $Q : \Pi_c \to F(T)$ by $Q(f) = \lim_{s\to 0} x_s, f \in \Pi_c$, then $Q(f)$ solves the variational inequality*

$$\langle (I-f)Q(f), j(Q(f)-p)\rangle \leq 0, \forall p \in F(T).$$

Lemma 7 ([23]). *Let $r > 0$. If X is a real smooth and uniformly convex Banach space, then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to R, g(0) = 0$ such that $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ for all $x, y \in B_r$.*

Lemma 8. *Let X be a real Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset and $A, B : E \to X$ be two nonlinear mappings. Suppose that Q_E is a sunny nonexpansive retraction. For $\forall \lambda, \mu > 0$ and $t \in [0,1]$, then the following assertions are equivalent:*
(a) $(x^\dagger, y^\dagger) \in E \times E$ is a solution of problem (2);
(b) Let $\Psi : E \to E$ be a mapping defined by

$$\Psi(x) = Q_E(I - \lambda A)[tx + (1-t)Q_E(I - \mu B)x],$$

then let $x^\dagger$ be the fixed point of Ψ, that is $x^\dagger = \Psi x^\dagger$.
where $x^\dagger = Q_E(I - \lambda A)[tx^\dagger + (1-t)y^\dagger]$, $y^\dagger = Q_E(I - \mu B)x^\dagger$. Assume that $A, B : E \to X$ are α-inverse strongly accretive operator and β-inverse strongly operator, respectively. If $0 < \lambda < \frac{2\alpha}{c}$, $0 < \mu < \frac{2\beta}{c}$, then Ψ is nonexpansive.

Proof. From Lemma 1 and the definition of the sunny nonexpansive retraction, we have that (2) is equivalent to

$$\begin{cases} x^\dagger = Q_E(I - \lambda A)[tx^\dagger + (1-t)y^\dagger]; \\ y^\dagger = Q_E(I - \mu B)x^\dagger, \end{cases}$$

which is a solution of problem (2). Hence $x^\dagger = Q_E(I - \lambda A)[tx^\dagger + (1-t)Q_E(I - \mu B)x^\dagger] = \Psi(x^\dagger)$.
From Lemma 3, for any $x, y \in E$, we find

$$\begin{aligned} \|\Psi(x) - \Psi(y)\| &= \|Q_E(I - \lambda A)[tx + (1-t)Q_E(I - \mu B)x] - Q_E(I - \lambda A)[ty + (1-t)Q_E(I - \mu B)y]\| \\ &\leq \|(I - \lambda A)[tx + (1-t)Q_E(I - \mu B)x] - (I - \lambda A)[ty + (1-t)Q_E(I - \mu B)y]\| \\ &\leq \|tx + (1-t)Q_E(I - \mu B)x - ty - (1-t)Q_E(I - \mu B)y\| \\ &\leq t\|x - y\| + (1-t)\|x - y\| \\ &= \|x - y\|. \end{aligned}$$

Thus, Ψ is nonexpansive. □

3. Main Results

Theorem 1. *Let X be a uniformly convex and uniformly smooth Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset. Suppose that $Q_E : X \to E$ is a sunny nonexpansive retraction and $T : E \to E$ is an asymptotically nonexpansive mapping satisfying the uniformly asymptotically regular condition. And $A, B : E \to X$ are an α-inverse strongly accretive operator and β-inverse strongly accretive operator, respectively. Let $f : E \to E$ be a contraction with coefficient $\rho \in (0,1)$ and Ψ be defined by Lemma 8. Assume that*

$\Omega = F(T) \bigcap F(\Psi) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$, the sequence $\{x_n\}$ defined by (3) satisfies the following conditions:

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1, \lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=0}^{\infty}\alpha_n = \infty, k_n - 1 = \epsilon\alpha_n, 0 < \epsilon < 1 - \rho;$$
$$(ii)\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0;$$
$$(iii)\ 0 \leq t < 1, 0 < \lambda < \frac{2\alpha}{c}, 0 < \mu < \frac{2\beta}{c};$$
$$(iv)\ \beta_n + \gamma_n k_n^2 < 1.$$

Then $\{x_n\}$ converges strongly to an element $x^\dagger \in \Omega$ which solves the variational inequality:

$$\langle (I - f)x^\dagger, j(x^\dagger - p)\rangle \leq 0, \forall p \in \Omega.$$

Proof. Let $x^\dagger \in \Omega$, from Lemma 8, we have $x^\dagger = Q_E(I - \lambda A)[tx^\dagger + (1-t)y^\dagger]$, $y^\dagger = Q_E(I - \mu B)x^\dagger$. It follows from (3) that

$$\begin{aligned}
\|u_n - x^\dagger\| &= \|\delta_n x_n + (1-\delta_n)z_n - x^\dagger\| \\
&\leq \delta_n \|x_n - x^\dagger\| + (1-\delta_n)\|z_n - x^\dagger\| \\
&= \delta_n \|x_n - x^\dagger\| + (1-\delta_n)\|\Psi(x_n) - x^\dagger\| \\
&\leq \delta_n \|x_n - x^\dagger\| + (1-\delta_n)\|x_n - x^\dagger\| \\
&= \|x_n - x^\dagger\|.
\end{aligned}$$

Then we compute:

$$\begin{aligned}
&\|x_{n+1} - x^\dagger\| \\
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n u_n - x^\dagger\| \\
&= \|\alpha_n (f(x_n) - f(x^\dagger)) + \alpha_n (f(x^\dagger) - x^\dagger) + \beta_n (x_n - x^\dagger) + \gamma_n (T^n u_n - x^\dagger)\| \\
&\leq \alpha_n \|f(x_n) - f(x^\dagger)\| + \alpha_n \|f(x^\dagger) - x^\dagger\| + \beta_n \|x_n - x^\dagger\| + \gamma_n \|T^n u_n - x^\dagger\| \\
&\leq \alpha_n \rho \|x_n - x^\dagger\| + \alpha_n \|f(x^\dagger) - x^\dagger\| + \beta_n \|x_n - x^\dagger\| + \gamma_n k_n \|u_n - x^\dagger\| \\
&= (\alpha_n \rho + \beta_n + \gamma_n k_n)\|x_n - x^\dagger\| + \alpha_n \|f(x^\dagger) - x^\dagger\| \\
&\leq [1 - (1 - \rho - \epsilon)\alpha_n]\|x_n - x^\dagger\| + \alpha_n \|f(x^\dagger) - x^\dagger\| \\
&\leq \max\{\|x_n - x^\dagger\|, \frac{1}{1 - \rho - \epsilon}\|f(x^\dagger) - x^\dagger\|\},
\end{aligned}$$

which implies that x_n is bounded, and so are z_n,u_n,$f(x_n)$, $T^n u_n$.

From (3) and Lemma 8, we observe that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|Q_E(I - \lambda A)(tx_{n+1} + (1-t)w_{n+1}) - Q_E(I - \lambda A)(tx_n + (1-t)w_n)\| \\
&= \|Q_E(I - \lambda A)(tx_{n+1} + (1-t)Q_E(I - \mu B)x_{n+1}) - Q_E(I - \lambda A)(tx_n + (1-t)Q_E(I - \mu B)x_n)\| \\
&= \|\Psi(x_{n+1}) - \Psi(x_n)\| \leq \|x_{n+1} - x_n\|,
\end{aligned}$$

then

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|\delta_{n+1} x_{n+1} + (1 - \delta_{n+1})z_{n+1} - \delta_n x_n - (1 - \delta_n)z_n\| \\
&= \|\delta_{n+1}(x_{n+1} - x_n) + (\delta_{n+1} - \delta_n)x_n + (1 - \delta_{n+1})(z_{n+1} - z_n) - (\delta_{n+1} - \delta_n)z_n\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|x_n - z_n\| + (1 - \delta_{n+1})\|z_{n+1} - z_n\| \\
&\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|x_n - z_n\|.
\end{aligned}$$

Set $p_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}, for\ all\ n \geq 0$, we obtain

$$\begin{aligned}
p_{n+1} - p_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T^{n+1} u_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n T^n u_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + (1-\beta_{n+1}-\alpha_{n+1}) T^{n+1} u_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n) T^n u_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[f(x_{n+1}) - f(x_n)] + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) f(x_n) \\
&- (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) T^n u_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}} (T^{n+1} u_{n+1} - T^n u_n) + T^{n+1} u_{n+1} - T^n u_n \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[f(x_{n+1}) - f(x_n)] + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})[f(x_n) - T^n u_n] \\
&+ (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})[T^{n+1} u_{n+1} - T^{n+1} u_n] + (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}})[T^{n+1} u_n - T^n u_n].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|p_{n+1} - p_n\| &\leq \frac{\rho\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|M \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{x\in E} \|T^{n+1} u_n - T^n u_n\| + (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}) k_{n+1} \|u_{n+1} - u_n\| \\
&\leq [\frac{\rho\alpha_{n+1}}{1-\beta_{n+1}} + (1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}) k_{n+1}] \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|M \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{x\in E} \|T^{n+1} u_n - T^n u_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} k_{n+1} |\delta_{n+1} - \delta_n| M \\
&\leq [1 - \frac{\alpha_{n+1}(1-\rho-\epsilon)}{1-\beta_{n+1}}] \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|M \\
&+ \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{x\in E} \|T^{n+1} u_n - T^n u_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} k_{n+1} |\delta_{n+1} - \delta_n| M,
\end{aligned}$$

where $M > 0$ is a constant satisfies:

$$M \geq \{sup_{n\geq 0}\|x_n - z_n\|, sup_{n\geq 0}\|f(x_n) - T^n u_n\|\}.$$

By (i), (ii) , we can find,

$$\limsup_{n\to\infty}(\|p_{n+1} - p_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Applying Lemma 4, we have

$$\lim_{n\to\infty} \|p_n - x_n\| = 0.$$

We know that

$$p_n - x_n = \frac{x_{n+1} - x_n}{1-\beta_n},$$

and we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4}$$

Next, we show that $\lim_{n\to\infty} \|x_n - \Psi x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

From (3), Lemma 3, and the non-expansiveness of Q_E, we have

$$\begin{aligned}\|w_n - y^\dagger\|^2 &= \|Q_E(I-\mu B)x_n - Q_E(I-\mu B)x^\dagger\|^2 \\ &\le \|(I-\mu B)x_n - (I-\mu B)x^\dagger\|^2 \\ &\le \|x_n - x^\dagger\|^2 - \mu(2\beta - c\mu)\|Bx_n - Bx^\dagger\|^2.\end{aligned}$$

Then

$$\begin{aligned}&\|z_n - x^\dagger\|^2 \\ &= \|Q_E(I-\lambda A)[tx_n + (1-t)w_n] - Q_E(I-\lambda A)[tx^\dagger + (1-t)y^\dagger]\|^2 \\ &\le \|(I-\lambda A)[tx_n + (1-t)w_n] - (I-\lambda A)[tx^\dagger + (1-t)y^\dagger]\|^2 \\ &\le \|tx_n + (1-t)w_n - (tx^\dagger + (1-t)y^\dagger)\|^2 - \lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2 \\ &\le t\|x_n - x^\dagger\|^2 + (1-t)\|w_n - y^\dagger\|^2 - \lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2 \\ &\le \|x_n - x^\dagger\|^2 - (1-t)\mu(2\beta - c\mu)\|Bx_n - Bx^\dagger\|^2 \\ &- \lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2,\end{aligned}$$

and

$$\begin{aligned}&\|u_n - x^\dagger\|^2 \\ &= \|\delta_n x_n + (1-\delta_n)z_n - x^\dagger\|^2 \\ &\le \delta_n\|x_n - x^\dagger\|^2 + (1-\delta_n)\|z_n - x^\dagger\|^2 \\ &\le \|x_n - x^\dagger\|^2 - (1-\delta_n)(1-t)\mu(2\beta - c\mu)\|Bx_n - Bx^\dagger\|^2 \\ &- (1-\delta_n)\lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2.\end{aligned}$$

Moreover, we know that

$$\begin{aligned}&\|x_{n+1} - x^\dagger\|^2 \\ &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n u_n - x^\dagger\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + \beta_n\|x_n - x^\dagger\|^2 + \gamma_n\|T^n u_n - x^\dagger\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + \beta_n\|x_n - x^\dagger\|^2 + \gamma_n k_n^2\|u_n - x^\dagger\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + \beta_n\|x_n - x^\dagger\|^2 + \gamma_n k_n^2\|x_n - x^\dagger\|^2 \\ &- \gamma_n k_n^2(1-\delta_n)(1-t)\mu(2\beta - c\mu)\|Bx_n - Bx^\dagger\|^2 \\ &- \gamma_n k_n^2(1-\delta_n)\lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\gamma_n k_n^2(1-\delta_n)(1-t)\mu(2\beta - c\mu)\|Bx_n - Bx^\dagger\|^2 \\ &+ \gamma_n k_n^2(1-\delta_n)\lambda(2\alpha - c\lambda)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + (\beta_n + \gamma_n k_n^2)\|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 \\ &\le \alpha_n\|f(x_n) - x^\dagger\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^\dagger\| + \|x_{n+1} - x^\dagger\|).\end{aligned}$$

By the conditions (i), (ii), and (4), we deduce

$$\begin{aligned}&\lim_{n\to\infty}\|Bx_n - Bx^\dagger\| = 0 \\ &\lim_{n\to\infty}\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\| = 0.\end{aligned} \tag{5}$$

Applying Lemmas 1 and 7 to find

$$\begin{aligned}
\|w_n - y^\dagger\|^2 &= \|Q_E(I-\mu B)x_n - Q_E(I-\mu B)x^\dagger\|^2 \\
&\leq \langle (I-\mu B)x_n - (I-\mu B)x^\dagger, j(w_n - y^\dagger)\rangle \\
&\leq \langle x_n - x^\dagger, j(w_n - y^\dagger)\rangle - \mu\langle Bx_n - Bx^\dagger, j(w_n - y^\dagger)\rangle \\
&\leq \frac{1}{2}[\|x_n - x^\dagger\|^2 + \|w_n - y^\dagger\|^2 - g(\|x_n - x^\dagger - (w_n - y^\dagger)\|)] - \mu\langle Bx_n - Bx^\dagger, j(w_n - y^\dagger)\rangle.
\end{aligned}$$

Hence, we have

$$\|w_n - y^\dagger\|^2 \leq \|x_n - x^\dagger\|^2 - g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) - 2\mu\langle Bx_n - Bx^\dagger, j(w_n - y^\dagger)\rangle.$$

Further, we estimate

$$\begin{aligned}
\|z_n - x^\dagger\|^2 &= \|Q_E(I-\lambda A)[tx_n + (1-t)w_n] - Q_E(I-\lambda A)[tx^\dagger + (1-t)y^\dagger]\|^2 \\
&\leq \langle (I-\lambda A)[tx_n + (1-t)w_n] - (I-\lambda A)[tx^\dagger + (1-t)y^\dagger], j(z_n - x^\dagger)\rangle \\
&\leq \langle tx_n + (1-t)w_n - (tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle \\
&\quad - \lambda\langle A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle \\
&\leq t\langle x_n - x^\dagger, j(z_n - x^\dagger)\rangle + (1-t)\langle w_n - y^\dagger, j(z_n - x^\dagger)\rangle \\
&\quad - \lambda\langle A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle \\
&\leq \frac{t}{2}[\|x_n - x^\dagger\|^2 + \|z_n - x^\dagger\|^2 - g(\|x_n - z_n\|)] \\
&\quad + \frac{1-t}{2}[\|w_n - y^\dagger\|^2 + \|z_n - x^\dagger\|^2 - g(\|w_n - y^\dagger - (z_n - x^\dagger)\|)] \\
&\quad - \lambda\langle A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle,
\end{aligned}$$

noting that $0 \leq t < 1$, so

$$\begin{aligned}
\|z_n - x^\dagger\|^2 &\leq t\|x_n - x^\dagger\|^2 + (1-t)\|w_n - y^\dagger\|^2 - tg(\|x_n - z_n\|) \\
&\quad - (1-t)g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&\quad - 2\lambda\langle A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle \\
&\leq \|x_n - x^\dagger\|^2 - (1-t)g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&\quad - 2\mu(1-t)\langle Bx_n - Bx^\dagger, j(w_n - y^\dagger)\rangle \\
&\quad - (1-t)g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&\quad - 2\lambda\langle A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger), j(z_n - x^\dagger)\rangle \\
&\leq \|x_n - x^\dagger\|^2 - (1-t)g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&\quad - 2\mu(1-t)\|Bx_n - Bx^\dagger\|\|w_n - y^\dagger\| \\
&\quad - (1-t)g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&\quad - 2\lambda\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|\|z_n - x^\dagger\|,
\end{aligned}$$

then

$$\begin{aligned}
&\|u_n - x^\dagger\|^2 \\
&\leq \delta_n\|x_n - x^\dagger\|^2 + (1-\delta_n)\|z_n - x^\dagger\|^2 \\
&\leq \|x_n - x^\dagger\|^2 - (1-\delta_n)(1-t)g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&- 2\mu(1-\delta_n)(1-t)\|Bx_n - Bx^\dagger\|\|w_n - y^\dagger\| \\
&- (1-t)(1-\delta_n)g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&- 2\lambda(1-\delta_n)\|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\|\|z_n - x^\dagger\|.
\end{aligned}$$

We know that

$$\begin{aligned}
&\|x_{n+1} - x^\dagger\|^2 \\
&\leq \alpha_n \|f(x_n) - x^\dagger\|^2 + \beta_n \|x_n - x^\dagger\|^2 + \gamma_n k_n^2 \|u_n - x^\dagger\|^2 \\
&\leq \alpha_n \|f(x_n) - x^\dagger\|^2 + \beta_n \|x_n - x^\dagger\|^2 + \gamma_n k_n^2 \|x_n - x^\dagger\|^2 \\
&- (1-\delta_n)(1-t)\gamma_n k_n^2 g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&- 2\mu(1-\delta_n)(1-t)\gamma_n k_n^2 \|Bx_n - Bx^\dagger\| \|w_n - y^\dagger\| \\
&- (1-t)(1-\delta_n)\gamma_n k_n^2 g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&- 2\lambda(1-\delta_n)\gamma_n k_n^2 \|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\| \|z_n - x^\dagger\| \\
&\leq \alpha_n \|f(x_n) - x^\dagger\|^2 + \|x_n - x^\dagger\|^2 \\
&- (1-\delta_n)(1-t)\gamma_n k_n^2 g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&- 2\mu(1-\delta_n)(1-t)\gamma_n k_n^2 \|Bx_n - Bx^\dagger\| \|w_n - y^\dagger\| \\
&- (1-t)(1-\delta_n)\gamma_n k_n^2 g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&- 2\lambda(1-\delta_n)\gamma_n k_n^2 \|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\| \|z_n - x^\dagger\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\delta_n)(1-t)\gamma_n k_n^2 g(\|x_n - x^\dagger - (w_n - y^\dagger)\|) \\
&+ (1-t)(1-\delta_n)\gamma_n k_n^2 g(\|w_n - y^\dagger - (z_n - x^\dagger)\|) \\
&\leq \alpha_n \|f(x_n) - x^\dagger\|^2 + \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 \\
&- 2\mu(1-\delta_n)(1-t)\gamma_n k_n^2 \|Bx_n - Bx^\dagger\| \|w_n - y^\dagger\| \\
&- 2\lambda(1-\delta_n)\gamma_n k_n^2 \|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\| \|z_n - x^\dagger\| \\
&\leq \alpha_n \|f(x_n) - x^\dagger\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^\dagger\| + \|x_{n+1} - x^\dagger\|) \\
&- 2\mu(1-\delta_n)(1-t)\gamma_n k_n^2 \|Bx_n - Bx^\dagger\| \|w_n - y^\dagger\| \\
&- 2\lambda(1-\delta_n)\gamma_n k_n^2 \|A(tx_n + (1-t)w_n) - A(tx^\dagger + (1-t)y^\dagger)\| \|z_n - x^\dagger\|.
\end{aligned}$$

It follows from (4), (5), condition (i), (iii), and the properties of g, that we have

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - x^\dagger - (w_n - y^\dagger)\| = 0; \\
&\lim_{n\to\infty} \|w_n - y^\dagger - (z_n - x^\dagger)\| = 0.
\end{aligned}$$

So,

$$\begin{aligned}
&\|x_n - z_n\| \\
&\leq \|x_n - x^\dagger - (w_n - y^\dagger)\| + \|w_n - y^\dagger - (z_n - x^\dagger)\| \\
&\to 0.
\end{aligned} \tag{6}$$

We can obtain

$$\|x_n - \Psi(x_n)\| = \|x_n - z_n\| \to 0, n \to \infty. \tag{7}$$

Moreover, we have

$$\begin{aligned}
&\|x_{n+1} - T^n u_n\| \\
&= \|\alpha_n f(x_n) + \beta_n x_n - \alpha_n T^n u_n - \beta_n T^n u_n\| \\
&= \|\alpha_n [f(x_n) - T^n u_n] + \beta_n [x_n - T^n u_n]\| \\
&\leq \beta_n \|x_n - x_{n+1}\| + \beta_n \|x_{n+1} - T^n u_n\| + \alpha_n \|f(x_n) - T^n u_n\|,
\end{aligned}$$

which implies that

$$(1-\beta_n)\|x_{n+1}-T^n u_n\| \leq \beta_n\|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-T^n u_n\|.$$

Therefore

$$\|x_{n+1}-T^n u_n\| \leq \frac{\beta_n}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-T^n u_n\|.$$

From conditions (i), (ii), and (4), we find

$$\|x_{n+1}-T^n u_n\| \rightarrow 0, (n \rightarrow \infty). \tag{8}$$

We obtain

$$\begin{aligned}\|x_n-T^n x_n\| &= \|x_n-x_{n+1}+x_{n+1}-T^n u_n+T^n u_n-T^n x_n\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^n u_n\|+k_n\|u_n-x_n\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^n u_n\|+k_n(1-\delta_n)\|z_n-x_n\|.\end{aligned}$$

By (4), (6), and (8), we have

$$\|x_n-T^n x_n\| \rightarrow 0, n \rightarrow \infty. \tag{9}$$

Since T is an asymptotically nonexpansive mapping, we have

$$\begin{aligned}\|x_n-Tx_n\| &= \|x_n-x_{n+1}+x_{n+1}-T^{n+1}x_{n+1}+T^{n+1}x_{n+1}-T^{n+1}x_n+T^{n+1}x_n-Tx_n\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+\|T^{n+1}x_{n+1}-T^{n+1}x_n\|+\|T^{n+1}x_n-Tx_n\| \\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+k_{n+1}\|x_{n+1}-x_n\|+k_1\|T^n x_n-x_n\| \\ &\leq (1+k_{n+1})\|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|+k_1\|T^n x_n-x_n\|.\end{aligned}$$

By (4) and (9), we have

$$\|x_n-Tx_n\| \rightarrow 0, n \rightarrow \infty. \tag{10}$$

Since E is a uniformly smooth Banach space and x_n is bounded, there exists a subsequence of x_n which converges weakly to ω. We know that Ψ is nonexpansive by Lemma 8. From (7) and Lemma 5, we deduce $\omega \in F(\Psi)$. It follows (10) and Lemma 5, and we deduce $w \in F(T)$. Therefore, $\omega \in \Omega$. From Lemma 6, the following holds:

$$\langle (I-f)x^{\dagger}, j(x^{\dagger}-\omega)\rangle \leq 0, \forall \omega \in \Omega.$$

Further, noticing that j is the weakly sequential continuous duality mapping, we have

$$limsup_{n\rightarrow\infty}\langle (I-f)x^{\dagger}, j(x^{\dagger}-x_n)\rangle = \lim_{k\rightarrow\infty}\langle (I-f)x^{\dagger}, j(x^{\dagger}-x_{n_k})\rangle = \langle (I-f)x^{\dagger}, j(x^{\dagger}-\omega)\rangle \leq 0. \tag{11}$$

Finally, we observe

$$
\begin{aligned}
\|x_{n+1}-x^{\dagger}\|^2 &= \langle x_{n+1}-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&= \langle \alpha_n(f(x_n)-x^{\dagger})+\beta_n(x_n-x^{\dagger})+\gamma_n(T^n u_n-x^{\dagger}), j(x_{n+1}-x^{\dagger})\rangle \\
&\le \alpha_n\langle f(x_n)-f(x^{\dagger}), j(x_{n+1}-x^{\dagger})\rangle+\beta_n\langle x_n-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&+\gamma_n\langle T^n u_n-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle+\alpha_n\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&\le \alpha_n\rho\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\|+\beta_n\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\| \\
&+\gamma_n k_n\|u_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\|+\alpha_n\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&\le \alpha_n\rho\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\|+\beta_n\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\| \\
&+\gamma_n k_n\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\|+\alpha_n\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&= [\alpha_n\rho+\beta_n+\gamma_n k_n]\|x_n-x^{\dagger}\|\|x_{n+1}-x^{\dagger}\|+\alpha_n\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&\le \frac{\alpha_n\rho+\beta_n+\gamma_n k_n}{2}(\|x_n-x^{\dagger}\|^2+\|x_{n+1}-x^{\dagger}\|^2)+\alpha_n\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|x_{n+1}-x^{\dagger}\|^2 \\
&\le \frac{\alpha_n\rho+\beta_n+\gamma_n k_n}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}\|x_n-x^{\dagger}\|^2+\frac{2\alpha_n}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&= [1-\frac{2(1-\alpha_n\rho-\beta_n-\gamma_n k_n)}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}]\|x_n-x^{\dagger}\|^2+\frac{2\alpha_n}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle \\
&\le [1-\frac{2\alpha_n((1-\rho-\epsilon)}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}]\|x_n-x^{\dagger}\|^2+\frac{2\alpha_n}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle.
\end{aligned}
\tag{12}
$$

We have $b_n=\frac{2\alpha_n((1-\rho-\epsilon)}{2-\alpha_n\rho-\beta_n-\gamma_n k_n}$ and $\sigma_n=\frac{\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle}{1-\rho-\epsilon}$, then by the condition (i) and (11), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} b_n &= \sum_{n=0}^{\infty}\frac{2\alpha_n((1-\rho-\epsilon)}{2-\alpha_n\rho-\beta_n-\gamma_n k_n} \ge \sum_{n=0}^{\infty}\alpha_n((1-\rho-\epsilon)=+\infty \\
\limsup_{n\to\infty}\sigma_n &= \limsup_{n\to\infty}\frac{\langle f(x^{\dagger})-x^{\dagger}, j(x_{n+1}-x^{\dagger})\rangle}{1-\rho-\epsilon} \le 0.
\end{aligned}
$$

Thus, applying Lemma 2 to (12), we have $\lim_{n\to\infty}\|x_n-x^{\dagger}\|=0$. This completes the proof. □

Corollary 1. *Let H be a real Hilbert space. Let $\emptyset \neq K \subset H$ be a closed convex subset. Suppose that $T: K \to K$ is a nonexpansive mapping. Let $A, B: K \to H$ be α-inverse strongly monotone operator and β-inverse strongly monotone operator. Let $f: K \to K$ be a contraction with coefficient $\rho \in (0,1)$ and Ψ be defined by Lemma 8. Assume that $\Omega = F(T)\bigcap F(\Psi) \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$. If the sequence $\{x_n\}$ is generated in the following manner:*

$$
\begin{cases}
w_n = P_K(I-\mu B)x_n, \\
z_n = P_K(I-\lambda A)(tx_n+(1-t)w_n), \\
u_n = \delta_n x_n+(1-\delta_n)z_n, \\
x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n T u_n.
\end{cases}
$$

satisfying the following conditions:

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1, \lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=0}^{\infty}\alpha_n = \infty;$$
$$(ii)\ 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0;$$
$$(iii)\ 0 \le t < 1, 0 < \lambda < \frac{2\alpha}{c}, 0 < \mu < \frac{2\beta}{c}.$$

Then $\{x_n\}$ converges strongly to $x^\dagger \in \Omega$ which solves the variational inequality:

$$\langle (I - f)x^\dagger, x^\dagger - p\rangle \le 0, \forall p \in \Omega.$$

Proof. In Theorem 1, we put $k_n \equiv 1$ for each $n \in N$ and replace Banach space X with Hilbert space H. □

Corollary 2. *Let X be a uniformly convex and uniformly smooth Banach space. Let $\emptyset \neq E \subset X$ be a closed convex subset. Suppose that $Q_E : X \to E$ is a sunny nonexpansive retraction and $T : E \to E$ is an asymptotically nonexpansive mapping satisfying the uniformly asymptotically regular condition, and $A, B : E \to X$ are an α-inverse strongly accretive operator and β-inverse strongly accretive operator, respectively. Let $f : E \to E$ be a contraction with coefficient $\rho \in (0, 1)$ and Ψ be defined by Lemma 8. Assume that $\Omega = F(T) \bigcap F(\Psi) \neq \emptyset$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$. If the sequence $\{x_n\}$ generated by the following manner:*

$$\begin{cases} w_n = Q_E(I - \mu B)x_n, \\ z_n = Q_E(I - \lambda A)w_n, \\ u_n = \delta_n x_n + (1 - \delta_n)z_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n u_n. \end{cases}$$

satisfies the following conditions:

$$(i)\ \alpha_n + \beta_n + \gamma_n = 1, \lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=0}^{\infty}\alpha_n = \infty, k_n - 1 = \epsilon\alpha_n, 0 < \epsilon < 1 - \rho;$$
$$(ii)\ 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0;$$
$$(iii)\ 0 < \lambda < \frac{2\alpha}{c}, 0 < \mu < \frac{2\beta}{c};$$
$$(iv)\ \beta_n + \gamma_n k_n^2 < 1,$$

then $\{x_n\}$ converges strongly to an element $x^\dagger \in \Omega$, which is also the solution of the variational inequality:

$$\langle (I - f)x^\dagger, j(x^\dagger - p)\rangle \le 0, \forall p \in \Omega.$$

Proof. In Theorem 1, if $t \equiv 0$, then we obtain the corollary. If $f = u$ and T is a nonexpansive mapping, it is the main result of Qin et al. [6]. □

4. Numerical Examples

In this section, we provide a numerical example to support the validity and feasibility of our proposed algorithm. The results are performed on a personal computer with Intel(R) Core(TM) i7-4710MQ CPU @ 2.50 GHz and RAM 8.00 GB.

Example 1. *In the real number field R, we put* $Bx = \frac{1}{6}x$ *and* $Ax = \frac{1}{4}x$ *where* $x \in R$*. Let* $k_n = 1 + \frac{1}{12n}$, $\delta_n = 1 - \frac{1}{3n}$, $\alpha_n = \frac{1}{3n}$, $\beta_n = \frac{1}{2} - \frac{1}{3n}$ *and* $\gamma_n = \frac{1}{2}$ *for all* $n \in N$*. We take* $t = \frac{1}{4}$, $\mu = 3, \lambda = 2$*. Let T and f be defined by* $Tx = \frac{1}{4}x$, $f(x) = \frac{1}{3}x$*. This implies that* $\epsilon = \frac{1}{4}$ *and* $\rho = \frac{1}{3}$*. Then, starting* $x_1 = 8$ *and* $x_1 = 25$ *in (3). We have*

$$x_{n+1} = \frac{144n + 144n \times (\frac{1}{4})^n - 33 \times (\frac{1}{4})^n - 64}{288n} x_n.$$

We obtain the following numerical results, as shown in Figures 1 and 2.

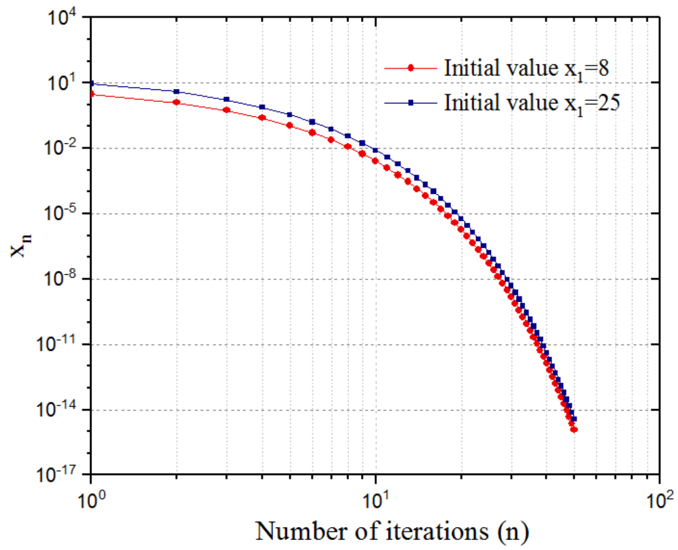

Figure 1. Exponential coordinate iteration.

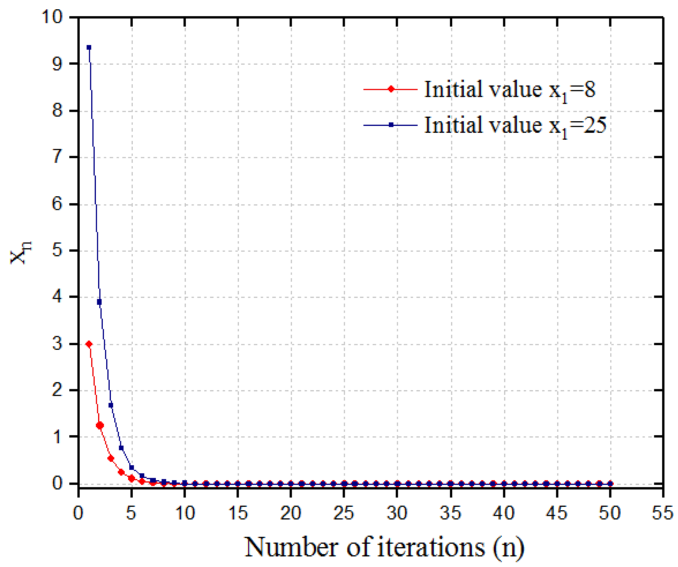

Figure 2. Real coordinate iteration.

Remark 1. *First of all, the parameters in Example 1 satisfy the conditions (i)–(iv) in Theorem 1, which shows that the coefficients in our Theorem 1 are obtained. From Figure 1, we can see the convergence value of the iterative sequence when the initial values are* $x_1 = 8$ *and* $x_1 = 25$*, respectively. From Figure 2, we can observe the convergence speed of the iterative algorithm. Figures 1 and 2 show that* $\{x_n\}$ *converges strongly to* 0*, where* $F(T) \bigcap F(\Psi) = 0$*. The convergence of* $\{x_n\}$ *in Example 1 shows the implementation and efficiency of our proposed algorithm.*

5. Conclusions

In this paper, we provide a viscosity approximation method for a general variational inequality system and fixed point problems in Banach spaces. Some strong convergence theorems are obtained and the numerical experiments can be guaranteed by Theorem 1. We give an extension to the general variational inequality system in Banach spaces and we generalize the Hilbert spaces to Banach spaces, and the nonexpansive mapping to the asymptotically nonexpansive mappings of Imnang [5] and Cai et al. [14], for the fixed point problem and variational inequality problem. In Theorem 1, if $t = 0$ $\delta_n = 0$ in Hilbert spaces, this is the main results of Ceng et al. [3]. The results and methods presented here also include some corresponding recent results of [6,9,10,12–14,17] as special cases.

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References

1. Moudafi, A. Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
2. Yao, Y.; Maruster, S. Strong convergence of an iterative algorithm for variational inequalities in Banach spaces. *Math. Comput. Modell.* **2011**, *54*, 325–329. [CrossRef]
3. Ceng, L.C.; Yao, J.C.; Muglia, L. An extragradient-like approximation method for variational inequality problems and fixed point problems. *Appl. Math. Comput.* **2007**, *190*, 205–215. [CrossRef]
4. Xie, S.; Imani, M.; Dougherty, E.R.; Braga-Neto, U.M. Nonstationary Linear Discriminant Analysis. In Proceedings of the 51st Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, USA, 29 October–1 November 2017; pp. 161–165.
5. Imnang, S. Viscosity iterative method for a new general system of variational inequalities in Banach spaces. *J. Inequalities Appl.* **2013**, *2013*, 249. [CrossRef]
6. Qin, X.; Cho, S.Y.; Kang, S.M. Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications. *J. Comput. Appl. Math.* **2009**, *233*, 231–240. [CrossRef]
7. Sen, M.D.L.; Muglia, L. Fixed and best proximity points of cyclic jointly accretive and contractive self-mappings. *J. Appl. Math.* **2012**, *2012*, 419–429.
8. Pan, C.; Wang, Y. Generalized viscosity implicit iterative process for asymptotically non-expansive mappings in Banach spaces. *Mathematics* **2019**, *7*, 379. [CrossRef]
9. Iiduka, H.; Takahashi, W.; Toyoda, M. Approximation of solutions of variational inequalities for monotone mappings. *Pan. Math. J.* **2004**, *14*, 49–61.
10. Yao, Y.; Noor, M.A.; Noor, K.I.; Liou, Y.C.; Yaqoob, H. Modified extragradient methods for a system of variational inequalities in Banach spaces. *Acta Appl. Math.* **2010**, *110*, 1211–1224. [CrossRef]
11. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. *Fixed Point Theory* **2019**, *20*, 113–134. [CrossRef]
12. Katchang, P.; Kumam, P. Convergence of iterative algorithm for finding common solution of fixed points and general system of variational inequalities for two accretive operators. *Positivity* **2012**, *15*, 281–295. [CrossRef]
13. Cai, G.; Shehu, Y.; Iyiola, O.S. Iterative algorithms for solving variational inequalities and fixed point problems for asymptotically nonexpansive mappings in Banach spaces. *Numer. Algorithms* **2016**, *73*, 869–906. [CrossRef]
14. Cai, G.; Shehu, Y.; Iyiola, O.S. Strong convergence theorems for fixed point problems for strict pseudo-contractions and variational inequalities for inverse-strongly accretive mappings in uniformly smooth Banach spaces. *J. Fixed Point Theory Appl.* **2019**, *21*, 41. [CrossRef]

15. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. *Fixed Point Theory* **2018**, *19*, 487–502. [CrossRef]
16. Sunthrayuth, P.; Kumam, P. Viscosity approximation methods base on generalized contraction mappings for a countable family of strict pseudo-contractions, a general system of variational inequalities and a generalized mixed equilibrium problem in Banach spaces. *Math. Comput. Model.* **2013**, *58*, 1814–1828. [CrossRef]
17. Luo, H.; Wang, Y. Iterative approximation for the common solutions of a infinite variational inequality system for inverse-strongly accretive mappings. *J. Math. Comput. Sci.* **2012**, *6*, 1660–1670.
18. Reich, S. Asymptotic behavior of contractions in Banach spaces. *J. Math. Anal. Appl.* **1973**, *44*, 57–70. [CrossRef]
19. Liu, L.S. Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces. *J. Math. Anal. Appl.* **1995**, *194*, 114–125. [CrossRef]
20. Yao, Y.; Shahzad, N.; Liou, Y.C. Modified semi-implicit midpoint rule for nonexpansive mappings. *Fixed Point Theory Appl.* **2015**, *2015*, 166. [CrossRef]
21. Lim, T.C.; Xu, H.K. Fixed point theorems for asymptotically nonexpansive mappings. *Nonlinear Anal.* **1994**, *22*, 1345–1355. [CrossRef]
22. Xu, H.K. Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.* **2004**, *298*, 279–291. [CrossRef]
23. Kamimura, S.; Takahashi, W. Strong convergence of a proximal-type algorithm in a Banach space. *SIAM J. Optim.* **2002**, *13*, 938–945. [CrossRef]