

Article

Particular Solutions of Ordinary Differential Equations Using Discrete Symmetry Groups

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Abstract: This article explains how discrete symmetry groups can be directly applied to obtain the particular solutions of nonlinear ordinary differential equations (ODEs). The particular solutions of some nonlinear ordinary differential equations have been generated by means of their discrete symmetry groups.

Keywords: discrete symmetry groups; nonlinear ODEs; particular solutions of ODEs

1. Introduction

Lie group analysis, founded by Sophus Lie, is an influential technique to solve differential equations, specially for the solutions of nonlinear differential equations. This method is basically based upon the invariance of differential equations under a group of continuous transformations. These transformations are called Lie point symmetry groups of a differential equation when this group of transformations leave the differential equation invariant [1–5]. Lie developed an algorithm to determine the symmetry groups associated with a given differential equation in a systematic way [2,5,6]. Once the symmetry group of a differential equation is discovered, it can be used to investigate the solutions of this differential equation in various ways. For example, the symmetry groups can be used (a) to develop new solutions from old ones [4,7], (b) to decrease the order of the given differential equation [1,7], (c) to decide whether a differential equation can be linearized and to build an explicit linearization if it exists [8–10] and (d) to find conserved quantities [7]. The Lie point symmetries of differential equations can be described by very small generators.

Symmetries which cannot be described by very small generators exist and one of them are discrete symmetries [1,11]. Discrete symmetries have many applications in differential equations like to make simple the numerical computation of solutions of partial differential equations and to generate new exact solutions from the known solutions [12]. Discrete symmetry groups also determine the nature of bifurcations in nonlinear dynamical systems [13].

Hydon developed a technique [1] to find all discrete point symmetries of the differential equations based on the result [11] that every Lie point symmetry generator of a Lie algebra \mathcal{L} of a differential equation induces an automorphism that preserves the commutator relation

$$[X_l, X_k] = c_{lk}^m X_m.$$

His method classifies all possible automorphisms of the Lie algebra \mathcal{L} , factoring out those automorphisms that are equivalent under the action of the continuous symmetry in the Lie group generated by the Lie algebra \mathcal{L} and give the most general realization of these automorphisms as point transformations. Finally, these point transformations are used to obtain the complete list of the discrete point symmetries of a differential equation.

The solutions of nonlinear ordinary differential equations are of great interest for physicists, engineers, mathematicians and biologists because many physical phenomena are modeled in the form of nonlinear ordinary differential equations e.g., The Legendre differential equation [14], that is used in Physics. Many techniques e.g., Laplace transformation [15], integral transform methods [16], Sumudu transform [17] and Lie group method [5], exist in the literature to solve nonlinear ODEs. Among these methods, the Lie group method is a more powerful method to obtain the solutions of nonlinear ordinary differential equations.

The Lie group of symmetry transformations of a nonlinear ordinary differential equations is used to find its general solutions by converting it into a linearizable form, by finding its invariants, or by getting new solutions from the old solutions, but a discrete symmetry group of transformations can be directly used to generate the particular solutions of the nonlinear ordinary differential equations. In this paper, we explain how discrete symmetry groups are used to obtain the particular solutions of ordinary differential equations. This paper is divided into three sections as follows: In Section 2, an algorithm to find the discrete symmetries of an ODE is described. In Section 3, the procedure to find the particular solutions of the ordinary differential equations using discrete symmetry groups is discussed and particular solutions of some nonlinear ODEs are obtained by using their discrete symmetries. A summary of the present work is given in the last section.

2. Method to Find the Discrete Point Symmetries of an ODE

An n th order ODE of the form

$$w^{(n)} = g(x, w, w', \dots, w^{(n-1)}), \quad n \geq 2, \quad (1)$$

has a *finite* dimensional Lie algebra \mathcal{L} of point symmetry generators (if it exists) with basis

$$X_i = \xi(x, w)\partial_x + \eta(x, w)\partial_w, \quad i = 1, 2, \dots, N, \quad (2)$$

where $N = \dim(\mathcal{L})$. Hydon [18] proposed a method to find a complete list of all discrete symmetry groups for a given ordinary differential Equation (1) with a Lie algebra \mathcal{L} of the symmetry generators given by (2). Each symmetry generator brings an automorphism of \mathcal{L} , of the basis generator X_i , preserving the following relation

$$[X_i, X_j] = c_{ij}^m X_m,$$

where c_{ij}^m are the structure constants for the basis $\{X_i\}$.

The method developed by Hydon classifies all conceivable automorphisms of \mathcal{L} , factoring out those which are equivalent under the action of any symmetry in Lie group generated by \mathcal{L} . Then it will be possible to attain the most general realization of the inequivalent automorphism as a point transformation. Finally, replacing these point transformations into the symmetry condition

$$\hat{w}^{(n)} = g(\hat{x}, \hat{w}, \dots, \hat{w}^{(n-1)}), \quad (3)$$

one can get a complete list of discrete symmetry groups of the ODE (1). The detailed aspect of this method can be seen in [18].

Example:

Consider a third order ODE, which was obtained by Whittaker [19]

$$w''' = 3w'' + \frac{w''w'}{w} - \frac{w'^2}{w} - 2w', \quad (4)$$

whose Lie algebra \mathcal{L} [20] is generated by

$$X_1 = \partial_x, \quad X_2 = e^{-x}\partial_x, \quad X_3 = w\partial_w. \quad (5)$$

This Lie algebra is non-abelian and its only nonzero structure constant is

$$c_{12}^2 = -1.$$

The elements of the non-singular matrix B satisfy the following system of nonlinear constraints

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n, \quad 1 \leq i < j \leq 3, \quad 1 \leq n \leq 3. \quad (6)$$

For $n = 1$, the above constraints become

$$c_{ij}^k b_k^1 = 0, \quad 1 \leq i < j \leq 3.$$

From above equation, only one value by setting $(i, j) = (1, 2)$, is obtained i.e.,

$$b_2^1 = 0.$$

Similarly for $n = 3$ in (6), it is found that

$$b_2^3 = 0.$$

For $n = 2$, the constraints (6) give the following values of the elements of B .

$$b_1^1 = 1, \quad b_3^1 = 0, \quad b_3^2 = 0.$$

Now the non-singular matrix B becomes

$$B = \begin{bmatrix} 1 & b_1^2 & b_1^3 \\ 0 & b_2^2 & 0 \\ 0 & 0 & b_3^3 \end{bmatrix}.$$

As X_3 is in the center of the Lie algebra (5), so the matrices corresponding to the automorphisms generated by X_1 and X_2 are

$$A(1, \epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A(2, \epsilon) = \begin{bmatrix} 1 & -\epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For further simplification of B , premultiply it by $A(2, b_1^2)$ to replace b_1^2 by zero and premultiply B by $A(1, -\ln |b_2^2|)$ to replace b_2^2 by α . Thus B takes the following form

$$B = \begin{bmatrix} 1 & 0 & b_1^3 \\ 0 & \alpha & 0 \\ 0 & 0 & b_3^3 \end{bmatrix}, \quad \alpha = \pm 1.$$

Now following determining equations have to be solved

$$\begin{bmatrix} \hat{x}_x & \hat{w}_x \\ e^{-x} \hat{x}_x & e^{-x} \hat{w}_x \\ w \hat{x}_w & w \hat{w}_w \end{bmatrix} = B \begin{bmatrix} 1 & 0 \\ e^{-\hat{x}} & 0 \\ 0 & \hat{w} \end{bmatrix} = \begin{bmatrix} 1 & b_1^3 \hat{w} \\ \alpha e^{-\hat{x}} & 0 \\ 0 & b_3^3 \hat{w} \end{bmatrix}.$$

The general solution of above determining equations is

$$\hat{x} = x, \quad \hat{w} = cw^{b_3^3}, \quad b_3^3 \neq 0, \quad (7)$$

where c is the constant of integration and determining equations require that $b_1^3 = 0$.

From (7), it is obtained that

$$\begin{aligned}\hat{w}' &= cb_3^3 w^{b_3^3-1} w', \\ \hat{w}'' &= cb_3^3 (b_3^3 - 1) w^{b_3^3-2} w'^2 + cb_3^3 w^{b_3^3-1} w'', \\ \hat{w}''' &= cb_3^3 (b_3^3 - 1) (b_3^3 - 2) w^{b_3^3-3} w'^3 + 3cb_3^3 (b_3^3 - 1) w^{b_3^3-2} w' w'' + cb_3^3 w^{b_3^3-1} w'''.\end{aligned}$$

Putting above results into the symmetry condition (3), it has been found that

$$(\hat{x}, \hat{w}) = (x, w),$$

and

$$(\hat{x}, \hat{w}) = (x, -w),$$

are discrete symmetries of (4). Thus the only non-trivial discrete symmetry group of (4) is generated by

$$Y_1 : (x, w) \mapsto (x, -w), \quad (8)$$

which is isomorphic to Z_2 .

3. Application of Discrete Symmetry Groups for Obtaining Particular Solutions of Nonlinear ODEs

In this section, the procedure to find particular solutions of nonlinear ordinary differential equations, by considering some examples, is explained.

3.1. Procedure to Find Particular Solutions Using Discrete Symmetry Groups

Let an n th order ODE given by (1) has the following discrete symmetry group

$$Y : (x, w) \mapsto (\hat{x}(x, w), \hat{w}(x, w)). \quad (9)$$

The discrete symmetry group (9) indicates that either

$$\hat{w}(x, w) = \hat{x}(x, w), \quad (10)$$

or

$$\hat{w}(x, w) = f(\hat{x}(x, w)), \quad (11)$$

can be a particular solution of ODE (1). By using Equations (10) and (11), the particular solutions of nonlinear ODEs via discrete symmetry groups, can be construct.

3.2. Particular Solutions of Some Nonlinear ODEs Using Discrete Symmetry Groups

Now particular solutions of some ordinary differential equations by using their discrete symmetry groups, are presented.

1. A third order nonlinear ODE, obtained by Whittaker [19]

$$w''' = 3w'' + \frac{w''w'}{w} - \frac{w'^2}{w} - 2w',$$

has the discrete symmetry group given in (8). This discrete symmetry group indicates that the particular solution of ODE can be of the form

$$-w = f(x).$$

If we take $f(x) = e^x$ then it is observed that $-w = e^x$ or $w = -e^x$ is the particular solution of (4). The graph of particular solution $w = -e^x$ of (4) is shown in Figure 1.

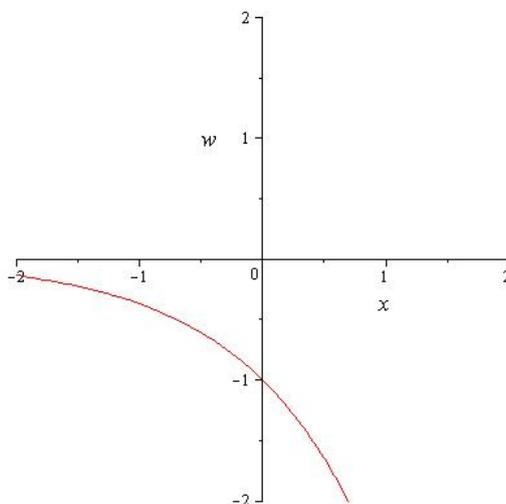


Figure 1. Graph for particular solution $w = -e^x$ of (4).

2. Consider the Blasius equation

$$w''' + \frac{1}{2}ww'' = 0, \quad (12)$$

which is invariant by two dimensional Lie algebra

$$\partial_x, x\partial_x - w\partial_w.$$

Following the method presented in [18], it is found that the discrete symmetry group of (12) is $Y : (x, w) \mapsto (-x, w)$, which shows that $w = -f(x)$ i.e., $w = -x$ is the particular solution of (12). The graph of particular solution $w = -x$ of (12) is presented in Figure 2.

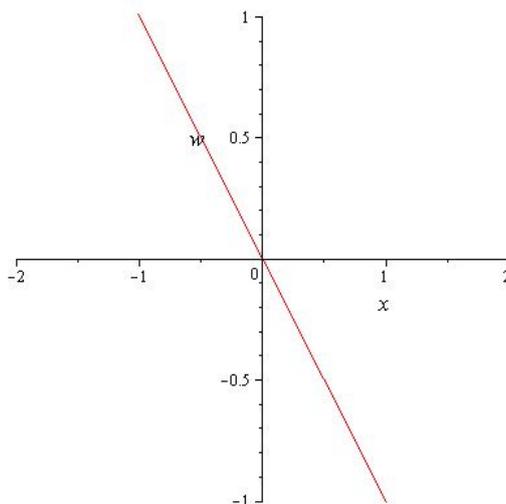


Figure 2. Graph for particular solution $w = -x$ of (12).

3. Consider the following ODE

$$w''' = \frac{3}{2} \frac{w''^2}{w'}, \quad (13)$$

which has a six dimensional Lie algebra [20]

$$\partial_x, x\partial_x, x^2\partial_x, \partial_w, w\partial_w, w^2\partial_w.$$

Using above Lie algebra and applying Hydon's method given in [18], it is found that

$$Y_1 : (x, w) \mapsto \left(\frac{1}{x}, w\right), \quad (14)$$

$$Y_2 : (x, w) \mapsto (-x, w), \quad (15)$$

are two discrete symmetry groups of ODE (13). Now using (14) and (15), two particular solutions of (13) can be found. From Y_1 and Y_2 , it has been deduced that $w = \frac{1}{x}$ and $w = -x$ are two particular solutions of (13), which are calculated by using its discrete symmetry groups. The graph of particular solution $w = \frac{1}{x}$ of (13) is shown in Figure 3.

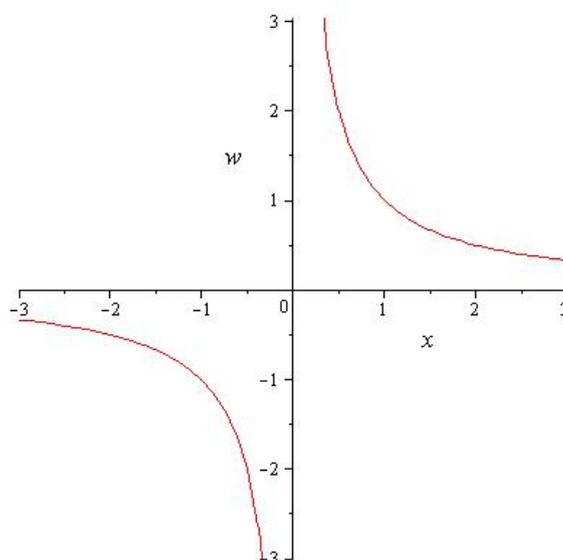


Figure 3. Graph for particular solution $w = \frac{1}{x}$ of (13).

4. The discrete symmetry group of nonlinear ODE [21]

$$w''' = w'' - (w''^2), \quad (16)$$

is

$$Y : (x, w) \mapsto \left(-x, \frac{1}{2}x^2 - w\right).$$

Now using (10), Y deduce that $\frac{1}{2}x^2 - w = -x$ or $w = \frac{1}{2}x^2 + x$ is a particular solution of (16). The graph of particular solution $w = \frac{1}{2}x^2 + x$ of (16) is presented in Figure 4.

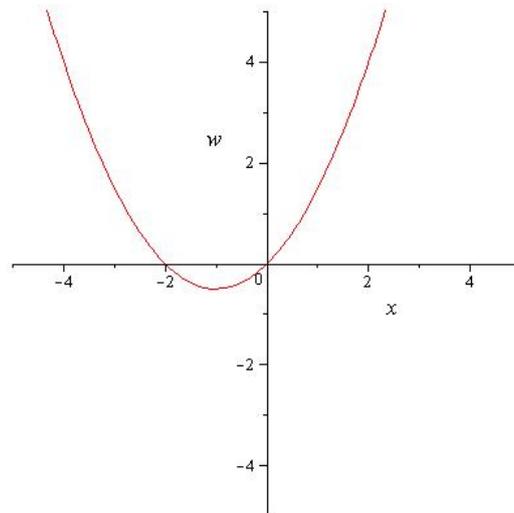


Figure 4. Graph for particular solution $w = \frac{1}{2}x^2 + x$ of (16).

5. Consider the following Chazy equation

$$w''' = 2ww'' - 3w'^2 + \lambda(6w' - w^2)^2, \quad (17)$$

whose discrete symmetries have been discussed in [18] and one of these discrete symmetry transformations is

$$(x, w) \mapsto \left(-\frac{1}{x}, x^2w + 6x\right),$$

which indicates that the solution of (17) is of the form

$$x^2w + 6x = c_1 f\left(-\frac{1}{x}\right).$$

or

$$x^2w + 6x = -c_1 \frac{1}{x}. \quad (18)$$

To seek the particular solution of (17), the value of c_1 has to be found. So the derivatives, w' , w'' and w''' by using Equation (18) are obtained and by putting values of these derivatives in Equation (17), it is obtained that $c_1 = 0$. Thus $x^2w + 6x = 0$ i.e., $w = -\frac{6}{x}$ is the particular solution of (17). The graph of this particular solution is shown in Figure 5.

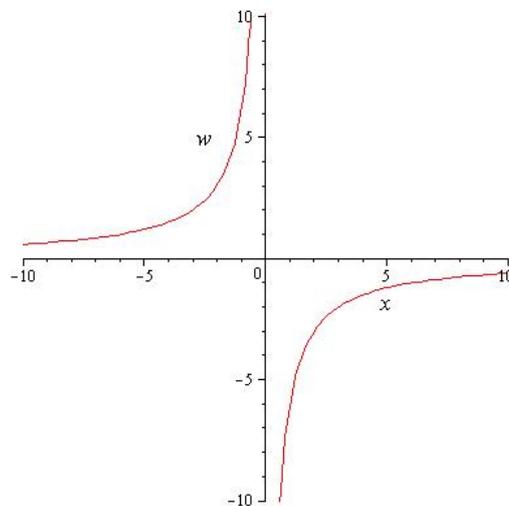


Figure 5. Graph for particular solution $w = -\frac{6}{x}$ of (17).

Remark:

All considered examples presented in the above section exemplify, how discrete symmetry groups are helpful to construct some particular solutions of nonlinear ordinary differential equations.

4. Summary

This article contains the application of the discrete point symmetries for ordinary differential equations. In this article, we have explained how discrete symmetry groups can be used to find particular solutions of nonlinear ordinary differential equations. Some nonlinear ODEs along their discrete symmetry transformations have been considered and particular solutions of these nonlinear ODEs have been obtained by means of their discrete symmetries.

Further application of the discrete symmetry groups that has to be investigated is to explore whether and how nonlinear ODEs can be linearized with the help of discrete symmetry groups.

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