

w - b -Cone Distance and Its Related Results: A Survey

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Abstract: In this work, we define the concept of a w - b -cone distance in tvs -cone b -metric spaces which differs from generalized c -distance in cone b -metric spaces, and we discuss its properties. Our results are significant, since all of the results in fixed point theory with respect to a generalized c -distance can be introduced in the version of w - b -cone distance. Moreover, using Minkowski functionals in topological vector spaces, we prove the equivalence between some fixed point results with respect to a w -distance in general b -metric spaces and a w - b -cone distance in tvs -cone b -metric spaces.

Keywords: w - b -cone distance; tvs -cone b -metric space; lower b -semicontinuous; Minkowski functional

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1. Introduction and Preliminaries

In 1931, Wilson [1] introduced symmetric spaces, as metric-like spaces lacking the triangle inequality. Thereinafter, b -metric spaces or metric type spaces as a new kind of spaces were defined by Bakhtin [2] and Boriceanu [3].

Definition 1 ([2,3]). Let X be a nonempty set. A real-valued function $D : X \times X \rightarrow [0, \infty)$ is said to be a b -metric if the following properties hold:

- (D₁) $D(x, y) = 0$ iff $x = y$ for all $x, y \in X$;
- (D₂) $D(x, y) = D(y, x)$ for all $x, y \in X$;
- (D₃) $D(x, z) \leq s[D(x, y) + D(y, z)]$ for all $x, y, z \in X$.

With this definition, (X, D) is named a b -metric space. Obviously, a b -metric space with $s = 1$ is a metric space.

Then, they introduced some definitions and topological properties in b -metric spaces and obtained some applications of these spaces in nonlinear analysis (also, see [4,5]). In 1996, Kada et al. [6] considered a w -distance in metric spaces for solving some non-convex minimization problems. Then some authors [7,8] applied this concept in fixed point theory.

Definition 2 ([6]). Consider a metric space (X, d) . Let $\rho : X \times X \rightarrow [0, +\infty)$ be a function satisfying the following conditions:

- (ρ_1) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$;
- (ρ_2) ρ is lower semicontinuous in its second variable; that is, if $x \in X$ and $y_n \rightarrow y$ in X then $\rho(x, y) \leq \liminf_n \rho(x, y_n)$;
- (ρ_3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(z, x) < \delta$ and $\rho(z, y) < \delta$ imply that $d(x, y) < \varepsilon$.

Then ρ is named a w -distance on X .

In 2014, Hussain et al. [9] generalized this w -distance to wt -distance on a b -metric space and obtained several its properties.

Definition 3 ([9]). Consider a b -metric space (X, D) with $s \geq 1$. Let $p : X \times X \rightarrow [0, +\infty)$ be a function satisfying the following conditions:

- (w_1) $p(x, z) \leq s[p(x, y) + p(y, z)]$ for all $x, y, z \in X$;
- (w_2) p is lower b -semicontinuous in its second variable; that is, if $x \in X$ and $y_n \rightarrow y$ in X then $p(x, y) \leq s \liminf_n p(x, y_n)$;
- (w_3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) < \delta$ and $p(z, y) < \delta$ imply that $D(x, y) < \varepsilon$.

Then p is named a wt -distance on X .

Obviously, a wt -distance in a b -metric space with $s = 1$ is a w -distance in a metric space in the sense of Definition 2.

Ordered normed spaces and cones have many applications in applied mathematics. Hence, fixed point theory in K -normed and K -metric spaces was extended in the 20th century (see, e.g., [10–12]). In 2007, Huang and Zhang [13] reintroduced these spaces under the name of cone metric spaces by substituting the set of real number by an ordered normed space. Also, topological vector space-valued version of this concept was discussed in [14–16]. In 2011, analogously with the definition of b -metric spaces and cone metric spaces, Hussain and Shah [17] and Ćvetković et al. [18] defined cone b -metric spaces (or cone metric type spaces) and proved some topological and analysis properties (also, see [19]). Moreover, topological vector space-valued version of cone b -metric spaces was defined in [20]. On the other hands, in 2011, Cho et al. [21] and Wang and Guo [22] introduced a cone version of the w -distance which is named c -distance. Then, some fixed point results under w -distance in metric spaces and under c -distance in cone metric spaces and tv s-cone metric spaces were obtained in [23,24] and the references cited therein. In 2015, Bao et al. [25] introduced a generalized c -distance in cone b -metric spaces and proved several fixed point theorems with respect to this generalized distance (also, see [26,27]).

We start with some preliminary definitions and results in [10,28,29].

Let E be a real Hausdorff topological vector space (for short, is named tv s) with the zero vector θ . A proper closed subset $P \neq \emptyset$ of E is said to be a cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{\theta\}$. For an arbitrary cone $P \subset E$, we consider a partial ordering \preceq with respect to P by $x \preceq y \iff y - x \in P$. We say $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \ll y$ if and only if $y - x \in \text{int } P$, where $\text{int } P$ is the interior of P . Also, the cone P is called solid if $\text{int } P \neq \emptyset$. Now, the pair (E, P) is an ordered tv s. For a pair of elements $x, y \in E$ with $x \preceq y$, set $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset A of E is called order-convex if $[x, y] \subset A$. By this definition, an ordered tv s (E, P) is order-convex if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, P is named a normal cone. If E is a normed space, this condition means that the unit ball is order-convex; that is, there is a number K such that for all $x, y \in E$, $\theta \preceq x \preceq y$ imply $\|x\| \leq K\|y\|$.

Theorem 1 ([11]). Let the underlying cone of an ordered tv s be normal and solid. Then tv s is an ordered normed space.

Below, we provide a full explanation of the main motivation behind this project:

In 2010, Du [14] studied the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in general metric spaces. Especially, he proved the equivalence between Banach contraction principle in general metric spaces and in tv s-cone metric spaces by a nonlinear scalarization function. Almost simultaneously with Du's work, in 2011, Kadelburg et al. [30] showed that the same results can be obtained more easily using a Minkowski

functional in topological vector spaces. In the meantime, Ćirić et al. [31] defined a w -cone distance in tv s-cone metric spaces, which is a generalization of the w -distance introduced by Kada et al. [6]. This definition is different of the definition of the c -distance introduced by Cho et al. [21] and Wang and Guo [22]. Indeed, each w -cone distance is a c -distance, but the converse is not true. Also, Ćirić et al. noted that the method of Du for cone contraction mappings cannot be applied for a w -cone distance contraction mappings. Then they proved some fixed point theorems with respect to a w -cone distance in tv s-cone metric spaces. Also, at the end of their paper, they posed an open question about the existence and uniqueness of fixed point of quasi-contraction mappings by applying a w -cone distance. In the same year, Kadelburg and Radenović [32] answered this question by using a similar technique in [30]. On the other hand, in 2013, Du and Karapinar [20] studied about the equivalence of vectorial versions of fixed point theorems in cone b -metric spaces and scalar versions of fixed point theorems in b -metric spaces. Now, let us we define a tv s-cone b -metric space, which is introduced by Kadelburg et al. [19] and Du and Karapinar [20].

Definition 4 ([19]). Let X be a nonempty set, $s \geq 1$ and (E, P) be an ordered tv s. A function $d : X \times X \rightarrow P$ is said to be a tv s-cone b -metric if the following properties hold:

- (d_1) $d(x, y) = \theta$ iff $x = y$ for all $x, y \in X$;
- (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d_3) $d(x, z) \preceq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

With this definition, (X, d) is named a tv s-cone b -metric space.

Now, by applying Definition 4, we can obtain other version of metric spaces as following.

Remark 1. Obviously, a tv s-cone b -metric space with $s = 1$ is a tv s-cone metric space in the sense of [14]. If we replace E by a real Banach space in Definition 4, we get the cone b -metric space in the sense of [17,18]. Now, if we consider $s = 1$ then we can obtain the classic definition of cone metric space introduced by Huang and Zhang [13] (also, see [33]). Moreover, it is evident that our definition coincides with the definition of b -metric spaces (Definition 1) if we replace E by the set of real numbers and P by the set of nonnegative real numbers. In this case, if we consider $s = 1$ then we obtain the same well-known results of metric spaces. Therefore, it seems Definition 4 is the most complete version of type metric spaces.

Definition 5 ([20,23,34]). Let $\{x_n\}$ be a sequence in a tv s-cone b -metric space (X, d) . Then

- $\{x_n\}$ converges to x if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $x_n \rightarrow x$ as $n \rightarrow \infty$.
- $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.
- X is said to be complete if every Cauchy sequence in X is convergent in X .

Also, we apply the following property several times in the proof of our theorems:

- (*) Let $v \preceq \lambda v$ with $v \in P$ and $0 \leq \lambda < 1$. Then $v = \theta$.

In [19], Kadelburg et al. showed that the same results of Du and Karapinar [20] can be obtained more easily using a Minkowski functional in tv s-cone b -metric spaces.

In this work, we first define a w - b -cone distance in the framework of tv s-cone b -metric spaces. Then we introduce several its preliminary properties. As an application, we obtain several fixed point and periodic point theorems for contractive mappings in such spaces with a non-normal cone under contractive conditions introduced in the terms of w - b -cone distance. Furthermore, we prove the equivalence of vectorial versions of fixed point theorems with respect to a w - b cone distance in cone b -metric spaces and scalar versions of fixed point theorems with respect to a w -distance in b -metric

spaces. In fact, this research is a continuation of the former articles such as [6,14,20–22,30–32] which extended, generalized and unified all of them in this literature.

2. Main Results

Let's start by introducing the following definition.

Definition 6. Let (X, d) be a tvs -cone b -metric space. A real-valued function $f : X \rightarrow P$ is said to be lower b -semicontinuous at a point $x \in X$ if for each $c \in \text{int } P$, there is $n_0 \in \mathbb{N}$ such that $f(x) \preceq sf(x_n) + c$ for all $n \geq n_0$, where $\{x_n\}$ is a sequence in X and $x_n \rightarrow x$.

Now, we introduce the following new definition of distance version in tvs -cone b -metric spaces. The idea of this definition comes from the definition of w -cone distance was introduced by Ćirić et al. [31] and the definition of generalized c -distance in cone b -metric spaces was defined by Bao et al. [25] and Soleimani Rad et al. [34].

Definition 7. Consider a tvs -cone b -metric space (X, d) with $s \geq 1$. Let $q : X \times X \rightarrow P$ be a function satisfying the following properties:

- (q_1) $q(x, z) \preceq s[q(x, y) + q(y, z)]$ for all $x, y, z \in X$;
- (q_2) $q(x, \cdot) : X \rightarrow P$ is lower b -semicontinuous for all $x \in X$;
- (q_3) for all $\theta \ll c$, there exists $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ implying $d(x, y) \ll c$.

Then q is said to be a w - b -cone distance on X .

Remark 2. Applying the definition of c -sequence, the condition (q_2) of above definition can be considered equivalently as follow:

- (q'_2) If $y_n, y \in X$, $y_n \rightarrow y$ as $n \rightarrow \infty$ and $g(y) = q(x, y)$, then $g(y) - g(y_n)$ is a c -sequence.

Remark 3. Each w -distance in a b -metric space (in the sense of Hussain et al. [9]) is w - b -cone distance in the tvs -cone b -metric space. Also, for $s = 1$, a w - b -cone distance is a w -cone distance of [31]. In this manner, let $E = \mathbb{R}$ and $P = [0, \infty)$. Then we have the same definition of w -distance of Kada et al. [6].

Remark 4. Note that the definition of generalized c -distance of Bao et al. [25] and the definition of generalized c -distance of Soleimani Rad et al. [34] differ from Definition 7 of w - b -cone distance. Also, it is evident that every w - b -cone distance is a c -distance, but the reverse is not true.

Now, we give some examples in the framework of tvs -cone b -metric spaces.

Lemma 1. If (X, d) be a cone b -metric space, then the cone b -metric d is a w - b -cone distance q .

Proof. Note that if $q = d$, then q satisfies (q_1). Thus we only prove (q_2) and (q_3). Let $x, y, y_n \in X$, $y_n \rightarrow y$, and $c \in E$ with $c \in \text{int } P$ be arbitrary. Since $y_n \rightarrow y$, there exists $N \in \mathbb{N}$ such that $d(y_n, y) \preceq \frac{c}{s}$ for each $n \geq N$. Now, if $G(y) = d(x, y)$, then

$$G(y) = d(x, y) \preceq s[d(x, y_n) + d(y_n, y)] \preceq sd(x, y_n) + c = sG(y_n) + c$$

for all $n \geq n_0$. Hence, $q(x, \cdot) = d(x, \cdot)$ is lower b -semicontinuous and (q_2) is established. For proving (q_3), for each $c \in \text{int } P$, set $e = \frac{c}{2s}$. Then $q(z, x) \ll e$ and $q(z, y) \ll e$ imply that $d(x, y) \ll c$. This completes the proof. \square

Example 1. Let (X, d) be a tvs -cone b -metric space and $u \in X$ fixed. Then $q(x, y) = \frac{1}{s}d(u, y)$ is a w - b -cone distance. Indeed, we have $sq(x, z) = d(u, z) \preceq s^2(d(u, y) + d(u, z))$, i.e., $q(x, z) \preceq sq(x, y) + sq(y, z)$. Thus,

(q_1) is true. Similar to Lemma 2, one can prove the validity of condition (q_2) . Finally, (q_3) is obtained by taking $e = \frac{c}{2s^2}$.

Example 2. Let $E = C_{\mathbb{R}}[0, 1]$, max-norm, $P = \{f \in E : f(t) \geq 0 \text{ for } t \in [0, 1]\}$, τ^* and X are as in same Example 3 of [23]. Also, define the cone b-metric $d : X \times X \rightarrow (E, \tau^*)$ by $d(x, y)(t) = |x - y|^s \varphi(t)$ with $s \in \{1, 2\}$, where $\varphi \in P$ is a fixed element. Then

$$q_1(x, y)(t) = d(x, y)(t), \quad q_2(x, y)(t) = y^s \varphi(t) \quad \text{and} \quad q_3(x, y)(t) = (x^s + y^s) \varphi(t).$$

are three examples of w -b-cone distances in a tvs -cone b-metric space.

From above examples, for all $x, y \in X$, we conclude that

- $q(x, y) = q(y, x)$ is not presently true;
- $q(x, y) = \theta$ does not certainly equivalent to $x = y$.

Lemma 2. Consider a tvs -cone b-metric space (X, d) with w -b-cone distance q on X . Also, let $\{\alpha_n\}$ be a sequence in X and $\alpha, \beta, \gamma \in X$, and $\{x_n\}$ and $\{y_n\}$ be two c -sequences in P . Then

- (I₁) If $q(\alpha_n, \beta) \preceq x_n$ and $q(\alpha_n, \gamma) \preceq y_n$ for $n \in \mathbb{N}$, then $\beta = \gamma$. Specially, if $q(\alpha, \beta) = \theta$ and $q(\alpha, \gamma) = \theta$, then $\beta = \gamma$.
- (I₂) If $q(\alpha_n, \alpha_m) \preceq x_n$ for $m > n > n_0$ (for some $n_0 \in \mathbb{N}$), then $\{\alpha_n\}$ is a Cauchy sequence.

Proof. Since the proof is easy and similar as in the case of w -distance in tvs -cone metric spaces in [31], we omit it. \square

Now, we prove some fixed and periodic point results with respect to a w -b-cone distance. Note that the scalarization method of Du and Karapinar [20] for linear contractive condition cannot be considered for a linear w -b-cone distance contractive condition. Thus, our results are substantial and new. In the following theorem, which extends and improves Theorem 9 of [31], Theorem 4.2 of [9] and Theorem 2 of [6], we obtain an estimate for a w -b-cone distance $q(x_n, z)$ of an approximate value x_n .

Theorem 2. Consider a complete tvs -cone b-metric space (X, d) with w -b-cone distance q on X . Assume that $f : X \rightarrow X$ is a mapping such that

$$q(fx, f^2x) \preceq kq(x, fx) \tag{1}$$

for all $x \in X$, where $0 \leq k < \frac{1}{s}$. Suppose that one of the following assertions holds:

- (i) If $fy \neq y$, there exists $c \in \text{int } P$ with $c \neq \theta$ such that $c \ll q(x, y) + q(x, fx)$ for each $x \in X$;
(ii) f is continuous.

Then $z \in X$ is a fixed point of f and

$$q(f^n x_0, z) \preceq \frac{s^2 k^n}{1 - sk} q(x_0, fx_0) \tag{2}$$

for all $n \in \mathbb{N}$. Moreover, if $fu = u$ for some $u \in X$, then $q(u, u) = \theta$.

Proof. Let $x_0 \in X$ and $x_n = fx_{n-1} = f^n x_0$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of f . Otherwise, we obtain from (1) that

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, f^2x_{n-1}) \preceq kq(x_{n-1}, fx_{n-1}) \\ &= kq(fx_{n-2}, f^2x_{n-2}) \preceq k^2q(x_{n-2}, fx_{n-2}) \preceq \cdots \preceq k^n q(x_0, x_1). \end{aligned} \tag{3}$$

Hence, for all $m, n \in \mathbb{N}$ with $m > n$, it follows from (3) and (q_1) that

$$\begin{aligned}
 q(x_n, x_m) &\preceq s[q(x_n, x_{n+1}) + q(x_{n+1}, x_m)] \\
 &\preceq sq(x_n, x_{n+1}) + s[s[q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_m)]] \\
 &\vdots \\
 &\preceq sq(x_n, x_{n+1}) + s^2q(x_{n+1}, x_{n+2}) + \dots + s^{m-n}q(x_{m-1}, x_m) \\
 &\preceq (sk^n + s^2k^{n+1} \dots + s^{m-n}k^{m-1})q(x_0, x_1) \\
 &\preceq \frac{sk^n}{1 - sk}q(x_0, x_1).
 \end{aligned} \tag{4}$$

Thus, by applying Lemma 2(l_2), $\{x_n\}$ is a Cauchy sequence in complete space X . Hence, there exists a point $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Now, we will prove the estimate (2). Consider a function $g : X \rightarrow P$ by $g(x) = q(x_n, x)$ for each $n \in \mathbb{N}$. Consider (q_2) and (1). Since $x_n \rightarrow z$ as $n \rightarrow \infty$, there is $M \in \mathbb{N}$ such that $q(x_n, z) \preceq sq(x_n, x_m) + c$ for $m > M$ and for each $c \in \text{int } P$. Now, it follows from (4) that

$$q(f^n x_0, z) = q(x_n, z) \preceq \frac{s^2k^n}{1 - sk}q(x_0, f x_0) + c. \tag{5}$$

If we consider $c = \frac{c}{j}$ for some $j \in \mathbb{N}$, then we obtain

$$\frac{s^2k^n}{1 - k}q(x_0, f x_0) + \frac{c}{j} - q(f^n x_0, z) \in P.$$

It is easy to show that

$$\lim_{j \rightarrow \infty} \left(\frac{s^2k^n}{1 - k}q(x_0, f x_0) + \frac{c}{j} - q(f^n x_0, z) \right) = \frac{s^2k^n}{1 - k}q(x_0, f x_0) - q(f^n x_0, z).$$

Since P is a closed set, we have

$$\frac{s^2k^n}{1 - k}q(x_0, f x_0) - q(f^n x_0, z) \in P,$$

which implies that $q(f^n x_0, z) \preceq \frac{s^2k^n}{1 - sk}q(x_0, f x_0)$. Thus, (2) is established.

At the first, suppose that (i) is satisfied. Then we show that $fz = z$. Let $fz \neq z$ (to the contrary). Then, from (i), we have

$$c \ll q(x, z) + q(x, f x) \tag{6}$$

for all $x \in X$, where $\theta \ll c$. Now, using (2), (4) and (5), there exists $n_0 \in \mathbb{N}$ such that

$$q(x_n, z) \ll \frac{c}{4}, \quad q(x_n, x_{n+1}) \ll \frac{c}{4} \tag{7}$$

for all $n > n_0$. Now, let $x = x_n$. It follows from (6), (7), (q_2) and the convergence x_n to z that

$$c \ll q(x_n, z) + q(x_n, x_{n+1}) \ll \frac{c}{4} + \frac{c}{4} = \frac{c}{2},$$

which is a contradiction (because $\theta \neq c \in \text{int } P$). Thus, our guess about $fz \neq z$ is false. Hence, $fz = z$.

Now, suppose that (ii) holds. By applying continuity of f , convergence x_n to z and the uniqueness of limit, we have $fu = z$. To complete the proof, let $fu = u$. It follows from (1) that

$$q(u, u) = q(fu, f^2u) \preceq kq(u, fu) = kq(u, u). \quad (8)$$

Since $0 \leq k < \frac{1}{s}$ and $s \geq 1$, then (8) and (*) imply that $q(u, u) = \theta$. This completes the proof. \square

Remark 5. Since all of other theorems in various version of distances can be obtained of Theorem 2, we claim our theorem is the most complete theorem with respect to a version of distance (see, Remarks 1 and 3). Also, using Theorem 2, we can obtain many fixed point results in tv -cone b -metric spaces and especially in tv -cone metric spaces by considering $s = 1$. Consequently, we can prove all of them in metric fixed point theory without direct proof and also without using Du's method [14] (by considering $E = \mathbb{R}$ and $P = [0, \infty)$).

Question: Can the conditions (i) and (ii) be replaced by another condition in Theorem 2?

Corollary 1. Consider a complete tv -cone b -metric space (X, d) with w - b -cone distance q on X . Assume that $f : X \rightarrow X$ is a mapping such that

$$q(fx, f^2x) \preceq hq(x, f^2x) \quad (9)$$

for all $x \in X$, where $0 \leq h < \frac{1}{s^2 + s}$. If one of the assertions (i) or (ii) in Theorem 2 holds, then $z \in X$ will be a fixed point of f . Further, if $fu = u$, then $q(u, u) = \theta$.

Proof. Let $x \in X$ is arbitrary. It follows from (9) and (q_1) that

$$q(fx, f^2x) \preceq hq(x, f^2x) \preceq hs[q(x, fx) + q(fx, f^2x)].$$

Thus, we have $(1 - hs)q(fx, f^2x) \preceq hsq(x, fx)$, which implies that $q(fx, f^2x) \preceq kq(x, fx)$ with $0 \leq k = \frac{hs}{1 - hs} < \frac{1}{s}$. The proof further follows by Theorem 2. \square

Corollary 2. Consider a complete tv -cone metric space (X, d) with w -cone distance q on X . Assume that $f : X \rightarrow X$ is a mapping such that $q(fx, f^2x) \preceq hq(x, f^2x)$ for all $x \in X$, where $0 \leq h < \frac{1}{2}$. If one of the assertions (i) or (ii) in Theorem 2 holds, then $z \in X$ will be a fixed point of f . Further, if $fu = u$, then $q(u, u) = \theta$.

Proof. In order to proof, set $s = 1$ in Corollary 1. \square

Clearly, if z is a fixed point of f , then z is a fixed point of f^n as well for all $n \in \mathbb{N}$, but the reverse is not true. If a mapping $f : X \rightarrow X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$, where $Fix(f)$ is the set of fixed points of f , then f has property (P) [35–37]. The following theorem extends and improves Theorem 12 of Ćirić et al. [31], Theorem 4 of Đorđević [23] and Theorem 2 of Abbas and Rhoades [35].

Theorem 3. Consider a complete tv -cone b -metric space (X, d) with w - b -cone distance q on X . Assume that the mapping $f : X \rightarrow X$ is subject to

$$q(fx, f^2x) \preceq kq(x, fx) \quad (10)$$

for all $x \in X$ with $0 \leq k < \frac{1}{s}$, then f has property (P).

Proof. Obviously, if f has a fixed point z , then z is a fixed point of f^n for each integer number n . Let $z \in \text{Fix}(f^n)$; that is, $f^n z = z$. Then, from (10), we have

$$\begin{aligned} q(z, fz) &= q(f^n z, f f^n z) = q(f f^{n-1} z, f^2 f^{n-1} z) \\ &\preceq k q(f^{n-1} z, f f^{n-1} z) = k q(f f^{n-2} z, f^2 f^{n-2} z) \\ &\preceq k^2 q(f^{n-2} z, f f^{n-2} z) \preceq \dots \preceq k^n q(z, fz), \end{aligned}$$

which implies that $q(z, fz) = \theta$ (by $(*)$ and $0 \leq k < \frac{1}{s}$). Hence, for each $m \in \{1, 2, \dots, n\}$, we get $q(f^m z, f^{m+1} z) \preceq k^m q(z, fz) = \theta$. Thus, $q(f^m z, f^{m+1} z) = \theta$. Using (q_1) , we have

$$q(z, f^2 z) \preceq s[q(z, fz) + q(fz, f^2 z)] = \theta,$$

which implies that $q(z, f^2 z) = \theta$. By following this process, we get $q(z, f^n z) = \theta$. Since $f^n z = z$, we have $q(z, z) = \theta$. Now, by applying Lemma 1. (l_1) , we conclude that $fz = z$; that is, z is also a fixed point of f . Hence, $z \in \text{Fix}(f)$ and so $\text{Fix}(f^n) \subseteq \text{Fix}(f)$. This ends to the proof. \square

Corollary 3. Consider a complete tvs-cone metric space (X, d) with w -cone distance q on X . Assume that the mapping $f : X \rightarrow X$ is subject to

$$q(fx, f^2x) \preceq kq(x, fx)$$

for all $x \in X$ with $0 \leq k < 1$, then f has property (P) .

In the following theorem, we prove the existence of fixed point with respect to a w - b -cone distance for a contraction, which is included Banach contraction version, Kannan contraction version and Cho contraction version as you will see in the sequel.

Theorem 4. Consider a complete tvs-cone b -metric space (X, d) with w - b -cone distance q on X . Assume that $f : X \rightarrow X$ is a mapping such that

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy) \tag{11}$$

for all $x, y \in X$, where $\alpha_i \geq 0$ for $i = 1, 2, 3, 4$ and

$$s(\alpha_1 + \alpha_2) + \alpha_3 + (s^2 + s)\alpha_4 < 1. \tag{12}$$

If either (i) f is continuous, or (ii) there exists $c \in \text{int}P$ with $c \neq \theta$ such that

$$c \ll q(x, y) + q(x, fx) + q(fx, y)$$

for each $x, y \in X$ with $y \neq fy$, then f has a fixed point. Further, if $fu = u$, then $q(u, u) = \theta$.

Proof. For arbitrary $x_0 \in X$, consider the sequence $\{x_n\}$ with $x_n = f^n x_0$, where $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some n , then x_n will be a fixed point of f . Hence, let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Consider $x = x_{n-1}$ and $y = x_n$ in (11). By a simple calculation, we have

$$q(x_n, x_{n+1}) = q(fx_{n-1}, fx_n) \preceq (\alpha_1 + \alpha_2 + s\alpha_4)q(x_{n-1}, x_n) + (\alpha_3 + s\alpha_4)q(x_n, x_{n+1}),$$

which implies that

$$q(x_n, x_{n+1}) \preceq \frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - (\alpha_3 + s\alpha_4)} q(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Now, by repeating the procedure, we obtain $q(x_n, x_{n+1}) \preceq \lambda^n q(x_0, x_1)$ for all $n \in \mathbb{N}$, where $0 \leq \lambda = \frac{\alpha_1 + \alpha_2 + s\alpha_4}{1 - (\alpha_3 + s\alpha_4)} < \frac{1}{s}$ by (12). By using a similar procedure in Theorem 2, we can conclude that f has a fixed point $z \in X$. Now, let $fu = u$. Then, from (11), we have

$$q(u, u) = q(fu, fu) \preceq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)q(u, u)$$

which means that $q(u, u) = \theta$ by (12) and (*). This completes the proof. \square

Remark 6. Note that we also can obtain the proof of the Theorem 4 by considering $y = fx$ in (11) and applying Theorem 2. Moreover, assume that from conditions (i) and (ii) in Theorem 4 only (i) is satisfied. By choosing proper α_i , respectively, we can obtain Banach-type, Kannan-type, Cho-type [21,31,34] fixed point results with respect to a w - b -cone distance in tv s-cone b -metric spaces as follows:

$$\begin{aligned} q(fx, fy) &\preceq \lambda q(x, y), & \lambda &\in [0, \frac{1}{s}); \\ q(fx, fy) &\preceq \lambda(q(x, fx) + q(y, fy)), & \lambda &\in [0, \frac{1}{s+1}); \\ q(fx, fy) &\preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy), & s(\alpha_1 + \alpha_2) + \alpha_3 &< 1. \end{aligned}$$

In Theorem 4 and Remark 6, let $s = 1$. We have the same results under a w -cone distance.

In the sequel, we discuss on the equivalence some w - b -cone distance and wt -distance introduced by Hussain et al. [9]. Let V be an absolutely convex and absorbing subset of E . Remember the Minkowski functional $q_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$ for $x \in E$ is a semi-norm on E [28]. Also, $W \subset V$ implies that $q_V(x) \leq q_W(x)$ for $x \in E$.

Now, if (E, P) is an ordered tv s with solid cone P and $e \in \text{int } P$, then $[-e, e] = (P - e) \cap (e - P)$ will be an absolutely convex neighborhood of θ [29], and q_e denotes its Minkowski functional $q_{[-e, e]}$. Also, $\text{int } [-e, e] = (\text{int } P - e) \cap (e - \text{int } P)$. Moreover, q_e is a non-decreasing function on P .

Theorem 5. Let P be a solid cone, $e \in \text{int } P$ and q_e be the corresponding Minkowski functional of $[-e, e]$. Consider a tv s-cone b -metric space (X, d) with a w - b -cone distance q on X , and a b -metric space (X, d_q) with $d_q = q_e \circ d$. Then $p = q_e \circ q$ is a wt -distance on X .

Proof. We must prove the conditions of Definition 3 are hold. Since q_e is a semi-norm and q is a w - b -cone distance, we have

$$q_e(q(x, z)) \leq s[q_e(q(x, y)) + q_e(q(y, z))]$$

for all $x, y, z \in X$, which implies that $p(x, z) \leq s[p(x, y) + p(y, z)]$. Thus, the condition (w_1) of the Definition 3 holds. Now, let $x \in X$ and denote $f(y) = p(x, y)$. We show that g is lower b -semicontinuous. Let $y, y_n \in X$ such that y_n convergent to y in d_q . It is equivalent to $y_n \rightarrow y$ in d . Since q is lower b -semicontinuous, for given $\epsilon > 0$, $q(x, y) \preceq sq(x, y_n) + \epsilon e$ for n large enough. Since q_e is a semi-norm, then $p(x, y) \leq sp(x, y_n) + \epsilon$; that is, $f = p(x, \cdot)$ is also lower b -semicontinuous. Thus, the condition (w_2) of the Definition 3 holds. In order to proof the validity of (w_3) , select arbitrary ϵ and ϵ' , and set $c = \epsilon e$. Clearly, $c \in \text{int } P$ and hence, by (q_3) , there exist $\theta \ll c' = \epsilon' e$ with $q(z, x) \ll c'$ and $q(z, y) \ll c'$ implying that $d(x, y) \ll c$. By Minkowski functional q_e , we have $p(z, x) < \epsilon'$ and $p(z, y) < \epsilon'$, which implies that $d_q(x, y) < \epsilon$. This completes the proof. \square

Note that all examples about the existence of fixed points in the framework of w -cone distance and generalized c -distance can be obtained in the version of w - b -cone distance by considering introduced distances in Example 2 and applying Theorem 5.

An order \subseteq is called a partial order or an order relation if it is reflexive, symmetric and transitive. Now, we consider the main theorem of Hussain et al. [9] in the framework of w - b -cone distance in ordered tv s cone b -metric spaces by applying the assertion of Theorem 5.

Theorem 6. Consider a complete partially ordered tv s-cone b -metric space (X, d) with parameter $s \geq 1$ and w - b -cone distance q on X . Assume that $f : X \rightarrow X$ is a non-decreasing mapping such that

$$q(fx, f^2x) \preceq rq(x, fx) \quad (13)$$

for all $x \subseteq fx$, where $r \in [0, 1/s)$. Also, suppose that the following conditions hold:

(i) for all $x \in X$ with $x \subseteq fx$, there exists $c \in \text{int } P$ such that

$$c \ll q(x, y) + q(x, fx) \quad (14)$$

for each y with $y \neq fy$;

(ii) there is $x_0 \in X$ such that $x_0 \subseteq fx_0$.

Then f has a fixed point in X .

Proof. Let $e \in \text{int } P$ and q_e be a Minkowski functional, $p = q_e \circ q$ be defined by Theorem 5 and $d_q = q_e \circ d$ be defined by Lemma 4 of Kadelburg et al. [19]. Then, by q_e and from (13), we have

$$p(fx, f^2x) \preceq rp(x, fx)$$

for all $x \in X$, where $r \in [0, 1/s)$. Also, by q_e and (14), we obtain $\inf\{p(x, y) + p(x, fx)\} > 0$ for all $x \in X$ with $x \subseteq fx$ and all y with $y \neq fy$. Thus, the conclusion follows from Theorem 4.2 of [9]. \square

Remark 7. Similar to the above theorem, one can obtain all results of Hussain et al.'s paper [9] in the framework of w - b -cone distance in ordered tv s cone b -metric spaces by applying the assertion of Theorem 5. Also, in Theorem 6, set $s = 1$. Then we can obtain this theorem in the framework of a w -cone distance.

3. Conclusions

In this paper, we defined the concept of a w - b -cone distance in tv s-cone b -metric spaces and discussed on its properties. Note that a w - b -cone distance with $s = 1$ is a w -cone distance. Hence, the class of w - b -cone distances in tv s-cone b -metric spaces is bigger than the class of w -cone distances in tv s-cone metric spaces. Also, a generalized c -distance is a w - b -cone distance, but the converse is not true. Hence, the class of w - b -cone distances in tv s-cone b -metric spaces is bigger than the class of generalized c -distances in cone b -metric spaces. Our results are significant, since all of the results in fixed point theory with respect to a generalized c -distance can be introduced in the version of w - b -cone distance. Also, by Theorem 5, if a theorem proved in the version of wt -distance, then this theorem is held in the version of w - b -cone distance. Thus, henceforth, the published results about fixed point theorems with respect to a w - b -cone distance or a generalized c -distance (a w -cone distance or a c -distance) in tv s-cone b -metric spaces (tv s-cone metric spaces) are not new and real unless they did not proved in the version of a wt -distance (a w -distance) in b -metric spaces (metric spaces). As a new subject, one can check this idea about distances in various cone b -metric spaces such as: generalized cone metric spaces, generalized cone b -metric spaces and etc.

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