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Hermite–Hadamard and Fejér Inequalities for Co-Ordinated (F, G)-Convex Functions on a Rectangle

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Abstract: We introduce the notion of a co-ordinated (F, G)-convex function defined on an interval in $\mathbb{R}^2$ and we prove the Hermite–Hadamard and Fejér type inequalities for such functions.

Keywords: Hermite–Hadamard inequality; Fejér inequality; approximate convexity

MSC: 26A51; 26B25

1. Introduction

The celebrated inequality states that, if $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Furthermore, if $p : [a, b] \to [0, \infty)$ is an integrable function symmetric with respect to $\frac{a+b}{2}$, that is

$$p(a+b-x) = p(x) \quad \text{for} \quad x \in [a, b],$$

then the following weighted generalization of the Hermite–Hadamard inequality is known as the Fejér inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx} \leq \frac{f(a)+f(b)}{2}.$$

Dragomir [1] established a counterpart of the Hermite–Hadamard inequality for co-ordinated convex functions, that is functions $f : [a, b] \times [c, d] \to \mathbb{R}$ which are convex with respect to each variable separately. It has been proven in [1] that for such functions, the following inequalities hold

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

Refinement versions of these inequalities have been presented in [1–3].

A counterpart of the Fejér inequality for co-ordinated convex functions has been formulated by Alomari and Darus [4]. They proved that if $p : [a,b] \times [c,d] \to [0,\infty)$ is an integrable function symmetric with respect to the lines $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, i.e.,

$$p(a+b-x,y) = p(x,y) \quad \text{for} \quad x \in [a,b],\ y \in [c,d] \tag{1}$$

and

$$p(x,c+d-y) = p(x,y) \quad \text{for} \quad x \in [a,b],\ y \in [c,d], \tag{2}$$

then for every co-ordinated convex function the following inequalities hold

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{\int_a^b \int_c^d f(x,y)p(x,y)dydx}{\int_a^b \int_c^d p(x,y)dydx} \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

In recent years, several modifications of the notion of convexity were studied by many authors (see e.g., [5–9]). The following general definition was introduced in [10].

Definition 1. *Let $F : [0,1] \times [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. A function $f : [a,b] \to \mathbb{R}$ is said to be convex with respect to F, or briefly F-convex, provided*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+F(t,x,y) \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1]. \tag{3}$$

In particular, if F is of the form

$$F(t,x,y) = Ct(1-t)|x-y| \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1], \tag{4}$$

where $C \in (0,\infty)$, then any function $f : [a,b] \to \mathbb{R}$ satisfying (3) is called approximately convex. Furthermore, if $f : [a,b] \to \mathbb{R}$ satisfies (3) with F given by

$$F(t,x,y) = -Ct(1-t)(x-y)^2 \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1], \tag{5}$$

where $C \in (0,\infty)$, then it is called strongly convex with modulus C. For some applications of F-convex functions in the optimization theory and in the theory of partial differential equations we refer to [11] and [12], respectively.

It should be noted here that, although a definition of the F-convex function does not require any additional properties of F, it is reasonable to assume that F is symmetric, that is

$$F(1-t,y,x) = F(t,x,y) \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1]. \tag{6}$$

In fact, if f is F-convex then there exists a symmetric function F_s such that f is F_s-convex and

$$F_s(t,x,y) \leq F(t,x,y) \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1].$$

To find this, one could take

$$F_s(t,x,y) := \min\{F(t,x,y),F(1-t,y,x)\} \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1].$$

Note that F given by (4) or (5) is symmetric. Moreover, a symmetry of F is a necessary condition for the existence of an F-affine function, i.e., a function satisfying equation

$$f(tx+(1-t)y) = tf(x)+(1-t)f(y)+F(t,x,y) \quad \text{for} \quad x,y \in [a,b],\ t \in [0,1].$$

In what follows we deal with the functions of two variables, which are F-convex with respect to each variable.

Definition 2. *Let $F : [c,d] \times [0,1] \times [a,b] \times [a,b] \to \mathbb{R}$, $G : [a,b] \times [0,1] \times [c,d] \times [c,d] \to \mathbb{R}$ be continuous functions. We call a function $f : [a,b] \times [c,d] \to \mathbb{R}$ co-ordinated (F,G)-convex, provided*

$$f(tx_1+(1-t)x_2,y) \le tf(x_1,y)+(1-t)f(x_2,y)+F(y,t,x_1,x_2),$$

$$f(x,ty_1+(1-t)y_2) \le tf(x,y_1)+(1-t)f(x,y_2)+G(x,t,y_1,y_2)$$

for $t \in [0,1]$, $x_1,x_2 \in [a,b]$, $y_1,y_2 \in [c,d]$, $x \in [a,b]$, $y \in [c,d]$.

Following the remark formulated above, we restrict our attention to the case where $F(y,\cdot,\cdot,\cdot)$ for $y \in [c,d]$ and $G(x,\cdot,\cdot,\cdot)$ for $x \in [a,b]$ are symmetric functions, i.e.,

$$F(y,1-t,x_2,x_1) = F(y,t,x_1,x_2) \quad \text{for} \quad x_1,x_2 \in [a,b],\ y \in [c,d],\ t \in [0,1]$$

and

$$G(x,1-t,y_2,y_1) = G(x,t,y_1,y_2) \quad \text{for} \quad x \in [a,b],\ y_1,y_2 \in [c,d],\ t \in [0,1],$$

respectively. This assumption will not be repeated. Our main aim is to present the Hermite–Hadamard and the Fejér type inequalities for co-ordinated (F,G)-convex functions.

2. Results

2.1. Hermite–Hadamard Type Inequalities

In this section, we prove the Hermite–Hadamard type inequalities for (F,G)-convex functions. Our proof is based on some methods used in [1,3]. We begin with the result establishing the Hermite–Hadamard type inequalities for F-convex functions. It will be useful in further considerations.

Theorem 1. *Let $F : [0,1] \times [a,b] \times [a,b] \to \mathbb{R}$ be a continuous symmetric function (cf. (6)). If $f : [a,b] \to \mathbb{R}$ is an integrable F-convex function then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b F\left(\frac{1}{2},x,a+b-x\right)dx \tag{7}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} + \int_0^1 F(t,a,b)dt. \tag{8}$$

Proof. Assume that $f : [a,b] \to \mathbb{R}$ is an integrable F-convex function. In view of (3), we obtain

$$\frac{1}{b-a}\int_a^b f(s)ds = \int_0^1 f(ta+(1-t)b)dt \le \frac{1}{2}f(a)+\frac{1}{2}f(b)+\int_0^1 F(t,a,b)dt,$$

which gives (8). Note also that, as f is F-convex, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + F\left(\frac{1}{2},x,y\right) \quad \text{for} \quad x,y \in [a,b]. \tag{9}$$

Setting in (9) $x = ta + (1-t)b$, $y = tb + (1-t)a$, where $t \in [0,1]$, and integrating obtained in this way inequality with respect to t, we obtain (7). □

Now, we are going to formulate and prove the Hermite–Hadamard type inequalities for co-ordinated (F,G)-convex functions.

Theorem 2. *Assume that $f : [a,b] \times [c,d] \to \mathbb{R}$ is an integrable co-ordinated (F,G)-convex function. Then:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] + R_1, \tag{10}$$

where

$$R_1 = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(\frac{c+d}{2}, \frac{1}{2}, x, a+b-x\right)dx + \frac{1}{d-c}\int_c^d G\left(\frac{a+b}{2}, \frac{1}{2}, y, c+d-y\right)dy\right];$$

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + R_2, \tag{11}$$

where

$$R_2 = \frac{1}{2(b-a)(d-c)}\left[\int_a^b\int_c^d G\left(x, \frac{1}{2}, y, c+d-y\right)dydx + \int_a^b\int_c^d F\left(y, \frac{1}{2}, x, a+b-x\right)dydx\right];$$

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ \leq\ &\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] + R_3,\end{aligned} \tag{12}$$

where

$$R_3 = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\int_0^1 G(x,t,c,d)dtdx + \frac{1}{d-c}\int_c^d\int_0^1 F(y,t,a,b)dtdy\right];$$

and

$$\begin{aligned}&\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + R_4,\end{aligned} \tag{13}$$

where

$$R_4 = \frac{1}{4}\left[\int_0^1 F(c,t,a,b)dt + \int_0^1 F(d,t,a,b)dt + \int_0^1 G(a,t,c,d)dt + \int_0^1 G(b,t,c,d)dt\right].$$

Proof. Note that, for every $x \in [a,b]$, the function $f(x,\cdot)$ is $G(x,\cdot,\cdot,\cdot)$-convex. Thus, applying Theorem 1, we obtain

$$f\left(x, \frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f(x,y)dy + \frac{1}{d-c}\int_c^d G\left(x, \frac{1}{2}, y, c+d-y\right)dy$$

$$\leq \frac{f(x,c)+f(x,d)}{2} + \int_0^1 G(x,t,c,d)dt + \frac{1}{d-c}\int_c^d G\left(x,\frac{1}{2},y,c+d-y\right)dy.$$

Integrating this inequality with respect to x, we find

$$\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx$$

$$\leq \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d f(x,y)dydx + \int_a^b\int_c^d G\left(x,\frac{1}{2},y,c+d-y\right)dydx\right]$$

$$\leq \frac{1}{2(b-a)}\left[\int_a^b f(x,c)dx + \int_a^b f(x,d)dx\right]$$

$$+\frac{1}{b-a}\int_a^b\int_0^1 G(x,t,c,d)dtdx + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d G\left(x,\frac{1}{2},y,c+d-y\right)dydx.$$

Moreover, since for every $y \in [c,d]$, $f(\cdot,y)$ is $F(y,\cdot,\cdot,\cdot)$-convex, using the similar arguments, we conclude that

$$\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$

$$\leq \frac{1}{(b-a)(d-c)}\left[\int_c^d\int_a^b f(x,y)dxdy + \int_c^d\int_a^b F\left(y,\frac{1}{2},x,a+b-x\right)dxdy\right]$$

$$\leq \frac{1}{2(d-c)}\left[\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right]$$

$$+\frac{1}{d-c}\int_c^d\int_0^1 F(y,t,a,b)dtdy + \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b F\left(y,\frac{1}{2},x,a+b-x\right)dxdy.$$

Adding up these inequalities, we obtain (11) and (12).

Since $f(\cdot,\frac{c+d}{2})$ is $F(\frac{c+d}{2},\cdot,\cdot,\cdot)$-convex and $f(\frac{a+b}{2},\cdot)$ is $G(\frac{a+b}{2},\cdot,\cdot,\cdot)$-convex, taking into account the first inequality in Theorem 1, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{b-a}\int_a^b F\left(\frac{c+d}{2},\frac{1}{2},x,a+b-x\right)dx$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy + \frac{1}{d-c}\int_c^d G\left(\frac{a+b}{2},\frac{1}{2},y,c+d-y\right)dy.$$

Adding them up we obtain (10).

Finally, as $f(\cdot,c)$, $f(\cdot,d)$, $f(a,\cdot)$ and $f(b,\cdot)$ are $F(c,\cdot,\cdot,\cdot)$-, $F(d,\cdot,\cdot,\cdot)$-, $G(a,\cdot,\cdot,\cdot)$- and $G(b,\cdot,\cdot,\cdot)$-convex, respectively, applying the second inequality in Theorem 1, we find

$$\frac{1}{b-a}\int_a^b f(x,c)dx \leq \frac{f(a,c)+f(b,c)}{2} + \int_0^1 F(c,t,a,b)dt,$$

$$\frac{1}{b-a}\int_a^b f(x,d)dx \leq \frac{f(a,d)+f(b,d)}{2} + \int_0^1 F(d,t,a,b)dt,$$

$$\frac{1}{d-c}\int_c^d f(a,y)dy \le \frac{f(a,c)+f(a,d)}{2} + \int_0^1 G(a,t,c,d)dt$$

and

$$\frac{1}{d-c}\int_c^d f(b,y)dy \le \frac{f(b,c)+f(b,d)}{2} + \int_0^1 G(b,t,c,d)dt.$$

Adding up these inequalities, we obtain (13). □

2.2. Fejér Type Inequalities

In order to prove the Fejér type inequalities for co-ordinated (F,G)-convex functions we need the following auxiliary result.

Lemma 1. *Assume that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a co-ordinated (F,G)-convex function.*

(i) If $[x_1,x_2] \subset [x_1',x_2'] \subset [a,b]$ and $x_1+x_2 = x_1'+x_2'$ then

$$f(x_1,y)+f(x_2,y) \le f(x_1',y)+f(x_2',y) + F\left(y, \frac{x_2'-x_1}{x_2'-x_1'}, x_1', x_2'\right) + F\left(y, \frac{x_2'-x_2}{x_2'-x_1'}, x_1', x_2'\right)$$

for $y \in [c,d]$.

(ii) If $[y_1,y_2] \subset [y_1',y_2'] \subset [c,d]$ and $y_1+y_2 = y_1'+y_2'$ then

$$f(x,y_1)+f(x,y_2) \le f(x,y_1')+f(x,y_2') + G\left(x, \frac{y_2'-y_1}{y_2'-y_1'}, y_1', y_2'\right) + G\left(x, \frac{y_2'-y_2}{y_2'-y_1'}, y_1', y_2'\right)$$

for $x \in [a,b]$.

Proof. We prove only the first part of the lemma since the proof of the second part is similar. Assume that $[x_1,x_2] \subset [x_1',x_2'] \subset [a,b]$ and $x_1+x_2 = x_1'+x_2'$. Since

$$x_1 = \frac{x_2'-x_1}{x_2'-x_1'}x_1' + \frac{x_1-x_1'}{x_2'-x_1'}x_2'$$

and

$$x_2 = \frac{x_2'-x_2}{x_2'-x_1'}x_1' + \frac{x_2-x_1'}{x_2'-x_1'}x_2',$$

for every $y \in [c,d]$, we obtain

$$f(x_1,y)+f(x_2,y) \le \frac{x_2'-x_1}{x_2'-x_1'}f(x_1',y) + \frac{x_1-x_1'}{x_2'-x_1'}f(x_2',y) + F\left(y, \frac{x_2'-x_1}{x_2'-x_1'}, x_1', x_2'\right)$$
$$+\frac{x_2'-x_2}{x_2'-x_1'}f(x_1',y) + \frac{x_2-x_1'}{x_2'-x_1'}f(x_2',y) + F\left(y, \frac{x_2'-x_2}{x_2'-x_1'}, x_1', x_2'\right)$$

$$= \frac{2x_2'-(x_1+x_2)}{x_2'-x_1'}f(x_1',y) + \frac{x_1+x_2-2x_1'}{x_2'-x_1'}f(x_2',y) + F\left(y, \frac{x_2'-x_1}{x_2'-x_1'}, x_1', x_2'\right) + F\left(y, \frac{x_2'-x_2}{x_2'-x_1'}, x_1', x_2'\right)$$

$$= f(x_1',y)+f(x_2',y) + F\left(y, \frac{x_2'-x_1}{x_2'-x_1'}, x_1', x_2'\right) + F\left(y, \frac{x_2'-x_2}{x_2'-x_1'}, x_1', x_2'\right).$$

□

In the next theorem we establish the Fejér type inequalities for (F,G)-convex functions.

Theorem 3. *Assume that $p : [a,b] \times [c,d] \to \mathbb{R}$ is a positive integrable function symmetric with respect to the lines $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ (cf. (1) and (2)). If $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous co-ordinated (F,G)-convex function such that fp is integrable on $[a,b] \times [c,d]$ then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{\int_a^b \int_c^d f(x,y)p(x,y)dydx + K}{\int_a^b \int_c^d p(x,y)dydx}, \tag{14}$$

where

$$K = 2\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} G\left(x, \frac{1}{2}, y, c+d-y\right) p(x,y)dydx$$
$$+2\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} G\left(a+b-x, \frac{1}{2}, y, c+d-y\right) p(x,y)dydx$$
$$+4\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} F\left(\frac{c+d}{2}, \frac{1}{2}, x, a+b-x\right) p(x,y)dydx$$

and

$$\frac{\int_a^b \int_c^d f(x,y)p(x,y)dydx - L}{\int_a^b \int_c^d p(x,y)dydx} \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}, \tag{15}$$

where

$$L = L_1 + L_2 + L_3$$

$$:= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[F\left(y, \tfrac{b-x}{b-a}, a, b\right) + F\left(y, \tfrac{x-a}{b-a}, a, b\right) + F\left(c+d-y, \tfrac{b-x}{b-a}, a, b\right) + F\left(c+d-y, \tfrac{x-a}{b-a}, a, b\right)\right] p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[G\left(a, \frac{d-y}{d-c}, c, d\right) + G\left(a, \frac{y-c}{d-c}, c, d\right)\right] p(x,y)dydx$$
$$+\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[G\left(b, \frac{d-y}{d-c}, c, d\right) + G\left(b, \frac{y-c}{d-c}, c, d\right)\right] p(x,y)dydx.$$

Proof. Assume that $f : [a,b] \times [c,d] \to \mathbb{R}$ is an integrable co-ordinated (F,G)-convex function such that fp is integrable. Then, for every $x \in [a,b]$ and $y \in [c,d]$, we have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}f\left(x, \frac{c+d}{2}\right) + \frac{1}{2}f\left(a+b-x, \frac{c+d}{2}\right) + F\left(\frac{c+d}{2}, \frac{1}{2}, x, a+b-x\right)$$
$$\leq \frac{1}{4}f(x,y) + \frac{1}{4}f(x,c+d-y) + \frac{1}{4}f(a+b-x,y) + \frac{1}{4}f(a+b-x,c+d-y)$$
$$+\frac{1}{2}G\left(x, \frac{1}{2}, y, c+d-y\right) + \frac{1}{2}G\left(a+b-x, \frac{1}{2}, y, c+d-y\right) + F\left(\frac{c+d}{2}, \frac{1}{2}, x, a+b-x\right).$$

Therefore, as p is symmetric with respect to the lines $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we obtain

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx = 4 \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) p(x,y)dydx$$

$$\leq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx$$

$$+ \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x, c+d-y) + f(a+b-x, y)]p(x,y)dydx + K$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx$$

$$+ \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} [f(a+b-x, c+d-y) + f(x,y)]p(a+b-x, y)dydx + K$$

$$= \int_a^b \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx + K$$

$$= \int_a^b \int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx + \int_a^b \int_c^{\frac{c+d}{2}} f(a+b-x, c+d-y)p(x,y)dydx + K$$

$$= \int_a^b \int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx + \int_a^b \int_{\frac{c+d}{2}}^d f(x,y)p(a+b-x, c+d-y)dydx + K$$

$$= \int_a^b \int_c^d f(x,y)p(x,y)dydx + K.$$

Thus, (14) holds.

Furthermore, using again the symmetry of p and applying Lemma 1 to $[y, c+d-y] \subset [c,d]$ and $[x, a+b-x] \subset [a,b]$, where $x \in [a, \frac{a+b}{2}]$, $y \in [c, \frac{c+d}{2}]$, we have

$$\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \int_a^b \int_c^d p(x,y)dydx$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(a,c) + f(a,d) + f(b,c) + f(b,d)]p(x,y)dydx$$

$$\geq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[f(a,y) + f(a, c+d-y) - G\left(a, \frac{d-y}{d-c}, c, d\right) - G\left(a, \frac{y-c}{d-c}, c, d\right)\right.$$

$$\left. + f(b,y) + f(b, c+d-y) - G\left(b, \frac{d-y}{d-c}, c, d\right) - G\left(b, \frac{y-c}{d-c}, c, d\right)\right] p(x,y)dydx$$

$$\geq \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left[f(x,y) + f(a+b-x, y) - F\left(y, \frac{b-x}{b-a}, a, b\right) - F\left(y, \frac{x-a}{b-a}, a, b\right)\right.$$

$$+ f(x, c+d-y) + f(a+b-x, c+d-y)$$

$$\left. - F\left(c+d-y, \frac{b-x}{b-a}, a, b\right) - F\left(c+d-y, \frac{x-a}{b-a}, a, b\right)\right] p(x,y)dydx - (L_2 + L_3)$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx$$

$$+ \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(a+b-x,y) + f(x,c+d-y)]p(x,y)dydx - (L_1 + L_2 + L_3)$$

$$= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx$$

$$+ \int_{\frac{a+b}{2}}^{b} \int_c^{\frac{c+d}{2}} [f(a+b-x,c+d-y) + f(x,y)]p(a+b-x,y)dydx - L$$

$$= \int_a^{b} \int_c^{\frac{c+d}{2}} [f(x,y) + f(a+b-x, c+d-y)]p(x,y)dydx - L$$

$$= \int_a^{b} \int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx + \int_a^{b} \int_c^{\frac{c+d}{2}} f(a+b-x,c+d-y)p(x,y)dydx - L$$

$$= \int_a^{b} \int_c^{\frac{c+d}{2}} f(x,y)p(x,y)dydx + \int_a^{b} \int_{\frac{c+d}{2}}^{c} f(x,y)p(a+b-x,c+d-y)dydx - L$$

$$= \int_a^{b} \int_c^{d} f(x,y)p(x,y)dydx - L,$$

which gives (15). □

3. Discussion

In this paper the Hermite–Hadamard and Fejér type inequalities for co-ordinated (F, G)-convex functions are proved. Since every co-ordinated convex function is co-ordinated (F, G)-convex (with F and G being identically 0), from our results, one can easily deduce the results by Dragomir [1] and Alomari and Darus [4]. Furthermore, applying Theorems 2 and 3, one can obtain the Hermite–Hadamard and Fejér type inequalities for co-ordinated (C, D)-approximately convex functions and co-ordinated (C, D)-strongly convex functions defined by

$$f(tx_1 + (1-t)x_2, y) \leq tf(x_1, y) + (1-t)f(x_2, y) + D(y)t(1-t)|x_1 - x_2|,$$

$$f(x, ty_1 + (1-t)y_2) \leq tf(x, y_1) + (1-t)f(x, y_2) + C(x)t(1-t)|y_1 - y_2|$$

for $t \in [0,1]$, $x_1, x_2 \in [a,b]$, $y_1, y_2 \in [c,d]$, $x \in [a,b]$, $y \in [c,d]$; and

$$f(tx_1 + (1-t)x_2, y) \leq tf(x_1, y) + (1-t)f(x_2, y) - D(y)t(1-t)(x_1 - x_2)^2,$$

$$f(x, ty_1 + (1-t)y_2) \leq tf(x, y_1) + (1-t)f(x, y_2) - C(x)t(1-t)(y_1 - y_2)^2$$

for $t \in [0,1]$, $x_1, x_2 \in [a,b]$, $y_1, y_2 \in [c,d]$, $x \in [a,b]$, $y \in [c,d]$, respectively, where $C : [a,b] \to (0, \infty)$ and $D : [c,d] \to (0, \infty)$ are given functions.

Note also that from Theorem 1 the Hermite–Hadamard inequalities for approximately convex functions and strongly convex functions can be derived. Finally, applying Theorem 1, with $F \equiv 0$, we obtain the classical Hermite–Hadamard inequality.

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