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On a General Extragradient Implicit Method and Its Applications to Optimization

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Abstract: Let X be a Banach space with both q-uniformly smooth and uniformly convex structures. This article introduces and considers a general extragradient implicit method for solving a general system of variational inequalities (GSVI) with the constraints of a common fixed point problem (CFPP) of a countable family of nonlinear mappings $\{S_n\}_{n=0}^{\infty}$ and a monotone variational inclusion, zero-point, problem. Here, the constraints are symmetrical and the general extragradient implicit method is based on Korpelevich's extragradient method, implicit viscosity approximation method, Mann's iteration method, and the W-mappings constructed by $\{S_n\}_{n=0}^{\infty}$.

Keywords: variational inclusions; general extragradient implicit method; variational systems; norm convergence.

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1. Introduction

Let X be Banach space and J be duality set-valued mapping on X. Let $A_1 : C \to X$ and $A_2 : C \to X$ be two nonlinear nonself mappings of accretive type. In this work, we investigate the following symmetrical system problem:

$$\begin{cases} \langle \mu_1 A_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 A_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1}$$

with two real constants μ_1 and $\mu_2 > 0$. This is called a symmetrical variational system. This system was first introduced and studied in [1]. The symmetry system is quite applicable in lots of convex optimizations and finds a lot of applications in applied sciences, such as intensity modulated radiation therapy, signal processing, image reconstruction, and so on. Indeed, the model of these problems can be rewritten as a variational inequality, which is a special case of the system that is, the unconstrained minimization problem

$$\min_{x \in H} \bar{f}(x) := f(x) + I_C(x),$$

where $f : H \to R$ is a real-valued convex function that is assumed to be continuously differentiable and $I_C(x)$ is the indicator of C:

$$I_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

There are lot of numerical techniques for dealing with it; see, e.g., [2–9]. In addition, $x^* = y^*$, $A_1 = A_2 = A$ yield that Equation (1) becomes the generalized variational inequality, which consists of numerically getting $x^* \in C$ with $\langle J(x - x^*), Ax^* \rangle \geq 0$, where x is any vector in its subset C. The (generalized) variational inequality models lots of real applications, such as image reconstruction in emission tomography. In addition, one knows that projection methods are efficient for such a problem [10]. In 2006, Aoyama, Iiduka, and Takahashi [11] proposed and focused on a process and proved the norm convergence of the sequences defined by their process with the aid of the weak topology.

In 2013, in order to solve the above symmetrical variational system with common fixed points of a family of non-expansive self-mappings $\{S_n\}_{n=0}^{\infty}$ on C, Ceng et al. [12] investigated an implicit two-step iterative process via a relaxed gradient technique in a class of Banach spaces with restricted geometry structures. Let Π_C be a sunny non-expansive retraction operator onto set C, A_1 be α_1-inverse-strongly accretive nonself operator, A_2 be α_2-inverse-strongly accretive nonself operator from C to X, and f be a contraction self operator on C. Under the restriction $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{Fix}(\Pi_C(I-\mu_1 A_1)\Pi_C(I-\mu_2 A_2)) \neq \emptyset$, let $\{x_n\}$ be the vector sequence devised by

$$\begin{cases} y_n = (1-\alpha_n)\Pi_C(I-\mu_1 A_1)\Pi_C(I-\mu_2 A_2)x_n + \alpha_n f(y_n), \\ x_{n+1} = (1-\beta_n)S_n y_n + \beta_n x_n \quad \forall n \geq 0, \end{cases} \tag{2}$$

with $0 < \kappa_2 \mu_i < 2\alpha_i$ for $i = 1, 2$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are number sequences in $(0, 1)$ satisfying the conditions: $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \beta_n > 0$. They proved norm convergence of $\{x_n\}$ to $x^* \in \Omega$. Recently, this problem has attracted much attention from the authors working on convex believel problems; see, e.g., [13–19]

Meantime, in order to solve the Equation (1) with the common fixed point problem constraint of a countable family of non-expansive self-mappings $\{S_n\}_{n=0}^{\infty}$ on C, Song and Ceng [20] found a general iterative scheme in a Banach space with both uniformly convex and q-uniformly smooth structures (whose smoothness constant is κ_q, where $1 < q \leq 2$). Let Π_C, A_1, A_2, G be the same operators as above. One lets $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \neq \emptyset$ and suppose that f is L-Lipschitzian nonself mapping with constant $L \geq 0$ and F is a k-Lipschitz η-strongly accretive singel-valued noself operator. Let $\tau = \rho(\eta - \frac{\kappa_q \rho^{q-1} k^q}{q})$ and assume $0 < \rho^{q-1} < \frac{q\eta}{\kappa_q k^q}$, $0 < \mu_i^{q-1} < \frac{q\alpha_i}{\kappa_q}$, $\tau > L\gamma > 0$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n \Pi_C(I-\mu_1 A_1)\Pi_C(I-\mu_2 A_2)x_n, \\ x_{n+1} = \Pi_C[\alpha_n \gamma f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)S_n y_n] \quad \forall n \geq 0, \end{cases}$$

where $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\}$ are real control sequences processing parameter conditions. They claimed convergence of $\{x_n\}$ to $x^* \in \Omega$ in the sense of norm.

Suppose that A is a q-order α-inverse-strongly accretive self operator on X and $B : X \to 2^X$ is an accretive operator with the range of $(I+B)^{-1}$ filling the full space. In 2017, in order to solve the variational inclusion (VI) of obtaining $x^* \in X$ such that $0 \in (A+B)x^*$, Chang et al. [21] suggested and devised a viscosity implicit generalized rule in the setting of smooth Banach spaces that also processes uniform convex structures. They claimed that $\{x_n\}$ converges to $x^* \in \Omega$ in norm. The method employed by Chang et al. [21] has been applied to popular equilibrium problems; see, e.g., [22–27]

Motivated by the above research results, the purpose of this research is to obtain, on the Banach space with uniform convexness and q-uniform smoothness, for example, L_p with $p > 1$, a feasibility point in the solution set of the Equation (1) involving a CFPP of nonlinear operator $\{S_n\}_{n=0}^{\infty}$ and a variational inclusion (VI). We suggest and investigate a general method of gradient implicit typ, which is based on Korpelevich's extragradient method, the implicit viscosity approximation method, and the W-mappings constructed by $\{S_n\}_{n=0}^{\infty}$. We then prove the vector sequences devised and generated by the proposed implicit method to a solution of the symmetrical variational Equation (1) with the VI and CFPP constraints in the norm. Finally, our results are applied for solving the CFPP of non-expansive and strict pseudocontractive operators, and convex minimization problems in Hilbert spaces. Our results improve and extend some related recent results in [12,20,21,28,29].

2. Preliminaries

Let $q > 1$ be a real number. The set-valued duality mapping $J_q : X \to 2^{X^*}$ is defined as

$$J_q(x) := \{\phi \in X^* : \|x\|^q = \langle x, \phi \rangle \text{ and } \|x\|^{q-1} = \|\phi\|\} \quad \forall x \in X.$$

It is known that the duality mapping J_q defined above from X into the family of nonempty (by Hahn–Banach's theorem) weak* compact subsets of X^* satisfies, for all $x \in X$, $J_q(-x) = -J_q(x)$. Under the structures of smoothness and uniform convexness, one knows that there exists a continuous convex and strictly increasing function $g : [0, 2r] \to \mathbf{R}$ such that $0 = g(0)$ and $g(\|x - y\|) \leq \|x\|^2 + \|y\|^2 - 2\langle x, J(y)\rangle$ for all $x, y \in B_r = \{y \in X : \|y\| \leq r\}$. We suppose that Π maps C into some subset D. One recalls that Π is called a sunny provided that $t(x - \Pi(x)) + \Pi(x) \in C$ for $x \in C$ and $t \geq 0$, $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$. Π is a retraction provided $\Pi = \Pi^2$.

Lemma 1. *[30] We suppose that $q > 1$ and X is a q-uniformly smooth Banach space with the generalized duality mapping J_q. Then, for any given $x, y \in X$, the inequality holds: $\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle$ $\forall j_q(x + y) \in J_q(x + y)$ and* $\mathrm{Fix}((I + \lambda B)^{-1}(I - \lambda A)) = (A + B)^{-1}0$ *$\forall \lambda > 0$. Let α, β, and γ be three position real constants with $\alpha + \beta + \gamma = 1$. In addition, if X is uniformly convex, then there exists a continuous convex and strictly increasing function $g : [0, \infty) \to [0, \infty)$ with the restraint that $\|\alpha x + \beta y + \gamma y\|^2 + \alpha\beta g(\|x - y\|) \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|y\|^2$ for all $\alpha, \beta, \gamma \in [0, 1]$.*

Proposition 1. *[31] We suppose that X is q-uniformly smooth space with $q \in (1, 2]$. Then, $\kappa_q\|y\|^q + \|x\|^q \geq \|x + y\|^q - q\langle y, J_q(x)\rangle$ for any vectors $x \in X$, $y \in X$. If $q = 2$, the special case, then $\kappa_2\|y\|^2 + \|x\|^2 \geq \|x + y\|^2 - 2\langle y, J_2(x)\rangle$ for any vectors $x \in X, y \in X$.*

From now on, one assumes that A is a set-valued operator from C to 2^X. A is called an accretive operator if $\langle j_q(x - y), u - v\rangle \geq 0$, where $j_q(x - y) \in J_q(x - y)$, $\forall u \in Ax, v \in Ay$. A is called an α-inverse-strongly accretive operator $\langle j_q(x - y), u - v\rangle \geq \alpha\|Ax - Ay\|^q$, where $j_q(x - y) \in J_q(x - y)$, $\alpha > 0, \forall u \in Ax, v \in Ay$. For all $\lambda > 0$, $X = (I + \lambda A)C$. Then, A is called m-accretive. On the class of m-accretive operators, one can get a back-ward operator $J^A_\lambda = (I + \lambda A)^{-1}$, which is commonly called the resolvent operator of A .

Lemma 2. *[32] In a Banach space X, one has $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)$, $\forall x \in X, \mu, \lambda > 0$. Let J^A_λ be the associated resolvent operator of A. Thus, J^A_λ is a single-valued Lipschitz continuous operator* $\mathrm{Fix}(J^A_\lambda) = A^{-1}0$, *where $A^{-1}0 = \{x \in C : 0 \in Ax\}$; if the setting is reduced to Hilbert spaces, m-accretiveness is equivalent to the maximal monotonicity.*

Proposition 2. *[33] Let X be a uniformly convex and q-uniformly smooth Banach space. Assume that $r > 0$ is some positive real number and A is a single-valued accretive with the inverse-strongly accretiveness and B is an accretive operator with $X = (I + \lambda B)C$. α-inverse-strongly accretive mapping of order q and $B : C \to 2^X$ is an m-accretive operator. Thus, there exists a continuous convex and strictly increasing function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\phi(0) = 0$ such that*

$$\|T_\lambda x - T_\lambda y\|^q + \lambda(\alpha q - \lambda^{q-1}\kappa_q)\|Ax - Ay\|^q \leq \|x - y\|^q - \phi(\|(I - J^B_\lambda)(I - \lambda A)y - (I - J^B_\lambda)(I - \lambda A)x\|),$$

for all $x, y \in \widetilde{B}_r$, a ball in C, where κ_q is the q-uniformly smooth constant of X. In particular, if $0 < \lambda^{q-1} \leq \frac{\alpha q}{\kappa_q}$, then T_λ is non-expansive.

Lemma 3. *[20] Let X be q-uniformly smooth and $A : C \to X$ be q-order α-inverse-strongly accretive. Then, the following inequality holds:*

$$\lambda(q\alpha - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q + \|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q.$$

In particular, if $0 < \lambda^{q-1} \leq \frac{q\alpha}{\kappa_q}$, then the complimentary operator of A processes the nonexpansivity. Suppose that Π_C is a non-expansive sunny retraction from X onto C. Let both the mapping $A_1 : C \to X$ and $A_2 : C \to X$ be inverse-strongly accretive and let G be a self-mapping on C defined by $Gx := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)\ \forall x \in C$. If $0 < \mu_i^{q-1} \leq \frac{q\alpha_i}{\kappa_q}$ for $i = 1, 2$, G is a non-expansive self-mapping C. (x^*, y^*), where both x^* and y^* are in C, solves the variational system (1) if and only if $x^* = \Pi_C(y^* - \mu_1 A_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 A_2 x^*)$.

Lemma 4. *[34] Let $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n\ \forall n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, where $\{\alpha_n\}$ is a sequence satisfying the condition $\limsup_{n\to\infty} \alpha_n < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$ and $\{x_n\}$ and $\{y_n\}$ are bounded sequences in a Banach space. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Let $\{\zeta_n\}$ be a real sequence in (0,1) and S_i a non-expansive mapping defined on C for each $i \in \{1, 2, \cdot, \}$. Next, one defines a mapping associated with n by

$$\begin{cases} U_{n,n} = \zeta_n S_n U_{n,n+1} + (1 - \zeta_n) I, \\ U_{n,n-1} = \zeta_{n-1} S_{n-1} U_{n,n} + (1 - \zeta_{n-1}) I, \\ \cdots \\ U_{n,k} = \zeta_k S_k U_{n,k+1} + (1 - \zeta_k) I, \\ \cdots \\ W_n = U_{n,0} = \zeta_0 S_0 U_{n,1} + (1 - \zeta_0) I, \end{cases} \tag{3}$$

where $U_{n,n+1} = I$. The W_n, called W-mapping, is a non-expansive mapping.

Lemma 5. *[35] Let $\{S_n\}_{n=0}^{\infty}$ be a countable family of non-expansive self-mappings on C, which is a subset of strictly convex space with $\cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \neq \emptyset$, and $\{\zeta_n\}_{n=0}^{\infty}$ be a real sequence such that $0 < \zeta_n \leq b < 1\ \forall n \geq 0$. Then, the following statements hold:*

- *(i) the limit $\lim_{n\to\infty} U_{n,k}x$ exists for all $x \in C$ and $k \geq 0$;*
- *(ii) W_n is non-expansive and $\mathrm{Fix}(W_n) = \cap_{i=0}^{n}\mathrm{Fix}(S_i)\ \forall n \geq 0$;*
- *(iii) the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0}x\ \forall x \in C$, is a non-expansive mapping satisfying $\mathrm{Fix}(W) = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n)$ and it is called the W-mapping generated by $S_0, S_1, \ldots$ and $\zeta_0, \zeta_1, \ldots$.*

Using the same arguments as in the proof of [[36], Lemma 4], we obtain the following.

Proposition 3. *Let $\{S_n\}_{n=0}^{\infty}$ and $\{\zeta_n\}_{n=0}^{\infty}$ be as in Lemma 5. Let D be any bounded set in C. One has $\lim_{n\to\infty} \sup_{x\in D} \|W_n x - Wx\| = 0$.*

Lemma 6. *[37] Let $a_{n+1} \leq a_n + \lambda_n \gamma_n - a_n \lambda_n\ \forall n \geq 0$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=0}^{\infty} |\lambda_n \gamma_n| < \infty$; $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$. Hence, $\lim_{n\to\infty} a_n = 0$.*

3. Convergence Results

Theorem 1. *Suppose that X is uniformly convex and q-uniformly, where $1 < q \leq 2$, smooth space. Suppose that B is a set-valued m-accretive operator and A is a single-valued α-inverse-strongly accretive operator. Suppose that A_1 is a single-valued α_1-inverse-strongly accretive operator and A_2 is a single-valued α_2-inverse-strongly accretive operator. Suppose that f is a contraction defined on set C with contractive efficient $\delta \in (0, 1)$ and $\{W_n\}$ is the sequence defined by Equation (3). Suppose that Π_C is a non-expansive sunny retraction from X onto set C and $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (B + A)^{-1}0 \neq \emptyset$, where*

GSVI(C, A_1, A_2) is the fixed point set of $G := \Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)$ with $0 < \mu_1^{q-1}\kappa_q < q\alpha_1$ and $0 < \mu_2^{q-1}\kappa_q < q\alpha_2$. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} u_n = \Pi_C(y_n - \mu_2 A_2 y_n), \\ v_n = \Pi_C(u_n - \mu_1 A_1 u_n), \\ y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1-t_n)J^B_{\lambda_n}(I - \lambda_n A)v_n) + \alpha_n f(x_n), \\ x_{n+1} = (1-\delta_n)W_n y_n + \delta_n x_n \quad n \ge 0, \end{cases} \tag{4}$$

where $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0,1)$ satisfy the following conditions:

- *(i)* $\alpha_n + \beta_n + \gamma_n = 1$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- *(ii)* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = \lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |t_n - t_{n-1}| = 0$;
- *(iii)* $\liminf_{n\to\infty} \gamma_n t_n(1-t_n) > 0$ *and* $\limsup_{n\to\infty} \gamma_n(1-t_n) < 1$;
- *(iv)* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
- *(v)* $0 < \bar{\lambda} \le \lambda_n \ \forall n \ge 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$.

Then, $x_n \to x^* \in \Omega$ strongly.

Proof. Re-write process Equation (4) as

$$\begin{cases} y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1-t_n)T_n G y_n) + \alpha_n f(x_n), \\ x_{n+1} = (1-\delta_n)W_n y_n + \delta_n x_n \quad n \ge 0, \end{cases} \tag{5}$$

where $T_n := J^B_{\lambda_n}(I - \lambda_n A) \ \forall n \ge 0$. From $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$ and Proposition 2, we observe, for each n, that T_n is a non-expansive self-mapping on C. Since $\alpha_n + \beta_n + \gamma_n = 1$, we know that

$$\alpha_n\delta + \beta_n + \gamma_n t_n + \gamma_n(1-t_n) = \alpha_n\delta + \gamma_n + \beta_n = 1 - \alpha_n(1-\delta).$$

For each n, one defines a self-mapping F_n on C by $F_n(x) = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n(t_n x_n + (1-t_n)T_n G x) \ \forall x \in C$. Thus,

$$\begin{aligned} \|F_n(x) - F_n(y)\| &= \gamma_n \|W_n(t_n x_n + (1-t_n)T_n Gx) - W_n(t_n x_n + (1-t_n)T_n Gy)\| \\ &\le \gamma_n(1-t_n)\|T_n Gx - T_n Gy\| \le \gamma_n(1-t_n)\|x - y\|. \end{aligned}$$

Since $0 < \gamma_n(1-t_n) < 1$, one has a unique vector $y_n \in C$ satisfying

$$y_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n(t_n x_n + (1-t_n)T_n G y_n).$$

The following proof is split to complete this conclusion. □

Step 1. We show that iterative sequence $\{x_n\}$ is bounded. Take a fixed $p \in \Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A+B)^{-1}0$ arbitrarily. Lemma 5 guarantees $W_n p = p$, $Gp = p$ and $T_n p = p$. Moreover, the nonexpansivity of T_n and G (due to Proposition 2) sends us to

$$\begin{aligned} &\|p - y_n\| = \|\beta_n(p - x_n) + \gamma_n[p - W_n(t_n x_n + (1-t_n)T_n G y_n)] + \alpha_n(p - f(x_n))\| \\ &\le \beta_n\|p - x_n\| + \gamma_n\|p - W_n(t_n x_n + (1-t_n)T_n G y_n)\| + \alpha_n(\|f(x_n) - f(p)\| + \|p - f(p)\|) \\ &\le \beta_n\|p - x_n\| + \gamma_n[t_n\|p - x_n\| + (1-t_n)\|p - T_n G y_n\|] + \alpha_n(\delta\|p - x_n\| + \|f(p) - p\|) \\ &\le \gamma_n(1-t_n)\|p - y_n\| + (\alpha_n\delta + \beta_n + \gamma_n t_n)\|p - x_n\| + \alpha_n\|f(p) - p\|, \end{aligned}$$

which therefore implies that

$$\begin{aligned}\|p-y_n\| &\leq \tfrac{\alpha_n\delta+\beta_n+\gamma_n t_n}{1-\gamma_n(1-t_n)}\|p-x_n\|+\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\| \\ &= \tfrac{1-\alpha_n(1-\delta)-\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}\|p-x_n\|+\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\| \\ &= (1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|p-x_n\|+\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|.\end{aligned} \tag{6}$$

Thus, from Equation (5), Equation (6), and Lemma 5 (i), we have

$$\begin{aligned}\|p-x_{n+1}\| &\leq (1-\delta_n)\|p-W_ny_n\|+\delta_n\|p-x_n\| \leq (1-\delta_n)\|p-y_n\|+\delta_n\|p-x_n\| \\ &\leq \delta_n\|p-x_n\|+(1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|p-x_n\|+\tfrac{\alpha_n}{1-\gamma_n(1-t_n)}\|f(p)-p\|\} \\ &= [1-\tfrac{(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\alpha_n]\|p-x_n\|+\tfrac{(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\alpha_n\tfrac{\|f(p)-p\|}{1-\delta} \leq \max\{\tfrac{\|f(p)-p\|}{1-\delta},\|p-x_n\|\}.\end{aligned}$$

One claims that all the iterative sequences are bounded.

Step 2. One proves that $\|x_{n+1}-x_n\|$ goes to 0 as n goes to ∞. By borrowing Equation (5), we have

$$z_n-z_{n-1}=t_n(x_n-x_{n-1})+(t_n-t_{n-1})(x_{n-1}-T_{n-1}Gy_{n-1})+(1-t_n)(T_nGy_n-T_{n-1}Gy_{n-1}),$$

and

$$\begin{aligned}y_n-y_{n-1} &= (\alpha_n-\alpha_{n-1})f(x_{n-1})+\alpha_n(f(x_n)-f(x_{n-1}))+\beta_n(x_n-x_{n-1}) \\ &\quad+(\beta_n-\beta_{n-1})x_{n-1}+\gamma_n(W_nz_n-W_{n-1}z_{n-1})+(\gamma_n-\gamma_{n-1})W_{n-1}z_{n-1}.\end{aligned} \tag{7}$$

By using Lemma 2 and Proposition 2, one deduces that

$$\begin{aligned}&\|T_nGy_n-T_{n-1}Gy_{n-1}\| \leq \|T_nGy_n-T_nGy_{n-1}\|+\|T_nGy_{n-1}-T_{n-1}Gy_{n-1}\| \\ &\leq \|y_n-y_{n-1}\|+\|J^B_{\lambda_n}(I-\lambda_nA)Gy_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)Gy_{n-1}\| \\ &\leq \|y_n-y_{n-1}\|+\|J^B_{\lambda_n}(I-\lambda_nA)Gy_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_nA)Gy_{n-1}\| \\ &\quad+\|J^B_{\lambda_{n-1}}(I-\lambda_nA)Gy_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)Gy_{n-1}\| \\ &= \|y_n-y_{n-1}\|+\|J^B_{\lambda_{n-1}}(\tfrac{\lambda_{n-1}}{\lambda_n}I+(1-\tfrac{\lambda_{n-1}}{\lambda_n})J^B_{\lambda_n})(I-\lambda_nA)Gy_{n-1} \\ &\quad-J^B_{\lambda_{n-1}}(I-\lambda_nA)Gy_{n-1}\|+\|J^B_{\lambda_{n-1}}(I-\lambda_nA)Gy_{n-1}-J^B_{\lambda_{n-1}}(I-\lambda_{n-1}A)Gy_{n-1}\| \\ &\leq \|y_n-y_{n-1}\|+|1-\tfrac{\lambda_{n-1}}{\lambda_n}|\|J^B_{\lambda_n}(I-\lambda_nA)Gy_{n-1}-(I-\lambda_nA)Gy_{n-1}\| \\ &\quad+|\lambda_n-\lambda_{n-1}|\|AGy_{n-1}\| \\ &\leq |\lambda_n-\lambda_{n-1}|M_1+\|y_n-y_{n-1}\|,\end{aligned} \tag{8}$$

where $\sup_{n\geq 1}\{\frac{1}{\lambda}\|J^B_{\lambda_n}(I-\lambda_nA)Gy_{n-1}-(I-\lambda_nA)Gy_{n-1}\|+\|AGy_{n-1}\|\} \leq M_1$ for some $M_1>0$. Thus, it follows from Equation (8) that

$$\begin{aligned}\|W_nz_n-W_{n-1}z_{n-1}\| &\leq \|W_nz_{n-1}-W_{n-1}z_{n-1}\|+\|W_nz_n-W_nz_{n-1}\| \\ &\leq |t_n-t_{n-1}|\|x_{n-1}-T_{n-1}Gy_{n-1}\|+t_n\|x_n-x_{n-1}\| \\ &\quad+(1-t_n)\|T_nGy_n-T_{n-1}Gy_{n-1}\|+\|W_nz_{n-1}-W_{n-1}z_{n-1}\| \\ &\leq |t_n-t_{n-1}|\|x_{n-1}-T_{n-1}Gy_{n-1}\|+t_n\|x_n-x_{n-1}\| \\ &\quad+(1-t_n)[\|y_n-y_{n-1}\|+|\lambda_n-\lambda_{n-1}|M_1]+\|W_nz_{n-1}-W_{n-1}z_{n-1}\|.\end{aligned}$$

This inequality together with Equation (7), implies that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \le |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \beta_n\|x_n - x_{n-1}\| + \alpha_n\|f(x_n) - f(x_{n-1})\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\|W_n z_n - W_{n-1}z_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|W_{n-1}z_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \alpha_n\delta\|x_n - x_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \gamma_n\{t_n\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\|x_{n-1} - T_{n-1}Gy_{n-1}\| \\
&\quad + (1 - t_n)[\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|M_1] + \|W_n z_{n-1} - W_{n-1}z_{n-1}\|\} \\
&\quad + |\gamma_n - \gamma_{n-1}|\|W_{n-1}z_{n-1}\| \\
&\le \gamma_n(1 - t_n)\|y_n - y_{n-1}\| + (\alpha_n\delta + \beta_n + \gamma_n t_n)\|x_{n-1} - x_n\| + (|\alpha_{n-1} - \alpha_n| \\
&\quad + |\beta_{n-1} - \beta_n| + |t_{n-1} - t_n| + |\gamma_{n-1} - \gamma_n| + |\lambda_{n-1} - \lambda_n|)M_2 + \|W_n z_{n-1} - W_{n-1}z_{n-1}\|,
\end{aligned}$$

where $\sup_{n\ge 0}\{\|f(x_n)\| + \|x_n\| + \|T_nGy_n\| + M_1 + \|W_n z_n\|\} \le M_2$ for some $M_2 > 0$. Then,

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \tfrac{\alpha_n\delta+\beta_n+\gamma_n t_n}{1-\gamma_n(1-t_n)}\|x_{n-1} - x_n\| + \tfrac{1}{1-\gamma_n(1-t_n)}[(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n| \\
&\quad + |t_{n-1} - t_n| + |\gamma_{n-1} - \gamma_n| + |\lambda_{n-1}| - \lambda_n)M_2 + \|W_{n-1}z_{n-1} - W_n z_{n-1}\|] \\
&= (1 - \tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n - x_{n-1}\| + \tfrac{1}{1-\gamma_n(1-t_n)}[(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\quad + |\gamma_n - \gamma_{n-1}| + |t_n - t_{n-1}| + |\lambda_n - \lambda_{n-1}|)M_2 + \|W_n z_{n-1} - W_{n-1}z_{n-1}\|] \\
&\le \|x_n - x_{n-1}\| + \tfrac{1}{1-\gamma_n(1-t_n)}[(|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| + |\alpha_{n-1} - \alpha_n| \\
&\quad + |t_{n-1} - t_n| + |\lambda_{n-1} - \lambda_n|)M_2 + \|W_n z_{n-1} - W_{n-1}z_{n-1}\|],
\end{aligned}$$

and hence

$$\begin{aligned}
&\|W_n y_n - W_{n-1}y_{n-1}\| \le \|W_n y_n - W_n y_{n-1}\| + \|W_n y_{n-1} - W_{n-1}y_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \tfrac{1}{1-\gamma_n(1-t_n)}[(|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| + |\alpha_{n-1} - \alpha_n| \\
&\quad + |t_{n-1} - t_n| + |\lambda_{n-1} - \lambda_n|)M_2 + \|W_n z_{n-1} - W_{n-1}z_{n-1}\|] + \|W_n y_{n-1} - W_{n-1}y_{n-1}\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|W_n y_n - W_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\| &\le \tfrac{1}{1-\gamma_n(1-t_n)}[(|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| + |\alpha_{n-1} - \alpha_n| \\
&\quad + |\lambda_{n-1} - \lambda_n| + |t_{n-1} - t_n|)M_2 + \|W_{n-1}z_{n-1} - W_n z_{n-1}\|] + \|W_n y_{n-1} - W_{n-1}y_{n-1}\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\sup_{x\in D}\|W_n x - Wx\| = 0$, where $D = \{y_n : n \ge 0\} \cup \{z_n : n \ge 0\}$ of C (due to Proposition 3), we know that

$$\lim_{n\to\infty}\|W_n y_{n-1} - W_{n-1}y_{n-1}\| = \lim_{n\to\infty}\|W_n z_{n-1} - W_{n-1}z_{n-1}\| = 0.$$

Note that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\lambda_n = \lambda$ and $\liminf_{n\to\infty}(1 - \gamma_n(1 - t_n)) > 0$. Since $|\beta_n - \beta_{n-1}|$, $|\gamma_n - \gamma_{n-1}|$ and $|t_n - t_{n-1}|$ all go to 0 as n goes to the infinity (due to conditions (ii), (iii)), one says

$$\limsup_{n\to\infty}(\|W_n y_n - W_{n-1}y_{n-1}\| - \|x_n - x_{n-1}\|) \le 0.$$

Lemma 4 guarantees $\lim_{n\to\infty}\|W_n y_n - x_n\| = 0$. Hence, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \delta_n)\|W_n y_n - x_n\| = 0. \tag{9}$$

Step 3. We show that $\|x_n - y_n\| \to 0$ and $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, for simplicity, we denote $\bar{p} := \Pi_C(I - \mu_2 A_2)p$. Note that $u_n = \Pi_C(I - \mu_2 A_2)y_n$ and $v_n = \Pi_C(I - \mu_1 A_1)u_n$. Then, $v_n = Gy_n$. From Lemma 3, we have

$$\begin{aligned}
\|u_n - \bar{p}\|^q &\le \|(I - \mu_2 A_2)y_n - (I - \mu_2 A_2)p\|^q \\
&\le \|y_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2 y_n - A_2 p\|^q.
\end{aligned} \tag{10}$$

In the same way, we get

$$\|v_n - p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q \le \|u_n - \bar{p}\|^q. \tag{11}$$

Substituting Equation (10) for Equation (11), we obtain

$$\|v_n - p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q \quad \le \|y_n - p\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2y_n - A_2p\|^q. \tag{12}$$

By Lemma 1, we infer from Equation (5) and Equation (12) that $\|z_n - p\|^q \le t_n\|x_n - p\|^q + (1 - t_n)\|v_n - p\|^q$, and hence

$$\begin{aligned}
&\|y_n - p\|^q = \|\beta_n(x_n - p) + \gamma_n(W_nz_n - p) + \alpha_n(f(p) - p) + \alpha_n(f(x_n) - f(p))\|^q \\
&\le \|\beta_n(x_n - p) + \gamma_n(W_nz_n - p) + \alpha_n(f(x_n) - f(p))\|^q + q\alpha_n\langle f(p) - p, J_q(y_n - p)\rangle \\
&\le \alpha_n\|f(x_n) - f(p)\|^q + \beta_n\|x_n - p\|^q + \gamma_n\|W_nz_n - p\|^q + q\alpha_n\langle f(p) - p, J_q(y_n - p)\rangle \\
&\le \alpha_n\delta\|x_n - p\|^q + \beta_n\|x_n - p\|^q + \gamma_n[t_n\|x_n - p\|^q + (1 - t_n)\|v_n - p\|^q] \\
&\quad + q\alpha_n\|f(p) - p\|\|y_n - p\|^{q-1} \\
&\le (\alpha_n\delta + \beta_n + \gamma_nt_n)\|x_n - p\|^q + \gamma_n(1 - t_n)[\|p - y_n\|^q - \mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2y_n - A_2p\|^q \\
&\quad - \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + q\alpha_n\|p - y_n\|^{q-1}\|p - f(p)\|.
\end{aligned}$$

It yields that

$$\begin{aligned}
\|y_n - p\|^q \quad &\le (1 - \tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n - p\|^q - \tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2y_n - A_2p\|^q \\
&\quad + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \tfrac{q\alpha_n}{1-\gamma_n(1-t_n)}\|p - y_n\|^{q-1}\|p - f(p)\|.
\end{aligned}$$

Combing this with Equation (5), one says

$$\begin{aligned}
&\|x_{n+1} - p\|^q \le \delta_n\|x_n - p\|^q + (1 - \delta_n)\|y_n - p\|^q \\
&\le \delta_n\|x_n - p\|^q + (1 - \delta_n)\{(1 - \tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n - p\|^q - \tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\times \\
&\quad \times \|A_2y_n - A_2p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \tfrac{q\alpha_n}{1-\gamma_n(1-t_n)}\|f(p) - p\|\|y_n - p\|^{q-1}\} \\
&= (1 - \tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n - p\|^q - \tfrac{\gamma_n(1-\delta_n)(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\times \\
&\quad \times \|A_2y_n - A_2p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \tfrac{q(1-\delta_n)\alpha_n}{1-\gamma_n(1-t_n)}\|f(p) - p\|\|y_n - p\|^{q-1} \\
&\le \|x_n - p\|^q - \tfrac{(1-\delta_n)\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2y_n - A_2p\|^q \\
&\quad + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] + \alpha_nM_3,
\end{aligned} \tag{13}$$

where $\sup_{n\ge0}\{\frac{q(1-\delta_n)}{1-\gamma_n(1-t_n)}\|p - y_n\|^{q-1}\|p - f(p)\|\} \le M_3$, where $M_3 > 0$ is a real. Thus, it follows from Equation (13) and Proposition 1 that

$$\begin{aligned}
&\tfrac{(1-\delta_n)\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[\mu_2(q\alpha_2 - \kappa_q\mu_2^{q-1})\|A_2y_n - A_2p\|^q + \mu_1(q\alpha_1 - \kappa_q\mu_1^{q-1})\|A_1u_n - A_1\bar{p}\|^q] \\
&\le \|x_n - p\|^q - \|x_{n+1} - p\|^q + \alpha_nM_3 \\
&\le q\|x_n - x_{n+1}\|\|x_{n+1} - p\|^{q-1} + \kappa_q\|x_n - x_{n+1}\|^q + \alpha_nM_3.
\end{aligned}$$

Since $0 < \mu_i^{q-1} < \frac{q\alpha_i}{\kappa_q}$ for $i = 1, 2$, from Equation (9), $\liminf_{n\to\infty}\gamma_n(1 - t_n) > 0$, $\liminf_{n\to\infty}(1 - \delta_n) > 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\|A_2y_n - A_2p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|A_1u_n - A_1\bar{p}\| = 0. \tag{14}$$

Utilizing Propositions 1 and 3, we have

$$\begin{aligned}
&\|u_n-\bar{p}\|^2=\|\Pi_C(I-\mu_2A_2)y_n-\Pi_C(I-\mu_2A_2)p\|^2\\
&\le\langle(I-\mu_2A_2)y_n-(I-\mu_2A_2)p,J(u_n-\bar{p})\rangle\\
&=\langle y_n-p,J(u_n-\bar{p})\rangle+\mu_2\langle A_2p-A_2y_n,J(u_n-\bar{p})\rangle\\
&\le\tfrac{1}{2}[\|y_n-p\|^2+\|u_n-\bar{p}\|^2-g_1(\|y_n-u_n-(p-\bar{p})\|)]+\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|,
\end{aligned}$$

which implies that

$$\|u_n-\bar{p}\|^2+g_1(\|y_n-u_n-(p-\bar{p})\|)\le\|y_n-p\|^2+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|. \tag{15}$$

Following the above line, one can derive

$$\|v_n-p\|^2+g_2(\|u_n-v_n+(p-\bar{p})\|)\le\|u_n-\bar{p}\|^2+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|. \tag{16}$$

Combining Equation (15) and Equation (16), one further derives

$$\begin{aligned}
&\|v_n-p\|^2+g_1(\|y_n-u_n-(p-\bar{p})\|)+g_2(\|u_n-v_n+(p-\bar{p})\|)\\
&\le\|y_n-p\|^2+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|.
\end{aligned} \tag{17}$$

Utilizing Lemma 1, we obtain from Equation (5) and Equation (17) that

$$\begin{aligned}
\|z_n-p\|^2 &\le(1-t_n)\|T_nGy_n-p\|^2-t_n(1-t_n)g_3(\|x_n-T_nGy_n\|)+t_n\|x_n-p\|^2\\
&\le(1-t_n)\|v_n-p\|^2-t_n(1-t_n)g_3(\|x_n-T_nGy_n\|)+t_n\|x_n-p\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|y_n-p\|^2\le+2\alpha_n\langle J(y_n-p),f(p)-p\rangle+\|\beta_n(x_n-p)+\gamma_n(W_nz_n-p)+\alpha_n(f(x_n)-f(p))\|^2\\
&\le\beta_n\|x_n-p\|^2+\gamma_n\|p-W_nz_n\|^2+\alpha_n\|f(p)-f(x_n)\|^2-\beta_n\gamma_ng_4(\|x_n-W_nz_n\|)\\
&\quad+2\alpha_n\langle f(p)-p,J(y_n-p)\rangle\\
&\le\alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[t_n\|x_n-p\|^2+(1-t_n)\|v_n-p\|^2\\
&\quad-t_n(1-t_n)g_3(\|x_n-T_nGy_n\|)]+2\alpha_n\|f(p)-p\|\|y_n-p\|-\beta_n\gamma_ng_4(\|x_n-W_nz_n\|)\\
&\le\alpha_n\delta\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\{t_n\|x_n-p\|^2+(1-t_n)[\|y_n-p\|^2\\
&\quad-g_1(\|y_n-u_n-(p-\bar{p})\|)-g_2(\|u_n-v_n+(p-\bar{p})\|)+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\
&\quad+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|]-t_n(1-t_n)g_3(\|x_n-T_nGy_n\|)\}+2\alpha_n\|f(p)-p\|\|y_n-p\|\\
&\quad-\beta_n\gamma_ng_4(\|x_n-W_nz_n\|)\\
&\le(\alpha_n\delta+\beta_n+\gamma_nt_n)\|x_n-p\|^2+\gamma_n(1-t_n)\|y_n-p\|^2-\gamma_n(1-t_n)[g_1(\|y_n-u_n-(p-\bar{p})\|)\\
&\quad+g_2(\|u_n-v_n+(p-\bar{p})\|)]+2\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|+2\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|\\
&\quad+2\alpha_n\|f(p)-p\|\|y_n-p\|-\gamma_nt_n(1-t_n)g_3(\|x_n-T_nGy_n\|)-\beta_n\gamma_ng_4(\|x_n-W_nz_n\|),
\end{aligned}$$

which immediately yields

$$\begin{aligned}
&\|y_n-p\|^2\le(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^2-\tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[g_1(\|y_n-u_n-(p-\bar{p})\|)\\
&\quad+g_2(\|u_n-v_n+(p-\bar{p})\|)]+\tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\|\\
&\quad+\mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\|+\alpha_n\|p-y_n\|\|f(p)-p\|]\\
&\quad-\tfrac{1}{1-\gamma_n(1-t_n)}[\gamma_nt_n(1-t_n)g_3(\|x_n-T_nGy_n\|)+\beta_n\gamma_ng_4(\|x_n-W_nz_n\|)].
\end{aligned}$$

This together with Equation (5) leads to

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \le \delta_n\|x_n-p\|^2 + (1-\delta_n)\|y_n-p\|^2 \\
&\le \delta_n\|x_n-p\|^2 + (1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^2 - \tfrac{\gamma_n(1-t_n)}{1-\gamma_n(1-t_n)}[g_1(\|y_n-u_n-(p-\bar{p})\|) \\
&\quad + g_2(\|u_n-v_n+(p-\bar{p})\|)] + \tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| \\
&\quad + \mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| + \alpha_n\|f(p)-p\|\|y_n-p\|] \\
&\quad - \tfrac{1}{1-\gamma_n(1-t_n)}[\gamma_n t_n(1-t_n)g_3(\|x_n-T_nGy_n\|) + \beta_n\gamma_n g_4(\|x_n-W_nz_n\|)]\} \\
&\le (1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-p\|^2 - \tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|) \\
&\quad + g_2(\|u_n-v_n+(p-\bar{p})\|)) + \gamma_n t_n(1-t_n)g_3(\|x_n-T_nGy_n\|) + \beta_n\gamma_n g_4(\|x_n-W_nz_n\|)] \\
&\quad + \tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| + \mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| \\
&\quad + \alpha_n\|f(p)-p\|\|y_n-p\|] \\
&\le \|x_n-p\|^2 - \tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|) + g_2(\|u_n-v_n+(p-\bar{p})\|)) \\
&\quad + \gamma_n t_n(1-t_n)g_3(\|x_n-T_nGy_n\|) + \beta_n\gamma_n g_4(\|x_n-W_nz_n\|)] + \tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\times \\
&\quad \times\|u_n-\bar{p}\| + \mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| + \alpha_n\|p-y_n\|\|p-f(p)\|].
\end{aligned}$$

It yields that

$$\begin{aligned}
&\tfrac{1-\delta_n}{1-\gamma_n(1-t_n)}[\gamma_n(1-t_n)(g_1(\|y_n-u_n-(p-\bar{p})\|) + g_2(\|u_n-v_n+(p-\bar{p})\|)) \\
&\quad + \gamma_n t_n(1-t_n)g_3(\|x_n-T_nGy_n\|) + \beta_n\gamma_n g_4(\|x_n-W_nz_n\|)] \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| \\
&\quad + \mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| + \alpha_n\|y_n-p\|\|f(p)-p\|] \\
&\le (\|x_{n+1}-p\| + \|x_n-p\|)\|x_n-x_{n+1}\| + \tfrac{2}{1-\gamma_n(1-t_n)}[\mu_2\|A_2p-A_2y_n\|\|u_n-\bar{p}\| \\
&\quad + \mu_1\|A_1\bar{p}-A_1u_n\|\|v_n-p\| + \alpha_n\|f(p)-p\|\|y_n-p\|].
\end{aligned}$$

Utilizing Equation (9) and Equation (14), from $\liminf_{n\to\infty}(1-\delta_n) > 0$, $\liminf_{n\to\infty}\gamma_n t_n(1-t_n) > 0$ and $\liminf_{n\to\infty}\beta_n\gamma_n > 0$, we conclude that $\lim_{n\to\infty} g_1(\|y_n-u_n-(p-\bar{p})\|) = 0$, $\lim_{n\to\infty} g_2(\|u_n-v_n+(p-\bar{p})\|) = 0$, $\lim_{n\to\infty} g_3(\|x_n-T_nGy_n\|) = 0$ and $\lim_{n\to\infty} g_4(\|x_n-W_nz_n\|) = 0$. Utilizing the properties of g_1, g_2, g_3 and g_4, we deduce that

$$\lim_{n\to\infty}\|y_n-u_n-(p-\bar{p})\| = \lim_{n\to\infty}\|u_n-v_n+(p-\bar{p})\| = \lim_{n\to\infty}\|x_n-T_nGy_n\| = \lim_{n\to\infty}\|x_n-W_nz_n\| = 0. \tag{18}$$

From Equation (18), we get

$$\|y_n-Gy_n\| = \|y_n-v_n\| \le \|y_n-u_n-(p-\bar{p})\| + \|u_n-v_n+(p-\bar{p})\| \to 0 \quad (n\to\infty). \tag{19}$$

In the meantime, again from Equation (5), we have $y_n - x_n = \alpha_n(f(x_n)-x_n) + \gamma_n(W_nz_n-x_n)$. Hence, from Equation (18), we get $\|y_n-x_n\| \le \alpha_n\|f(x_n)-x_n\| + \|W_nz_n-x_n\| \to 0$ $(n\to\infty)$. This together with Equation (19) implies that

$$\begin{aligned}
\|x_n-Gx_n\| &\le \|x_n-y_n\| + \|y_n-Gy_n\| + \|Gy_n-Gx_n\| \\
&\le 2\|x_n-y_n\| + \|y_n-Gy_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{20}$$

Step 4. We show that $\|x_n-Wx_n\| \to 0$, $\|x_n-T_\lambda x_n\| \to 0$ and $\|x_n-\Gamma x_n\| \to 0$ as $n\to\infty$, where $Wx = \lim_{n\to\infty} W_nx$ $\forall x \in C$, $T_\lambda = J_\lambda^B(I-\lambda A)$ and $\Gamma x = \theta_1 Wx + \theta_2 Gx + \theta_3 T_\lambda x$ $\forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ satisfying $\theta_1+\theta_2+\theta_3 = 1$. Indeed, since $x_{n+1}-x_n+x_n-y_n = \delta_n(x_n-y_n) + (1-\delta_n)(W_ny_n-y_n)$, from $x_n-x_{n+1}\to 0$ and $x_n-y_n\to 0$, we obtain

$$\|W_ny_n-y_n\| = \frac{1}{1-\delta_n}\|x_{n+1}-x_n+(1-\delta_n)(x_n-y_n)\| \le \frac{\|x_{n+1}-x_n\| + \|x_n-y_n\|}{1-\delta_n} \to 0 \quad (n\to\infty),$$

which together with Proposition 3 and $x_n - y_n \to 0$ implies that

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{21}$$

Furthermore, utilizing the same method used for Equation (8), one arrives at

$$\begin{aligned}\|T_n y_n - T_\lambda y_n\| &\le |1 - \tfrac{\lambda}{\lambda_n}|\|J^B_{\lambda_n}(I-\lambda_n A)y_n - (I-\lambda_n A)y_n\| + |\lambda_n - \lambda|\|Ay_n\| \\ &= |1 - \tfrac{\lambda}{\lambda_n}|\|T_n y_n - (I-\lambda_n A)y_n\| + |\lambda_n - \lambda|\|Ay_n\|.\end{aligned}$$

Since $\lim_{n\to\infty} \lambda_n = \lambda$ and the sequences $\{y_n\}, \{T_n y_n\}, \{Ay_n\}$ are bounded, we get

$$\lim_{n\to\infty} \|T_n y_n - T_\lambda y_n\| = 0. \tag{22}$$

By utilizing Lemma 1, we deduce from Equation (18), Equation (19), Equation (22), and $x_n - y_n \to 0$ that

$$\lim_{n\to\infty} \|T_\lambda x_n - x_n\| = 0. \tag{23}$$

We now define the mapping $\Gamma x = \theta_1 Wx + \theta_2 Gx + \theta_3 T_\lambda x \ \forall x \in C$ for constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ satisfying $\theta_1 + \theta_2 + \theta_3 = 1$. Lemma 4 further sends us to

$$\begin{aligned}\|x_n - \Gamma x_n\| &= \|\theta_1(x_n - Wx_n) + \theta_2(x_n - Gx_n) + \theta_3(x_n - T_\lambda x_n)\| \\ &\le \theta_1\|x_n - Wx_n\| + \theta_2\|x_n - Gx_n\| + \theta_3\|x_n - T_\lambda x_n\|.\end{aligned} \tag{24}$$

From Equation (20), Equation (21), Equation (23), and Equation (24), we get

$$\lim_{n\to\infty} \|x_n - \Gamma x_n\| = 0. \tag{25}$$

Step 5. We show that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J(x_n - x^*)\rangle \le 0, \tag{26}$$

where $x^* =$s-$\lim_{n\to\infty} x_t$ with x_t being a fixed point of the contraction $(1-t)\Gamma + tf$ for each $t \in (0,1)$. By Lemma 1, we conclude that

$$\|x_n - x_t\|^2 \le f_n(t) + 2t\|x_t - x_n\|^2 + (1 + t^2 - 2t)\|x_t - x_n\|^2 + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle, \tag{27}$$

where

$$f_n(t) = (\|\Gamma x_n - x_n\| + 2\|x_t - x_n\|)\|x_n - \Gamma x_n\|(1-t)^2 \to 0 \quad (n \to \infty). \tag{28}$$

Equation (27) yields that

$$2t\langle J(x_t - x_n), x_t - f(x_t)\rangle \le f_n(t) + t^2\|x_n - x_t\|^2. \tag{29}$$

Letting $n \to \infty$ in Equation (29), one arrives at

$$\limsup_{n\to\infty} 2\langle x_t - f(x_t), J(x_t - x_n)\rangle \le tM_4, \tag{30}$$

where $\sup\{\|x_t - x_n\|^2\} \le M_4$, where $M_4 > 0$. Further letting t go to 0 in Equation (30), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

Thus,

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, J(x_n-x^*)\rangle \le \limsup_{n\to\infty}\langle f(x^*)-x^*, J(x_n-x^*)-J(x_n-x_t)\rangle$$
$$+(1+\delta)\|x_t-x^*\|\limsup_{n\to\infty}\|x_n-x_t\|+\limsup_{n\to\infty}\langle f(x_t)-x_t, J(x_n-x_t)\rangle.$$

Taking into account that $\lim_{t\to 0}\|x_t-x^*\|=0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*)-x^*, J(x_n-x^*)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*, J(x_n-x^*)\rangle \\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(x^*)-x^*, J(x_n-x^*)-J(x_n-x_t)\rangle.\end{aligned} \tag{31}$$

Since the space is smooth, we conclude from Equation (26) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(x^*)-x^*, J(y_n-x^*)\rangle = \limsup_{n\to\infty}\{\langle J(x_n-x^*), f(x^*)-x^*\rangle \\ &+\langle J(y_n-x^*)-J(x_n-x^*), f(x^*)-x^*\rangle\} = \limsup_{n\to\infty}\langle J(x_n-x^*), f(x^*)-x^*\rangle \le 0.\end{aligned} \tag{32}$$

Step 6. We show that $\|x_n-x^*\|\to 0$ as $n\to\infty$. Indeed, we observe that

$$\begin{aligned}&\|y_n-x^*\|^2 = \|\alpha_n(f(x_n)-f(x^*))+\beta_n(x_n-x^*)+\gamma_n(W_nz_n-x^*)+\alpha_n(f(x^*)-x^*)\|^2 \\ &\le \alpha_n\|f(x_n)-f(x^*)\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|z_n-x^*\|^2+2\alpha_n\langle f(x^*)-x^*, J(y_n-x^*)\rangle \\ &\le \alpha_n\delta\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n(t_n\|x_n-x^*\|^2+(1-t_n)\|y_n-x^*\|^2) \\ &\quad +2\alpha_n\langle f(x^*)-x^*, J(y_n-x^*)\rangle,\end{aligned}$$

which hence yields

$$\|y_n-x^*\|^2 \quad \le (1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-x^*\|^2+\tfrac{2\alpha_n}{1-\gamma_n(1-t_n)}\langle f(x^*)-x^*, J(y_n-x^*)\rangle. \tag{33}$$

Thus,

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\|W_ny_n-x^*\|^2 \\ &\le \delta_n\|x_n-x^*\|^2+(1-\delta_n)\{(1-\tfrac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)})\|x_n-x^*\|^2 \\ &\quad +\tfrac{2\alpha_n}{1-\gamma_n(1-t_n)}\langle f(x^*)-x^*, J(y_n-x^*)\rangle\} \\ &= [1-\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}]\|x_n-x^*\|^2+\tfrac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\cdot\tfrac{2\langle f(x^*)-x^*, J(y_n-x^*)\rangle}{1-\delta}.\end{aligned} \tag{34}$$

Since $\liminf_{n\to\infty}\frac{(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)} > 0$, $\{\frac{\alpha_n(1-\delta)}{1-\gamma_n(1-t_n)}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, we know that $\{\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}\}\subset(0,1)$ and $\sum_{n=0}^{\infty}\frac{\alpha_n(1-\delta_n)(1-\delta)}{1-\gamma_n(1-t_n)}=\infty$. Utilizing Lemma 6 and Equation (32), one from Equation (34) gets that $\|x_n-x^*\|\to 0$ as n tends to the infinity. This completes the proof.

Let $q>1$. A mapping $T: C\to C$ is said to be η-strictly pseudocontractive of order q if for each $x,y\in C$, there exists $j_q(x-y)\in J_q(x-y)$ such that $\langle Tx-Ty, j_q(x-y)\rangle \le \|x-y\|^q-\eta\|x-y-(Tx-Ty)\|^q$ for some $\eta\in(0,1)$. It is clear that $T: C\to C$ is η-strictly pseudocontractive of order q iff $I-T$ is q-order η-inverse-strongly accretive.

Corollary 1. *Let X be uniformly convex and q-uniformly, where $1<q\le 2$, smooth space. Let $B: C\to 2^X$ be an m-accretive operator and $A: C\to X$ be a q-order α-inverse-strongly accretive operator Let Π_C be a non-expansive sunny retraction onto C and let T be a q-order η-strictly pseudocontractive self-mapping defined on C such that $\Omega=\cap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\mathrm{Fix}(T)\cap(A+B)^{-1}0\neq\emptyset$. Let f be a δ-contractive self-mapping defined*

on C with constant $\delta \in (0,1)$ and $\{W_n\}$ be the vector sequence defined by Equation (3). Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1-t_n)J^B_{\lambda_n}(I-\lambda_n A)((1-l)I + lT)y_n) + \alpha_n f(x_n), \\ x_{n+1} = (1-\delta_n)W_n y_n + \delta_n x_n \quad n \geq 0, \end{cases} \tag{35}$$

where $0 < l < \min\{1, (\frac{q\eta}{\kappa_q})^{\frac{1}{q-1}}\}$, $\{\lambda_n\} \subset (0, (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}})$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = \lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |t_n - t_{n-1}| = 0$;
(iii) $\limsup_{n\to\infty} \gamma_n t_n(1-t_n) > 0$ *and* $\liminf_{n\to\infty} \gamma_n(1-t_n) < 1$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
(v) $0 < \bar{\lambda} \leq \lambda_n \ \forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$.

Then, $x_n \to x^* \in \Omega$ strongly.

Proof. In Theorem 1, we put $A_1 = I - T$, $A_2 = 0$ and $\mu_1 = l$, where $0 < l < \min\{1, (\frac{q\eta}{\kappa_q})^{\frac{1}{q-1}}\}$. Then, GSVI (1) is equivalent to the variational inequality: $\langle A_1 x^*, J(x - x^*)\rangle \geq 0 \ \forall x \in C$. In this case, $A_1 : C \to X$ is q-order η-inverse-strongly accretive. It is not hard to see that $\mathrm{Fix}(T) = \mathrm{VI}(C, A_1)$. Indeed, for $l \in (0,1)$, we observe that

$$\begin{aligned} p \in \mathrm{VI}(C, A_1) \quad &\Leftrightarrow \langle A_1 p, J(x-p)\rangle \geq 0 \ \forall x \in C \Leftrightarrow p = \Pi_C(p - lA_1 p) \\ &\Leftrightarrow p = \Pi_C(p - l(I-T)p) = (1-l)p + lTp \Leftrightarrow p \in \mathrm{Fix}(T). \end{aligned}$$

Thus, we obtain that $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, A_1, A_2) \cap (A+B)^{-1}0 = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{Fix}(T) \cap (A+B)^{-1}0$, and $\Pi_C(I - \mu_1 A_1)\Pi_C(I - \mu_2 A_2)y_n = \Pi_C(I - \mu_1 A_1)y_n = ((1-l)I + lT)y_n$. Thus, Equation (4) reduces to Equation (35). Therefore, the desired result follows from Theorem 3.1. □

4. Subresults

4.1. Variational Inequality Problem

The framework of potential spaces will be restricted into a Hilbert space H in this section. Let $A : C \to H$, where C is a nonempty subset, be a single-valued operator. Let us recall the classical variational inequality problem (VIP): $\langle Ax^*, x - x^*\rangle \geq 0$ for any $x \in C$. The set of solutions of the VIP is denoted by the notation $\mathrm{VI}(C, A)$. Let I_C be an indicator operator of C given by

$$I_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

One finds that I_C is a proper convex and lower semicontinuous function and ∂I_C, the subdifferential, is a maximally monotone operator. For $\lambda > 0$, the resolvent of ∂I_C is denoted by $J^{\partial I_C}_{\lambda}$, i.e., $J^{\partial I_C}_{\lambda} = (I + \lambda\partial I_C)^{-1}$. We denote the normal cone of C at u by $N_C(u)$, i.e., $N_C(u) = \{w \in H : \langle w, v - u\rangle \leq 0 \ \forall v \in C\}$. Note that

$$\begin{aligned} \partial I_C(u) \quad &= \{w \in H : I_C(v) + \langle w, v-u\rangle \leq I_C(u)\} \\ &= \{w \in H : \langle w, v-u\rangle \leq 0\} = N_C(u). \end{aligned}$$

Thus, we know that $x - u \in \lambda N_C(u) \ \Leftrightarrow \ u = J^{\partial I_C}_{\lambda}(x) \ \Leftrightarrow \ u = P_C(x) \ \forall v \in C \ \Leftrightarrow \ \langle x - u, v - u\rangle \leq 0$. Hence, we get $\mathrm{VI}(C, A) = (A + \partial I_C)^{-1}0$.

Next, putting $B = \partial I_C$ in Corollary 1, we can obtain the following convergence theorem.

Theorem 2. *Let the mapping $A : C \to H$ be α-inverse-strongly monotone, and $T : C \to C$ be a 2-order η-strictly pseudocontractive mapping such that $\Omega = \cap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(T) \neq \emptyset$. Let $\{W_n\}$ be the mapping sequence defined by (2.1) and f be a δ-contractive self-mapping with contractive constant $\delta \in (0, 1)$. Define a sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1 - t_n)P_C(I - \lambda_n A)((1 - l)I + lT)y_n) + \alpha_n f(x_n), \\ x_{n+1} = (1 - \delta_n)W_n y_n + \delta_n x_n \quad n \geq 0, \end{cases}$$

where $0 < l < \min\{1, 2\eta\}$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = \lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |t_n - t_{n-1}| = 0$;
(iii) $\limsup_{n\to\infty} \gamma_n t_n(1 - t_n) \geq 0$ *and* $\liminf_{n\to\infty} \gamma_n(1 - t_n) < 1$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
(v) $0 < \bar{\lambda} \leq \lambda_n \ \forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < 2\alpha$.

Then, $x_n \to x^* \in \Omega$ strongly.

4.2. Convex Minimization Problem

Let $g : C \to \mathbf{R}$ be a convex smooth function and $h : C \to \mathbf{R}$ be a proper convex and lower semicontinuous function. The convex minimization problem is

$$g(x^*) + h(x^*) = \min_{x \in C}\{g(x) + h(x)\}. \tag{36}$$

This is equivalent to the problem $0 \in \partial h(x^*) + \nabla g(x^*)$, where ∂h is the subdifferential of h and ∇g is the gradient of g. Next, setting $A = \nabla g$ and $B = \partial h$ in Corollary 1, we can obtain the following.

Theorem 3. *Let $g : C \to \mathbf{R}$ be a convex and differentiable function with $\frac{1}{\alpha}$-Lipschitz continuous gradient ∇g and $h : C \to \mathbf{R}$ be a convex and lower semicontinuous function. Let f be a δ-contractive self-mapping defined on C and $\{W_n\}$ be the sequence defined by Equation (3). Let T be an η-strictly pseudocontractive self-mapping defined on C with order 2 such that $\Omega = \cap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(T) \cap (\nabla g + \partial h)^{-1}0 \neq \emptyset$, where $(\nabla g + \partial h)^{-1}0$ is the set of minimizers attained by $g + h$. Define a sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1 - t_n)J_{\lambda_n}^{\partial h}(I - \lambda_n \nabla g)((1 - l)I + lT)y_n) + \alpha_n f(x_n), \\ x_{n+1} = (1 - \delta_n)W_n y_n + \delta_n x_n \quad n \geq 0, \end{cases}$$

where $0 < l < \min\{1, 2\eta\}$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0, 1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}| = \lim_{n\to\infty} |\gamma_n - \gamma_{n-1}| = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |t_n - t_{n-1}| = 0$;
(iii) $\limsup_{n\to\infty} \gamma_n t_n(1 - t_n) < 1$ *and* $\liminf_{n\to\infty} \gamma_n(1 - t_n) > 0$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
(v) $0 < \bar{\lambda} \leq \lambda_n \ \forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < 2\alpha$.

Then, $x_n \to x^* \in \Omega$ strongly. Indeed, * also solves the inequality: $\langle (I - f)x^*, x^* - p\rangle \leq 0 \ \ \forall p \in \Omega$ uniquely.

4.3. Split Feasibility Problem

Let $\Gamma : H_1 \to H_2$ be a linear bounded operator with its adjoint Γ^*. Let C and Q be convex closed sets in Hilbert spaces H_1 and H_2, respectively. One considers the split feasibility problem (SFP):

$x^* \in C \quad$ and $\quad \Gamma x^* \in Q$. The solution set of the SFP is $C \cap \Gamma^{-1}Q$. To solve the SFP, one can set it as the following convexly minimization problem:

$$\min_{x \in C} g(x) := \frac{1}{2}\|\Gamma x - P_Q \Gamma x\|^2.$$

Here, g has a Lipschitz gradient given by $\nabla g = \Gamma^*(I - P_Q)\Gamma$. In addition, ∇g is $\frac{1}{\|\Gamma\|^2}$-inverse-strongly monotone, where $\|\Gamma\|^2$ is the spectral radius of $\Gamma^*\Gamma$. Thus, x^* solves the SFP iff x^* satisfies the inclusion problem:

$$\begin{aligned} 0 \in \partial I_C(x^*) + \nabla g(x^*) \quad &\Leftrightarrow \ x^* - \lambda \nabla g(x^*) \in (I + \lambda \partial I_C)x^* \\ &\Leftrightarrow \ x^* = J_\lambda^{\partial I_C}(x^* - \lambda \nabla g(x^*)) \\ &\Leftrightarrow \ x^* = P_C(x^* - \lambda \nabla g(x^*)). \end{aligned}$$

Theorem 4. *Let $\Gamma : H_1 \to H_2$ be a linear bounded operator with its adjoint Γ^*, and T be an η-strictly pseudocontractive self-mapping defined on C with order 2 such that $\Omega = \cap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{Fix}(T) \cap (C \cap \Gamma^{-1}Q) \neq \emptyset$. Let f be a δ-contractive self-mapping defined on C and $\{W_n\}$ be the sequence defined by (2.1). Define a sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n x_n + \gamma_n W_n(t_n x_n + (1 - t_n)P_C(I - \lambda_n \Gamma^*(I - P_Q)\Gamma)((1 - l)I + lT)y_n) + \alpha_n f(x_n), \\ x_{n+1} = (1 - \delta_n)W_n y_n + \delta_n x_n \quad n \geq 0, \end{cases}$$

where $0 < l < \min\{1, 2\eta\}$, $\{\lambda_n\} \subset (0, \frac{2}{\|\Gamma\|^2})$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{t_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} |\beta_{n-1} - \beta_n| = \lim_{n\to\infty} |\gamma_{n-1} - \gamma_n| = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} |t_{n-1} - t_n| = 0$;
(iii) $\limsup_{n\to\infty} \gamma_n t_n(1 - t_n) < 1$ *and* $\liminf_{n\to\infty} \gamma_n(1 - t_n) > 0$;
(iv) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, $\liminf_{n\to\infty} \delta_n > 0$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
(v) $0 < \bar{\lambda} \leq \lambda_n \ \forall n \geq 0$ *and* $\lim_{n\to\infty} \lambda_n = \lambda < \frac{2}{\|\Gamma\|^2}$.

Then, $x_n \to x^* \in \Omega$ strongly.

5. Conclusions

In this paper, we established norm convergence theorems of solutions for a general symmetrical variational system, which can be acted as a framework for many real world problems arising in engineering and medical imaging, which involves some convex optimization subproblems. There is no compact assumption on the operators of accretive type and the sets in the whole space. The restrictions, which are also mild, imposed on the control parameters. Our results provide an outlet for viscosity type algorithms without compact assumptions in infinite-dimensional spaces. From the space frameworks' point of view, the space in our convergence theorems is still not general; however, it is Banach now. It is of interest to further relax the convex restrictions in the future research.

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