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A Solution of Fredholm Integral Equation by Using the Cyclic η_s^q-Rational Contractive Mappings Technique in b-Metric-Like Spaces

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Abstract: In this article, the notion of cyclic η_s^q-rational contractive mappings is discussed and some fixed point theorems in the context of complete b-metric-like spaces are showed. Here, the obtained consequences unify, extend and generalize various comparable known results. Furthermore, new common fixed point outcomes in a directed graph are demonstrated. Moreover, some useful examples are discussed to justify our theoretical results and finding a solution of Fredholm integral equation was discussed as enforcement.

Keywords: cyclic η_s^q-rational contractive mapping; a directed graph; fixed point technique; Fredholm integral equation; b-metric-like spaces

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1. Introduction

Problems in the nonlinear analysis are solved by a popular tool called the Banach contraction principle, which appeared in Banach's thesis [1] and was used to find a solution (existence and uniqueness) for some integral equations; this stated: consider (Ω, d) is a metric space and Γ is a nonlinear self-mapping defined on it; then, the mapping Γ is called a Banach contraction if there exists $0 \leq \Lambda < 1$ such that

$$d(\Gamma\kappa, \Gamma\mu) \leq \Lambda d(\kappa, \mu), \ \forall \kappa, \mu \in \Omega. \tag{1}$$

Note that, for all $\kappa, \mu \in \Omega$, inequality (1) holds, which imposes the nonlinear mapping Γ to be continuous, while it is not applicable in the case of discontinuity. The major drawback of this principle is how we apply this contraction in case of discontinuity. This problem was overcome in the past by Kannan [2] where it proved a fixed point result without continuity, while, recently many authors attempted to solve this problem (see, for example, [3–6]).

In 1989, one of the interesting extensions of this basic principle was presented by Bakhtin [7] (and also Czerwik [8], 1993) by introducing the notion of b-metric spaces. For fixed point results in mention spaces, see, for example, [9–16]).

In 2010, the concept of b-metric-like was initiated by Alghamdi et al. [17] as an extension of a b-metric. They studied some important fixed point consequences concerned with this space. Recently, a lot of contributions on fixed points consequences via certain contractive conditions in the mentioned spaces are made (for example, see [18–24] and references therein).

In 2012, on a complete metric space Ω, the concept of η-admissible mapping is given by Samet et al. [6] as the following:

Definition 1. *Let $\Omega \neq \emptyset$. A nonlinear mapping $\Gamma : \Omega \to \Omega$ is said to be an η-admissible mapping if*

$$\eta(\kappa, \mu) \geq 1 \text{ implies } \eta(\Gamma\kappa, \Gamma\mu) \geq 1$$

$\forall \kappa, \mu \in \Omega$ and $\eta : \Omega \times \Omega \to [0, +\infty)$.

By using this concept, they introduced the following theorem:

Theorem 1. *Suppose that (Ω, d) is a complete metric space and $\Gamma : \Omega \to \Omega$ is an η-admissible mapping. Consider that the following hypotheses are realized:*

(a) *$\forall \kappa, \mu \in \Omega$, we get*
$$\eta(\kappa, \mu) d(\Gamma\kappa, \Gamma\mu) \leq \theta(d(\kappa, \mu)), \tag{2}$$
where $\theta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that
$$\sum_{n=1}^{+\infty} \theta^n(t) < +\infty, \text{ for all } t > 0;$$
(b) *there exists $\kappa_\circ \in \Omega$ such that $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$;*
(c) *Γ has a fixed point if one conditions holds: Either Γ is continuous or for any sequence $\{\kappa_n\} \in \Omega$ such that $\eta(\kappa_n, \kappa_{n+1}) \geq 1$, $\forall n \geq 0$ and $\lim_{n \to +\infty} \kappa_n = \kappa$, we obtain $\eta(\kappa_n, \kappa) \geq 1$.*

If we take $\eta(\kappa, \mu) = 1$ in the inequality (2) for all $\kappa, \mu \in \Omega$, then Γ reduces to inequality (1).

One can note that, If Γ does not satisfy the contractive condition (1), in this situation, the mapping Γ is a weaker version of Banach principle according to Theorem 1. Therefore, the researchers turned to some important generalizations of the metric space such as metric like space, b-metric-like spaces, and others.

The shell of this article is as follows. In Section 2, we present some known consequences about b-metric-like spaces and some useful lemmas which will be used in the sequel. In Section 3, we introduce a cyclic η_s^q-rational contractive mapping and we obtain some results in related fixed points on it in the context of b-metric-like spaces and we support our theoretical results by some examples. In Section 4, we use the same contractive mapping without an admissible concept to connect the graph theory with a fixed point theory by getting some fixed point results in a directed graph. In the final section, Section 5, an application to find an analytical solution of the Fredholm integral equation of the second kind is presented and a numerical example to justify it is discussed.

2. Preliminaries and Known Results

This section is devoted to discuss some basic notions in metric-like and b-metric-like spaces.

Definition 2 ([8])**.** *Let $\Omega \neq \emptyset$. A mapping $\omega : \Omega \times \Omega \to \mathbb{R}^+$ is called a metric-like if the following three conditions hold for all $\kappa, \mu, \tau \in \Omega$:*

(ω_1) $\omega(\kappa, \mu) = 0 \Rightarrow \kappa = \mu$;
(ω_2) $\omega(\kappa, \mu) = \omega(\mu, \kappa)$;
(ω_3) $\omega(\kappa, \tau) \leq \omega(\kappa, \mu) + \omega(\tau, \mu)$.

In this case, the pair (Ω, ω) is called a metric-like space.

Definition 3 ([17])**.** *A b-dislocated on a nonempty set Ω is a function $\varpi : \Omega \times \Omega \to [0, +\infty)$ such that, for all $\kappa, \mu, \tau \in \Omega$ and coefficient $s \geq 1$, the following three conditions hold:*

(ϖ_1) $\varpi(\kappa, \mu) = 0 \Rightarrow \kappa = \mu$;
(ϖ_2) $\varpi(\kappa, \mu) = \varpi(\mu, \kappa)$;
(ϖ_3) $\varpi(\kappa, \tau) \leq s[\varpi(\kappa, \mu) + \varpi(\mu, \tau)]$.

In this case, the pair (Ω, ϖ) is called a b-metric-like space (with coefficient s).

It should be noted that the class of b-metric-like spaces is larger than the class of metric-like spaces, since a b-metric-like is a metric-like with $s = 1$.

Since a b-metric-like on Ω is an ordinary metric except that $\varpi(\kappa, \mu)$ may be positive for $\kappa \in \Omega$, we can thus generate a new topology $\Re_\varpi$ on Ω as follows:

$$ß_\varpi(\kappa, \epsilon) = \{\mu \in \Omega : |\varpi(\kappa, \mu) - \varpi(\kappa, \kappa)| < \frac{\epsilon}{s}\} \text{ for all } \kappa \in \Omega,\ s \geq 1 \text{ and } \epsilon > 0,$$

where $ß_\varpi(\kappa, \epsilon)$ is the family of open ϖ-balls that have a base of topology.

According to a topology $\Re_\varpi$, we can present the following results:

Definition 4. *Let (Ω, ϖ) be a b-metric-like space and $\chi \subseteq \Omega$. It is said that χ is a ϖ-open subset of Ω, if $\forall \kappa \in \Omega$, $\exists \epsilon > 0$ such that $ß_\varpi(\kappa, \epsilon) \subseteq \chi$. Moreover, $\sigma \subseteq \Omega$ is a ϖ-closed subset of Ω if $\Omega \backslash \chi$ is a ϖ-open subset of Ω.*

Lemma 1. *Let (Ω, ϖ) be a b-metric-like space and σ be a ϖ-closed subset in Ω. Let $\{\kappa_n\}$ be a sequence of σ such that $\lim_{n\to\infty} \kappa_n = \kappa$. Then, $\kappa \in \sigma$.*

Proof. Let $\kappa \notin \sigma$ by Definition 4, and $(\Omega \backslash \sigma)$ is a ϖ-open set. Then, there exists $\epsilon > 0$ such that $ß_\varpi(\kappa, \epsilon) \subseteq \Omega \backslash \sigma$. On the other hand, we have $\lim_{n\to\infty} |\varpi(\kappa_n, \kappa) - \varpi(\kappa, \kappa)| = 0$ since $\lim_{n\to\infty} \kappa_n = \kappa$. Hence, there exists $n_\circ \in N$ such that

$$|\varpi(\kappa_n, \kappa) - \varpi(\kappa, \kappa)| < \frac{\epsilon}{s}$$

for all $n \geq n_\circ$. Thus, we conclude that $\{\kappa_n\} \subseteq ß_\varpi(\kappa, \epsilon) \subseteq \Omega \backslash \sigma$ for all $n \geq n_\circ$, a contradiction due to $\{\kappa_n\} \subseteq \sigma$ for all $n_\circ \in N$. $\square$

Lemma 2. *Let (Ω, ϖ) be a b-metric-like space and $\{\kappa_n\}$ be a sequence of Ω such that $\lim_{n\to\infty} \kappa_n = \kappa$ and $\varpi(\kappa, \kappa) = 0$. Then, $\lim_{n\to\infty} \varpi(\kappa_n, \mu) = s\varpi(\kappa, \mu)$ for all $\mu \in \Omega$.*

Proof. Applying (ϖ_3), we get

$$s[\varpi(\kappa, \mu) - \varpi(\kappa_n, \kappa)] \leq \varpi(\kappa_n, \mu) \leq s[\varpi(\kappa_n, \kappa) + \varpi(\kappa, \mu)].$$

Passing the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \varpi(\kappa_n, \mu) = s\varpi(\kappa, \mu).$$

In a b-metric-like space (Ω, ϖ), if $\kappa, \tau \in \Omega$ and $\varpi(\kappa, \tau) = 0$, then $\kappa = \tau$, but the converse not true in general. $\square$

Example 1. *Let $\Omega = \{0, 1, 2, 3, 4\}$ and let*

$$\varpi(\kappa, \tau) = \begin{cases} 5, & \kappa = \tau = 0, \\ \frac{1}{5}, & \text{otherwise.} \end{cases}$$

Then, (Ω, ϖ) is a b-metric-like space with a coefficient $s = 5$.

For new examples in metric-like and b-metric-like spaces, see, [24,25].

Definition 5 ([17])**.** *Assume that $\{\kappa_n\}$ is a sequence on a metric-like space (Ω, ϖ) with a coefficient s:*

(a) If $\lim_{m,n\to\infty} \varpi(\kappa_n, \kappa) = \varpi(\kappa, \kappa)$, then the sequence $\{\kappa_n\}$ is said to be convergent to κ;
(b) a sequence $\{\kappa_n\}$ is called a Cauchy sequence in (Ω, ϖ) if $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n)$ finite and exists;
(c) if for every Cauchy sequence $\{\kappa_n\}$ in Ω, there exists $\kappa \in \Omega$, such that $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n) = \varpi(\kappa, \kappa) = \lim_{n\to\infty} \varpi(\kappa_n, \kappa)$, then the pair (Ω, ϖ) is called a complete b-metric-like space.

Remark 1. *The limit of the sequence need not be unique and a convergent sequence need not be a Cauchy in the context of b-metric-like.*

To show this remark, the authors [25] gave the following example:

Example 2 ([25])**.** *Consider $\Omega = [0, +\infty)$. Define a function $\varpi : [0, +\infty) \times [0, +\infty) \to \Omega$ by $\varpi(\kappa, \mu) = (\max\{\kappa, \mu\})^2$. Then, (Ω, ϖ) is a b-metric-like space with a coefficient $s = 2$. Suppose that*

$$\{\kappa_n\} = \begin{cases} 0 & \text{when} \quad n \text{ is odd,} \\ 1 & \text{when} \quad n \text{ is even.} \end{cases}$$

For $\kappa \geq 1$, $\lim_{n\to\infty} \varpi(\kappa_n, \kappa) = \lim_{n\to\infty} (\max\{\kappa_n, \kappa\})^2 = \kappa^2 = \varpi(\kappa, \kappa)$. Therefore, it is a convergent sequence and $\kappa_n \to \kappa$ for all $\kappa \geq 1$. That is, the limit of the sequence is not unique. In addition, $\lim_{m,n\to\infty} \varpi(\kappa_m, \kappa_n)$ does not exist. Thus, it is not a Cauchy sequence.

Lemma 3 ([26])**.** *Let $\Gamma : \Omega \to \Omega$ be a nonlinear self-mapping on a b-metric-like space (Ω, ϖ) with coefficient s. Consider that Γ is continuous at $\delta \in \Omega$. Then, for all sequences $\{\kappa_n\}$ in Ω such that $\kappa_n \to \delta$, we have $\Gamma\kappa_n \to \Gamma\delta$ that is*

$$\lim_{n\to\infty} \varpi(\Gamma\kappa_n, \Gamma\delta) = \varpi(\Gamma\delta, \Gamma\delta).$$

The proof of the following lemma is clear.

Lemma 4. *Let (Ω, ϖ) be a b-metric-like space with a coefficient $s \geq 1$. Then,*

(i) If $\varpi(\kappa, \mu) = 0$, then $\varpi(\kappa, \kappa) = \varpi(\kappa, \mu) = 0$;
(ii) If $\{\kappa_n\}$ is a sequence such that $\lim_{n\to\infty} \varpi(\kappa_n, \kappa_{n+1}) = 0$, then we can write

$$\lim_{n\to\infty} \varpi(\kappa_n, \kappa_n) = \lim_{n\to\infty} \varpi(\kappa_{n+1}, \kappa_{n+1}) = 0,$$

(iii) $\varpi(\kappa, \mu) > 0$, if $\kappa \neq \mu$.

Lemma 5 ([27])**.** *Let $\{\kappa_n\}$ be a sequence on a complete b-metric-like space (Ω, ϖ) with parameter $s \geq 1$ such that*

$$\lim_{n\to\infty} \varpi(\kappa_n, \kappa_{n+1}) = 0.$$

If $\lim_{n,m\to\infty} \varpi(\kappa_n, \kappa_m) \neq 0$, there exists $\varepsilon > 0$ and two sequences $\{m(\ell)\}_{\ell=1}^{\infty}$, $\{n(\ell)\}_{\ell=1}^{\infty}$ of positive integers with $n_\ell > m_\ell > \ell$ such that

$$\begin{aligned} \varpi(\kappa_{n_\ell}, \kappa_{m_\ell}) &\geq \varepsilon, \ \varpi(\kappa_{m_\ell}, \kappa_{n_\ell - 1}) < \varepsilon, \ \frac{\varepsilon}{s^2} \leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell - 1}, \kappa_{m_\ell - 1}) \leq \varepsilon s, \\ \frac{\varepsilon}{s} &\leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell - 1}, \kappa_{m_\ell}) \leq \varepsilon, \ \frac{\varepsilon}{s} \leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell}, \kappa_{m_\ell - 1}) \leq \varepsilon s^2. \end{aligned}$$

3. Cyclic η_s^q-Rational Contractive Mappings

This part is devoted to define a cyclic η_s^q-rational contractive mapping, and some new fixed point results via this contractive on the context of complete b-metric-like spaces are presented.

Let Θ refers to the class of all functions $\theta : [0,\infty) \to [0,\infty)$, satisfying the following:

(i) for each $t > 0$, θ non-decreasing and continuous such that $\theta(t) < t$;
(ii) $\lim_{n\to\infty} \theta^n(t) = 0$ for all $t > 0$.

We begin with the following new definition:

Definition 6. *Let (Ω,ϖ) be a b-metric-like space, $l \in \mathbb{N}$, $B_1, B_2, ...B_l$ be ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0,\infty)$ be a nonlinear mapping. The mapping Γ is called a cyclic η_s^q-rational contractive if*

(a)

$$\Gamma(B_j) \subseteq B_{j+1},\ j = 1,2,...,l,\ \ \text{where}\ \ B_{l+1} = B_1,$$

(b) for any $\kappa \in B_i$ and $\tau \in B_{i+1}$, $i = 1,2,...,l$, such that $\eta(\kappa,\Gamma\kappa)\eta(\mu,\Gamma\mu) \geq 1$, we get

$$2s^q\varpi(\Gamma\kappa,\Gamma\mu) \leq \theta(N(\kappa,\mu)), \tag{3}$$

for all $q > 1$ and

$$N(\kappa,\mu) = \max\left\{\varpi(\kappa,\mu), \varpi(\mu,\Gamma\kappa), \frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}, \frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s}\right\}.$$

If we take $\Omega = B_i$, $i = 1,2,...,l$, in the above definition, then the mapping Γ reduce to η_s^q-rational contraction mapping.

Here, we refer to the set of all fixed points of Γ by ∇, that is, $\nabla = \{\kappa \in \Omega : \Gamma\kappa = \kappa\}$.

It is noted that the class of η_s^q-rational contraction mapping is a strictly larger class than (s,q)-Dass and Gupta contraction and hence a larger class than Dass and Gupta and Jaggi contraction [28].

Example 3. *Let $\Omega = \mathbb{R}$ be equipped with a b-metric-like mapping $\varpi(\kappa,\mu) = (\kappa^2 + \mu^2)^q$ for all $\kappa,\mu \in \Omega$, with $s = q = 2$. Suppose $B_1 = [-\frac{\pi}{2}, 0]$ and $B_2 = [0, \frac{\pi}{2}]$ and $\Upsilon = B_1 \cup B_2$. Define $\Gamma : \Omega \to \Omega$ and $\eta : \Omega \times \Omega \to [0,\infty)$ by*

$$\Gamma\kappa = \begin{cases} -\frac{\kappa}{4}\left|\sin(\frac{1}{\kappa})\right|, & \text{if } \kappa \in [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}], \\ 0, & \text{if } \kappa = 0, \end{cases} \quad \text{and } \eta(\kappa,\mu) = \begin{cases} 1, & \text{if } \kappa,\mu \in [-\frac{\pi}{2},\frac{\pi}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

In addition, define $\theta : [0,\infty) \to [0,\infty)$ by $\theta(t) = \frac{1}{32}t$ for all $t > 0$. Clearly, $\Gamma(B_1) \subseteq B_2$ and $\Gamma(B_2) \subseteq B_1$. Let $\kappa \in B_1$, $\mu \in B_2$ and $\eta(\kappa,\Gamma\kappa)\eta(\mu,\Gamma\mu) \geq 1$. Now, if $\kappa \notin [-\frac{\pi}{2},\frac{\pi}{2}]$ or $\mu \notin [-\frac{\pi}{2},\frac{\pi}{2}]$, then $\eta(\kappa,\Gamma\kappa) = 0$ or $\eta(\mu,\Gamma\mu) = 0$. That is, $\eta(\kappa,\Gamma\kappa)\eta(\mu,\Gamma\mu) = 0 < 1$, which is a contradiction. Hence, $\kappa \in B_1$, $\mu \in B_2$ and $\kappa,\mu \in [-\frac{\pi}{2},\frac{\pi}{2}]$. This implies that $\kappa \in [-\frac{\pi}{2},0]$ and $\mu \in [0,\frac{\pi}{2}]$. We discuss the following positions:

(i) $\kappa = 0$ and $\mu = 0$ with $(\kappa,\mu) \in B_1 \times B_2$ or $(\kappa,\mu) \in B_2 \times B_1$. (Trivial).
(ii) $\kappa \neq 0$ and $\mu = 0$ with $(\kappa,\mu) \in B_1 \times B_2$ or $(\kappa,\mu) \in B_2 \times B_1$. Then,

$$\begin{aligned} 2s^q\varpi(\Gamma\kappa,\Gamma\mu) &= 8\left(0 + \left(-\frac{\kappa}{4}\left|\sin\left(\frac{1}{\kappa}\right)\right|\right)^2\right)^2 = 8\left(\frac{\kappa^2}{16}\left|\sin\left(\frac{1}{\kappa}\right)\right|^2\right)^2 \\ &\leq \frac{\kappa^4}{32} = \frac{1}{32}\varpi(\kappa,0) \leq \theta(M_\varpi(\kappa,0)). \end{aligned}$$

(iii) $\kappa = 0$ *and* $\mu \neq 0$ *with* $(\kappa,\mu) \in B_1 \times B_2$ *or* $(\kappa,\mu) \in B_2 \times B_1$. *Then,*

$$\begin{aligned} 2s^q\varpi(\Gamma\kappa,\Gamma\mu) &= 8\left(0+\left(-\frac{\mu}{4}\left|\sin\left(\frac{1}{\mu}\right)\right|\right)^2\right)^2 = 8\left(\frac{\mu^2}{16}\left|\sin\left(\frac{1}{\mu}\right)\right|^2\right)^2 \\ &\leq \frac{\mu^4}{32} = \frac{1}{32}\varpi(0,\mu) \leq \theta(M_\varpi(0,\mu)). \end{aligned}$$

(iv) $\kappa \neq 0$ *and* $\mu \neq 0$ *with* $(\kappa,\mu) \in B_1 \times B_2$ *or* $(\kappa,\mu) \in B_2 \times B_1$. *Then,*

$$\begin{aligned} 2s^q\varpi(\Gamma\kappa,\Gamma\mu) &= 8\left(\left(-\frac{\kappa}{4}\left|\sin\left(\frac{1}{\kappa}\right)\right|\right)^2+\left(-\frac{\mu}{4}\left|\sin\left(\frac{1}{\mu}\right)\right|\right)^2\right)^2 \\ &\leq 8\left(\frac{\kappa^2}{16}+\frac{\mu^2}{16}\right)^2 \\ &= \frac{1}{32}(\kappa^2+\mu^2)^2 = \frac{1}{32}\varpi(\kappa,\mu) \leq \theta(M_\varpi(\kappa,\mu)). \end{aligned}$$

By the four positions, one can deduce that the mapping Γ *is a cyclic* η_s^q*-rational contractive.*

Remark 2. *If* $\Gamma : \Omega \to \Omega$ *is a cyclic* η_s^q*-rational contractive mapping,* $\kappa \in \nabla$ *and* $\eta(\kappa,\kappa) \geq 1$*, then* $\varpi(\kappa,\kappa) = 0$.

Proof. Let $\varpi(\kappa,\kappa) > 0$, then $N(\kappa,\kappa) = \varpi(\kappa,\kappa)$ and, by (3), we can write

$$2s^q\varpi(\kappa,\kappa) = 2s^q\varpi(\Gamma\kappa,\Gamma\kappa) \leq \theta(\varpi(\kappa,\kappa)) < \varpi(\kappa,\kappa).$$

Thus, $s^q < \frac{1}{2}$ for all $q \geq 1$, a contradiction, so $\varpi(\kappa,\kappa) = 0$. □

Definition 7 ([29]). *Let* (Ω,ϖ) *be a metric-like space and* $\eta : \Omega \times \Omega \to [0,\infty)$ *be admissible mapping. It is said that* $\Gamma : \Omega \to \Omega$ *is* η*-continuous on* (Ω,ϖ)*, if*

$$\lim_{n\to\infty} \kappa_n = \kappa,\ \eta(\kappa_n,\kappa_{n+1}) \geq 1 \Longrightarrow \Gamma\kappa_n \to \Gamma\kappa,\ \forall n \in \mathbb{N}.$$

Example 4. *Assume that* $\Omega = [0,+\infty)$ *and* $\varpi(\kappa,\eta) = (\kappa+\mu)^2$ *is a b-metric-like on* Ω*. Let an* η*-admissible mapping* $\Gamma : \Omega \to \Omega$ *and* $\eta : \Omega \times \Omega \to \Omega$ *be defined by*

$$\Gamma\kappa = \begin{cases} \kappa^2, & \text{if } \kappa \in [0,\frac{1}{2}], \\ \tan\frac{\pi}{2}\kappa + 2, & \text{if } \kappa > \frac{1}{2}, \end{cases} \qquad \alpha(\kappa,\mu) = \begin{cases} \kappa^2+\mu^2+3, & \text{if } \kappa,\mu \in [0,\frac{1}{2}], \\ \frac{1}{5}, & \text{if } \kappa > \frac{1}{2}. \end{cases}$$

Clearly, Γ *is not continuous. Let* $\kappa_\circ = 0 \in \Omega$ *so* $\Gamma\kappa_\circ = 0$ *and* $\eta(\kappa_\circ,\Gamma\kappa_\circ) = \eta(0,0) = 3 > 1$*. Let* $\{\kappa_n\} \subset [0,\frac{1}{2}]$ *for all* $n \in \mathbb{N}$*. If* $\lim_{n\to\infty}\kappa_n = \kappa$*, then we can get* $\kappa \in [0,\frac{1}{2}]$ *and* $\eta(\kappa_n,\Gamma\kappa_{n+1}) = 3 > 1$*. Hence,*

$$\lim_{n\to\infty}\Gamma\kappa_n = \lim_{n\to\infty}\kappa_n^2 = \kappa^2 = \Gamma\kappa.$$

That is, Γ *is* η*-continuous on* (Ω,ϖ).

Now, we are ready to present our first results.

Theorem 2. *Let* (Ω,ϖ) *be a complete b-metric-like space,* l *be a positive integer,* $B_1, B_2, ...B_l$ *be non-empty* ϖ*-closed subsets of* Ω*,* $\Upsilon = \cup_{i=1}^{l} B_i$ *and* $\eta : \Upsilon \times \Upsilon \to [0,\infty)$ *be a mapping. Assume that* $\Gamma : \Upsilon \to \Upsilon$ *is a cyclic* η_s^q*-rational contractive mapping satisfying the following conditions:*

(i) *a nonlinear mapping* Γ *is an* η*-admissible;*
(ii) $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$ *if there exists* $\kappa_\circ \in \Upsilon$*;*
(iii) (a) *a nonlinear mapping* Γ *is* η*-continuous, or*

(b) *for any sequence* $\{\kappa_n\}$ *in* Ω *and for all* $n \geq 0$*, if* $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ *and* $\lim \kappa_n = \kappa$*,*

then, $\eta(\kappa, \Gamma\kappa) \geq 1$*. Therefore,* Γ *has a fixed point* $\kappa \in \cap_{i=1}^{l} B_i$*. Furthermore, if*
(iv) $\forall \kappa \in \nabla$*, we get* $\eta(\kappa, \kappa) \geq 1$*.*

Then, the uniqueness of the fixed point is realized.

Proof. Let $\kappa_\circ \in \Upsilon$ be an arbitrary point such that $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$. Thus, there exists some $i_\circ$ such that $\kappa_\circ \in B_{i_\circ}$. Now, $\Gamma(B_{i_\circ}) \subseteq B_{i_\circ+1}$ implies that $\Gamma\kappa_\circ \in B_{i_\circ+1}$. Thus, find κ_1 in $B_{i_\circ+1}$ such that $\Gamma\kappa_\circ = \kappa_1$. By a similar way, $\Gamma\kappa_n = \kappa_{n+1}$, where $\kappa_n \in B_{i_n}$. Hence, for $n \geq 0$, there exists $i_n \in \{1, 2, ..., l\}$ such that $\kappa_n \in B_{i_n}$ and $\Gamma\kappa_n = \kappa_{n+1}$. Otherwise, a mapping Γ is an η-admissible, we get

$$\eta(\kappa_1, \Gamma\kappa_1) = \eta(\Gamma\kappa_\circ, \Gamma^2\kappa_\circ) \geq 1.$$

Again, since Γ is an η-admissible mapping, then

$$\eta(\kappa_2, \Gamma\kappa_2) = \eta(\Gamma\kappa_1, \Gamma^2\kappa_1) \geq 1.$$

Continuing this manner, we can get

$$\eta(\kappa_n, \Gamma\kappa_n) \geq 1, \forall n \geq 0,$$

so

$$\eta(\kappa_n, \Gamma\kappa_n)\eta(\kappa_{n-1}, \Gamma\kappa_{n-1}) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Therefore, $\kappa_{n_\circ}$ is a fixed point of Γ if $\kappa_{n_\circ} = \kappa_{n_\circ+1}$ for some $n_\circ = 0, 1, 2, ...$ Now, consider $\kappa_n \neq \kappa_{n+1}$ for all n. Thus, by Lemma 3 (iii), we obtain that $\varpi(\kappa_{n+1}, \kappa_n) > 0$ for all n. Now, we shall prove that the sequence $\{\varpi(\kappa_n, \kappa_{n+1})\}$ is non-increasing. By (3), we have

$$2s\varpi(\kappa_n, \kappa_{n+1}) \leq 2s^q\varpi(\kappa_n, \kappa_{n+1}) = 2s^q\varpi(\Gamma\kappa_{n-1}, \Gamma\kappa_n) \leq \theta(N_\varpi(\kappa_{n-1}, \kappa_n)), \tag{4}$$

where

$$\begin{aligned} N_\varpi(\kappa_{n-1}, \kappa_n) &= \max \left\{ \begin{array}{c} \varpi(\kappa_{n-1}, \kappa_n), \varpi(\kappa_n, \Gamma\kappa_{n-1}), \frac{\varpi(\kappa_{n-1}, \Gamma\kappa_{n-1})\varpi(\kappa_n, \Gamma\kappa_n)}{\varpi(\kappa_{n-1}, \kappa_n)}, \frac{\varpi(\kappa_n, \Gamma\kappa_n)(\varpi(\kappa_{n-1}, \Gamma\kappa_{n-1})+1)}{1+\varpi(\kappa_{n-1}, \kappa_n)}, \\ \frac{\varpi(\kappa_{n-1}, \Gamma\kappa_n)+\varpi(\kappa_n, \Gamma\kappa_{n-1})}{4s} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} \varpi(\kappa_{n-1}, \kappa_n), \varpi(\kappa_n, \kappa_n), \frac{\varpi(\kappa_{n-1}, \kappa_n)\varpi(\kappa_n, \kappa_{n+1})}{\varpi(\kappa_{n-1}, \kappa_n)}, \frac{\varpi(\kappa_n, \kappa_{n+1})(\varpi(\kappa_{n-1}, \kappa_n)+1)}{1+\varpi(\kappa_{n-1}, \kappa_n)}, \\ \frac{\varpi(\kappa_{n-1}, \kappa_{n+1})+\varpi(\kappa_n, \kappa_n)}{4s} \end{array} \right\} \\ &= \max \left\{ \varpi(\kappa_{n-1}, \kappa_n), \varpi(\kappa_n, \kappa_{n+1}), \varpi(\kappa_n, \kappa_n), \frac{\varpi(\kappa_{n-1}, \kappa_{n+1}) + \varpi(\kappa_n, \kappa_n)}{4s} \right\}. \end{aligned}$$

On the other hand, from (ϖ_3), we get

$$\varpi(\kappa_n, \kappa_n) \leq 2s\omega(\kappa_n, \kappa_{n-1}),$$

and

$$\varpi(\kappa_{n-1}, \kappa_{n+1}) \leq s[\varpi(\kappa_{n-1}, \kappa_n) + \varpi(\kappa_n, \kappa_{n+1})].$$

That is,

$$\begin{aligned} N_{\varpi}(\kappa_{n-1},\kappa_n) &\leq \max\left\{\varpi(\kappa_n,\kappa_{n+1}), 2s\omega(\kappa_n,\kappa_{n-1}), \frac{s[\varpi(\kappa_{n-1},\kappa_n)+\varpi(\kappa_n,\kappa_{n+1})]+2s\omega(\kappa_n,\kappa_{n-1})}{4s}\right\} \\ &= \max\left\{\varpi(\kappa_n,\kappa_{n+1}), 2s\omega(\kappa_n,\kappa_{n-1}), \frac{1}{4}\varpi(\kappa_n,\kappa_{n+1}), \frac{3}{4}\varpi(\kappa_{n-1},\kappa_n)\right\} \\ &= \max\left\{\varpi(\kappa_n,\kappa_{n+1}), 2s\omega(\kappa_n,\kappa_{n-1})\right\}. \end{aligned}$$

It follows from (4) that

$$2s\varpi(\kappa_n,\kappa_{n+1}) \leq 2s^q\varpi(\kappa_n,\kappa_{n+1}) \leq \theta(\max\{\varpi(\kappa_n,\kappa_{n+1}), 2s\omega(\kappa_n,\kappa_{n-1})\}).$$

Now, if $\varpi(\kappa_{n-1},\kappa_n) \leq \varpi(\kappa_n,\kappa_{n+1})$ for some $n \in \mathbb{N}$, then

$$2s\varpi(\kappa_n,\kappa_{n+1}) \leq 2s^q\varpi(\kappa_n,\kappa_{n+1}) \leq \theta(\omega(\kappa_n,\kappa_{n+1})) < \varpi(\kappa_n,\kappa_{n+1}).$$

By the above inequality, we deduce that $\varpi(\kappa_n,\kappa_{n+1}) = 0$, which is a contradiction. Since we get supposed $\varpi(\kappa_n,\kappa_{n+1}) > 0$, thus, for all $n \in \mathbb{N}$,

$$2s\varpi(\kappa_n,\kappa_{n+1}) \leq 2s^q\varpi(\kappa_n,\kappa_{n+1}) \leq \theta(2s\varpi(\kappa_{n-1},\kappa_n)). \tag{5}$$

Hence,

$$\varpi(\kappa_n,\kappa_{n+1}) < \varpi(\kappa_{n-1},\kappa_n),$$

that is, a sequence $\{\varpi(\kappa_n,\kappa_{n+1})\}$ is decreasing and bounded below. Thus, there exists $\hbar \geq 0$ such that

$$\lim_{n\to\infty} \varpi(\kappa_n,\kappa_{n+1}) = \hbar. \tag{6}$$

We shall demonstrate that $\hbar = 0$. By a way of contradiction, thus, assume that $\hbar > 0$. By (5) and (6) and the properties of θ, one can write

$$2s\hbar \leq 2s^q\hbar \leq \theta(2s\hbar) < 2s\hbar.$$

This is a contradiction again. Hence,

$$\lim_{n\to\infty} \varpi(\kappa_n,\kappa_{n+1}) = 0. \tag{7}$$

In the next step, we claim that

$$\lim_{n,m\to\infty} \omega(\kappa_n,\kappa_m) = 0.$$

Let, if possible, $\lim_{n,m\to\infty}\varpi(\kappa_n,\kappa_m) \neq 0$. Then, by Remark 5, there exists $\varepsilon > 0$ and sequences $\{m(\ell)\}_{\ell=1}^{\infty}$ and $\{n(\ell)\}_{\ell=1}^{\infty}$ of positive integers with $n_\ell > m_\ell > \ell$ such that $\varpi(\kappa_{n_\ell},\kappa_{m_\ell}) \geq \varepsilon$, $\varpi(\kappa_{m_\ell},\kappa_{n_\ell-1}) < \varepsilon$ and

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1}) \leq \varepsilon s, \\ \frac{\varepsilon}{s} &\leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell-1},\kappa_{m_\ell}) \leq \varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{n\to\infty} \varpi(\kappa_{n_\ell},\kappa_{m_\ell-1}) \leq \varepsilon s^2. \end{aligned} \tag{8}$$

Applying condition (3), one can get

$$2s^2\varpi(\kappa_{n_\ell},\kappa_{m_\ell}) = 2s^q\varpi(\kappa_{n_\ell},\kappa_{m_\ell}) = 2s^q\varpi(\Gamma\kappa_{n_\ell-1},\Gamma\kappa_{m_\ell-1}) \leq \theta\left(N_{\varpi}(\kappa_{n_\ell-1},\kappa_{m_\ell-1})\right), \tag{9}$$

where

$$\begin{aligned} N_{\varpi}(\kappa_{n_\ell-1},\kappa_{m_\ell-1}) &= \max\left\{ \begin{array}{c} \varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1}),\varpi(\kappa_{m_\ell-1},\Gamma\kappa_{n_\ell-1}),\frac{\varpi(\kappa_{n_\ell-1},\Gamma\kappa_{n_\ell-1})\varpi(\kappa_{m_\ell-1},\Gamma\kappa_{m_\ell-1})}{\varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1})}, \\ \frac{\varpi(\kappa_{m_\ell-1},\Gamma\kappa_{m_\ell-1})[1+\varpi(\kappa_{n_\ell-1},\Gamma\kappa_{n_\ell-1})]}{1+\varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1})},\frac{\varpi(\kappa_{n_\ell-1},\Gamma\kappa_{m_\ell-1})+\varpi(\kappa_{m_\ell-1},\Gamma\kappa_{n_\ell-1})}{4s} \end{array}\right\} \\ &= \max\left\{ \begin{array}{c} \varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1}),\varpi(\kappa_{m_\ell-1},\kappa_{n_\ell}),\frac{\varpi(\kappa_{n_\ell-1},\kappa_{n_\ell})\varpi(\kappa_{m_\ell-1},\kappa_{m_\ell})}{\varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1})}, \\ \frac{\varpi(\kappa_{m_\ell-1},\kappa_{m_\ell})[1+\varpi(\kappa_{n_\ell-1},\kappa_{n_\ell})]}{1+\varpi(\kappa_{n_\ell-1},\kappa_{m_\ell-1})},\frac{\varpi(\kappa_{n_\ell-1},\kappa_{m_\ell})+\varpi(\kappa_{m_\ell-1},\kappa_{n_\ell})}{4s} \end{array}\right\}. \end{aligned} \tag{10}$$

Passing the upper limit as $\ell \to \infty$ in (10) and using (7) and (8), one can write

$$\limsup_{\ell\to\infty} N_{\varpi}(\kappa_{n_\ell-1},\kappa_{m_\ell-1}) = \max\left\{\varepsilon s,\varepsilon s^2,0,0,\frac{\varepsilon+\varepsilon s^2}{4s}\right\} = \varepsilon s^2. \tag{11}$$

Again, passing the upper limit as $\ell \to \infty$ in (9) and applying (11), we can get $2s^2 \le \theta(\varepsilon s^2)$. This contradiction proves that $\lim_{n,m\to\infty}\varpi(\kappa_n,\kappa_m) = 0$, hence, $\{\kappa_n\}$ is a ϖ-Cauchy sequence. Since Y is ϖ-closed in (Ω,ϖ), there exists $\kappa \in \mathrm{Y} = \cup_{i=1}^{l} B_i$ such that $\lim_{n\to\infty}\kappa_n = \kappa$ in (Y,ϖ),

$$\varpi(\kappa,\kappa) = \lim_{n\to\infty}\varpi(\kappa,\kappa_n) = \lim_{n,m\to\infty}\varpi(\kappa,\kappa_m) = 0. \tag{12}$$

Firstly, suppose that (a) of (iii) is satisfied, it is mean that, Γ is η-continuous. Thus, we can write,

$$\kappa = \lim_{n\to\infty}\kappa_{n+1} = \lim_{n\to\infty}\Gamma\kappa_n = \Gamma\kappa.$$

Otherwise, if (b) of (iii) is realized and $\varpi(\kappa,\Gamma\kappa) > 0$, then, we get $\eta(\kappa,\Gamma\kappa) \ge 1$ and so

$$\eta(\kappa,\Gamma\kappa)\eta(\kappa_{n(k)},\Gamma\kappa_{n(k)}) \ge 1.$$

Secondly, we shall show that a fixed point of Γ exists and is named κ. Since $\lim_{n\to\infty}\kappa_n = \kappa$ and $\Gamma(B_j) \subseteq B_{j+1}$, $j = 1,2,...,l$, where $B_{l+1} = B_1$, the sequence $\{\kappa_n\}$ has infinitely many terms in each B_i for $i \in \{1,2,...,l\}$. Assume that $\kappa \in B_i$. Therefore, $\Gamma\kappa \in B_{i+1}$ and we select a subsequence $\{\kappa_{n(k)}\}$ of $\{\kappa_n\}$ with $\kappa_{n(k)} \in B_{i-1}$ (by the above-mentioned comment, this subsequence exists). By using the contractive condition, we have

$$\varpi(\Gamma\kappa,\kappa) \le 2s^q\varpi(\Gamma\kappa,\Gamma\kappa_{n(k)}) \le \theta\left(\max\left\{ \begin{array}{c} \varpi(\kappa,\kappa_{n(k)}),\varpi(\kappa_{n(k)},\Gamma\kappa),\frac{\varpi(\kappa,\Gamma\kappa)\varpi(\kappa_{n(k)},\Gamma\kappa_{n(k)})}{\varpi(\kappa,\kappa_{n(k)})}, \\ \frac{\varpi(\kappa_{n(k)},\Gamma\kappa_{n(k)})[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\kappa_{n(k)})},\frac{\varpi(\kappa,\Gamma\kappa_{n(k)})+\varpi(\kappa_{n(k)},\Gamma\kappa)}{4s} \end{array}\right\}\right),$$

taking the limit as $k \to \infty$ and using (12), we can get

$$\begin{aligned} \varpi(\Gamma\kappa,\kappa) &\le \theta\left(\max\left\{ \begin{array}{c} \varpi(\kappa,\kappa),\varpi(\kappa,\Gamma\kappa),\frac{\varpi(\kappa,\Gamma\kappa)\varpi(\kappa,\kappa)}{\varpi(\kappa,\kappa)}, \\ \frac{\varpi(\kappa,\kappa)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\kappa)},\frac{\varpi(\kappa,\kappa)+\varpi(\kappa,\Gamma\kappa)}{4s} \end{array}\right\}\right) \\ &= \theta(\max\left\{0,\varpi(\kappa,\Gamma\kappa),0,0,\frac{\varpi(\kappa,\Gamma\kappa)}{4s}\right\}) = \theta(\varpi(\kappa,\Gamma\kappa)), \end{aligned}$$

a contradiction, so $\varpi(\kappa,\Gamma\kappa) = 0$, that is, $\Gamma\kappa = \kappa$. The cyclic character of Γ and the fact that $\kappa \in \Omega$ is a fixed point of Γ, which leads to $\kappa \in \bigcap_{i=1}^{l} B_i$.

Finally, we will demonstrate the uniqueness of the fixed point, let $\kappa, \mu \in \bigcap_{i=1}^{l} B_i$ be two fixed points of Γ such that $\kappa \neq \mu$ and (iv) is realized. Then, we obtain that $\varpi(\kappa,\mu) > 0$, $\varpi(\kappa,\kappa) \geq 1$, $\varpi(\mu,\mu) \geq 1$ and, by (3), we have

$$\begin{aligned} \varpi(\kappa,\mu) &\leq 2s^q \varpi(\Gamma\kappa,\Gamma\mu) \leq \theta\left(\max\left\{\begin{array}{c} \varpi(\kappa,\mu), \varpi(\mu,\Gamma\kappa), \frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}, \\ \frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s} \end{array}\right\}\right) \\ &= \theta\left(\max\left\{\begin{array}{c} \varpi(\kappa,\mu), \varpi(\mu,\kappa), \frac{\varpi(\kappa,\kappa)\varpi(\mu,\mu)}{\varpi(\kappa,\mu)}, \\ \frac{\varpi(\mu,\mu)[1+\varpi(\kappa,\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\mu)}{2s} \end{array}\right\}\right). \end{aligned} \tag{13}$$

Applying Remark 2, we have $\varpi(\kappa,\kappa) = \omega(\mu,\mu) = 0$; therefore, it follows from (13) that

$$\varpi(\kappa,\mu) \leq \theta\left(\max\left\{\varpi(\kappa,\mu), \varpi(\mu,\kappa), 0, 0, \frac{\varpi(\kappa,\mu)}{2s}\right\}\right) = \theta(\varpi(\kappa,\mu)) < \varpi(\kappa,\mu),$$

a contradiction again. Hence, $\varpi(\kappa,\mu) = 0$, that is, $\kappa = \mu$. The proof is finished. □

To display the validity results of Theorem 2, we present two examples as follows:

Example 5. *Suppose that $\Omega = \mathbb{R}$ is equipped with the b-metric-like mapping $\varpi(\kappa,\mu) = |\kappa| + |\mu|$ for all $\kappa,\mu \in \Omega$ with $s \geq 1$ and $q > 1$. Suppose that $B_1 = (-\infty, 0]$ and $B_2 = [0,\infty)$ and $\Upsilon = B_1 \cup B_2$. Define $\Gamma : \Upsilon \to \Upsilon$ and $\eta : \Upsilon \times \Upsilon \to [0,\infty)$ by*

$$\Gamma\kappa = \begin{cases} -2\kappa, & \text{if } \kappa \in (-\infty,-1), \\ \frac{-\kappa}{16s^q}, & \text{if } \kappa \in [-1,0], \\ \frac{-\kappa^3}{8s^q}, & \text{if } \kappa \in [0,1], \\ -4\kappa, & \text{if } \kappa \in (1,\infty), \end{cases} \quad \text{and } \eta(\kappa,\mu) = \begin{cases} \kappa^2 + \mu^2 + 2, & \text{if } \kappa,\mu \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

In addition, define $\theta : [0,\infty) \to [0,\infty)$ by $\theta(t) = \frac{1}{4}t$. Clearly, $\Gamma(B_1) \subseteq B_2$ and $\Gamma(B_2) \subseteq B_1$. Let $\kappa \in B_1$, $\mu \in B_2$ and $\eta(\kappa,\Gamma\kappa)\eta(\mu,\Gamma\mu) \geq 1$. Now, if $\kappa \notin [-1,1]$ or $\mu \notin [-1,1]$, then $\eta(\kappa,\Gamma\kappa) = 0$ or $\eta(\mu,\Gamma\mu) = 0$. That is, $\eta(\kappa,\Gamma\kappa)\eta(\mu,\Gamma\mu) = 0 < 1$, which is a contradiction. Hence, $\kappa \in B_1$, $\mu \in B_2$ and $\kappa,\mu \in [-1,1]$. This implies that $\kappa \in [-1,0]$ and $\mu \in [0,1]$. Then,

$$\begin{aligned} 2s^q\varpi(\Gamma\kappa,\Gamma\mu) &= 2s^q\left(\left|\frac{-\kappa}{16s^q}\right| + \left|\frac{-\mu^3}{8s^q}\right|\right) \\ &= \frac{|\kappa|}{8} + \frac{|\mu^3|}{4} \\ &\leq 2 \times \max\left\{\frac{|\kappa|}{8}, \frac{|\mu^3|}{4}\right\} \text{ since } \frac{|a|+|b|}{2} \leq \max\{|a|,|b|\}, \forall a,b \in \mathbb{R} \\ &\leq 2 \times \max\left\{\frac{|\kappa|}{8}, \frac{|\mu|}{8}\right\} \\ &= \frac{1}{4} \times \max\{|\kappa|,|\mu|\} \leq \frac{1}{4} \times (|\kappa| + |\mu|) \text{ since } \max\{|a|,|b|\} \leq |a| + |b|, \forall a,b \in \mathbb{R} \\ &= \frac{1}{4}\varpi(\kappa,\mu) \leq \theta(M_\varpi(\kappa,\mu)). \end{aligned}$$

Then, the mapping Γ is a cyclic η_s^q-rational contractive. It is clear that, $\eta(0,\Gamma 0) \geq 1$, and this proves the condition (ii) of Theorem 2. If $\eta(\kappa,\mu) \geq 1$, then $\kappa,\mu \in [-1,1]$ which leads to $\eta(\Gamma\kappa,\Gamma\mu) \geq 1$, that is, the mapping Γ is an η-admissible. Consider a sequence $\{\kappa_n\} \in \Omega$ such that $\eta(\kappa_n,\Gamma\kappa_n) \geq 1$ and $\kappa_n \to \kappa$ as $n \to \infty$. Then, we should have $\kappa_n \in [-1,1]$ and so $\kappa \in [-1,1]$, that is, $\eta(\kappa,\Gamma\kappa) \geq 1$. Hence, all the hypotheses of Theorem 2 are realized and Γ has a fixed point $\kappa = 0 \in B_1 \cap B_2$.

Definition 8. *Let (Ω, d) be a metric space, $l \in \mathbb{N}$, $B_1, B_2, ...B_l$ closed non-empty subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ be a mapping. The mapping Γ is called a cyclic η_s^q-rational contractive if*

(a)

$$\Gamma(B_j) \subseteq B_{j+1},\ j = 1, 2, ..., l,\ \text{ where }\ B_{l+1} = B_1.$$

(b) For any $\kappa \in B_i$ and $\tau \in B_{i+1}$, $i = 1, 2, ..., l$, such that $\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1$, we get

$$2s^q d(\Gamma\kappa, \Gamma\mu) \leq \theta(N(\kappa, \mu)),$$

for all $q \geq 1$ and

$$N(\kappa, \mu) = \max\left\{d(\kappa, \mu), d(\mu, \Gamma\kappa), \frac{d(\kappa, \Gamma\kappa)d(\mu, \Gamma\mu)}{d(\kappa, \mu)}, \frac{d(\mu, \Gamma\mu)[1 + d(\kappa, \Gamma\kappa)]}{1 + d(\kappa, \mu)}, \frac{d(\kappa, \Gamma\mu) + d(\mu, \Gamma\kappa)}{4s}\right\}.$$

The following corollary follows immediately by Theorem 2 in the ordinary metric space:

Corollary 1. *Suppose that (Ω, d) is a complete metric space, l be a positive integer, $B_1, B_2, ...B_l$ be non-empty closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ be a mapping. Let the mapping $\Gamma : \Upsilon \to \Upsilon$ be a cyclic η_s^q-rational contractive such that the following conditions hold:*

(i) a nonlinear mapping Γ is an η-admissible;
(ii) $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$ if there exists $\kappa_\circ \in \Upsilon$;
(iii) (a) a nonlinear mapping Γ is η-continuous, or

(b) for any sequence $\{\kappa_n\}$ in Ω and for all $n \geq 0$, if $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ and $\lim \kappa_n = \kappa$, then, $\eta(\kappa, \Gamma\kappa) \geq 1$. Therefore, Γ has a fixed point $\kappa \in \cap_{i=1}^{l} B_i$. Furthermore, if
(iv) $\forall \kappa \in \nabla$, we get $\eta(\kappa, \kappa) \geq 1$.

Then, the uniqueness of the fixed point is realized.

If we put $\theta(t) = \frac{1}{2}t$, in Theorem 2, we get the following consequence.

Corollary 2. *Consider that (Ω, ϖ) is a complete b-metric-like space, l is a positive integer, $B_1, B_2, ...B_l$ is nonempty ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ is a mapping. Let the mapping $\Gamma : \Upsilon \to \Upsilon$ be a cyclic η_s^q-rational contractive that verifies the following hypotheses:*

(i) $\Gamma(B_j) \subseteq B_{j+1}$, $j = 1, 2, .., l$, where $B_{l+1} = B_1$;
(ii) a nonlinear mapping Γ is an η-admissible;
(iii) $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$ if there exists $\kappa_\circ \in \Upsilon$;
(iv) (a) a nonlinear mapping Γ is η-continuous, or

(b) for any sequence $\{\kappa_n\}$ in Ω and for all $n \geq 0$, if $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ and $\lim \kappa_n = \kappa$, then, $\eta(\kappa, \Gamma\kappa) \geq 1$.
(v) there exists $h \in (0, \frac{1}{4}]$ such that $\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1 \Longrightarrow$

$$s^q \varpi(\Gamma\kappa, \Gamma\mu) \leq h \max\left\{\varpi(\kappa, \mu), \varpi(\mu, \Gamma\kappa), \tfrac{\varpi(\kappa, \Gamma\kappa)\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)}, \tfrac{\varpi(\mu, \Gamma\mu)[1 + \varpi(\kappa, \Gamma\kappa)]}{1 + \varpi(\kappa, \mu)}, \tfrac{\varpi(\kappa, \Gamma\mu) + \varpi(\mu, \Gamma\kappa)}{4s}\right\},$$

for any $\kappa \in B_i$ and $\mu \in B_{i+1}$, $i = 1, 2, .., l$.
Therefore, Γ has a fixed point $\kappa \in \cap_{i=1}^{l} B_i$. Furthermore, if
(iv) $\forall \kappa \in \nabla$, we get $\eta(\kappa, \kappa) \geq 1$.

Then, the uniqueness of the fixed point is realized.

In the line of exponential type, the following consequence holds:

Corollary 3. *Suppose that (Ω, ϖ) is a complete b-metric-like space, l is a positive integer, $B_1, B_2, ...B_l$ are nonempty ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ is a mapping. Let the mapping $\Gamma : \Upsilon \to \Upsilon$ be a cyclic η_s^q-rational contractive that fulfills the following hypotheses:*

(i) $\Gamma(B_j) \subseteq B_{j+1}$, $j = 1, 2, .., l,$, *where* $B_{l+1} = B_1$;
(ii) *a nonlinear mapping* Γ *is an* η*-admissible;*
(iii) $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$ *if there exists* $\kappa_\circ \in \Upsilon$;
(iv) (a) *a nonlinear mapping* Γ *is* η*-continuous, or*

(b) *for any sequence* $\{\kappa_n\}$ *in* Ω *and for all* $n \geq 0$, *if* $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ *and* $\lim \kappa_n = \kappa$, *then,* $\eta(\kappa, \Gamma\kappa) \geq 1$.
(v) *there exists* $h \in (0, \frac{1}{4}]$ *such that*

$\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1 \Longrightarrow$

$$e^{s^q \varpi(\Gamma\kappa, \Gamma\mu)} \leq h e^{\max\left\{\varpi(\kappa,\mu), \varpi(\mu,\Gamma\kappa), \frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}, \frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s}\right\}},$$

for any $\kappa \in B_i$ *and* $\mu \in B_{i+1}$, $i = 1, 2, .., l$, *where* $e : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping satisfying* $e^\epsilon > 0$ *for* $\epsilon > 0$.
Therefore, Γ *has a fixed point* $\kappa \in \cap_{i=1}^{l} B_i$. *Furthermore, if*
(iv) $\forall \kappa \in \nabla$, *we get* $\eta(\kappa, \kappa) \geq 1$.

Then, the uniqueness of the fixed point is realized.

Note that the results of Corollary 3 also hold if we take $\Omega = B_i$, $i = 1, 2, .., l$, as meaning the nonlinear mapping Γ is not cyclic.

The following theorem follows immediately from Theorem 2 by taking $\Omega = B_i$, $i = 1, 2, .., l$.

Theorem 3. *Let (Ω, ϖ) be a complete b-metric-like space, and $\eta : \Omega \times \Omega \to [0, \infty)$ be a mapping. Suppose that the mapping $\Gamma : \Omega \to \Omega$ is a cyclic η_s^q-rational contractive that fulfills the following hypotheses:*

(i) *a nonlinear mapping* Γ *is an* η*-admissible;*
(ii) $\eta(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$ *if there exists* $\kappa_\circ \in \Upsilon$;
(iii) (a) *a nonlinear mapping* Γ *is* η*-continuous, or*

(b) *for any sequence* $\{\kappa_n\}$ *in* Ω *and for all* $n \geq 0$, *if* $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ *and* $\lim \kappa_n = \kappa$, *then,* $\eta(\kappa, \Gamma\kappa) \geq 1$.
Therefore, Γ *has a fixed point* $\kappa \in \cap_{i=1}^{l} B_i$. *Furthermore, if*
(iv) $\forall \kappa \in \nabla$, *we get* $\eta(\kappa, \kappa) \geq 1$.

Then, the uniqueness of the fixed point is realized.

To illustrate the usefulness of Theorem 3, we give the following example:

Example 6. *Let $\Omega = \mathbb{R}^+$ be equipped with the metric-like mapping $\omega(x, y) = (\max\{x, y\})^2$ for all $\kappa, \mu \in \Omega$ with $s = q = 2$. Let $\Gamma : \Omega \to \Omega$ and $\eta : \Omega \times \Omega \to [0, \infty)$ be defined by*

$$\Gamma\kappa = \begin{cases} \frac{1}{4}\kappa^3, & \text{when } 0 \leq \kappa < \frac{1}{2}, \\ \frac{1}{6}\kappa^2, & \text{when } \frac{1}{2} \leq \kappa \leq 1, \\ \frac{1}{20}\kappa, & \text{when } 1 < \kappa \leq 2, \\ 2\kappa^3 + 1, & \text{when } \kappa > 2, \end{cases} \quad \text{and } \eta(\kappa, \mu) = \begin{cases} 4, & \text{if } \kappa, \mu \in [0, 2], \\ 0, & \text{otherwise.} \end{cases}$$

In addition, define $\theta : [0, \infty) \to [0, \infty)$ as in the cases below. Let $\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1$, then $\kappa, \mu \in [0, 2]$. Now, we discuss the following cases:

- *Consider* $0 \leq \kappa, \mu < \frac{1}{2}$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa^3}{4}, \frac{\mu^3}{4}\right\}\right)^2 = \frac{1}{2}\left(\max\left\{\kappa^3, \mu^3\right\}\right)^2 \leq \frac{1}{2}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{1}{2}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{1}{2}t. \end{aligned}$$

- *Consider* $\frac{1}{2} \leq \kappa, \mu \leq 1$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa^2}{6}, \frac{\mu^2}{6}\right\}\right)^2 = \frac{1}{8}\left(\max\left\{\kappa^2, \mu^2\right\}\right)^2 \leq \frac{1}{8}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{1}{8}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{1}{8}t. \end{aligned}$$

- *Consider* $1 < \kappa, \mu \leq 2$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa}{20}, \frac{\mu}{20}\right\}\right)^2 = \frac{2}{25}\left(\max\left\{\frac{\kappa}{2}, \frac{\mu}{2}\right\}\right)^2 \leq \frac{2}{25}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{2}{25}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{2}{25}t. \end{aligned}$$

- *Consider* $0 \leq \kappa < \frac{1}{2}$ *and* $\frac{1}{2} \leq \mu < 1$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa^3}{4}, \frac{\mu^2}{6}\right\}\right)^2 = \frac{1}{2}\left(\max\left\{\kappa^3, \frac{2\mu^2}{3}\right\}\right)^2 \leq \frac{1}{2}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{1}{2}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{1}{2}t. \end{aligned}$$

- *Consider* $0 \leq \kappa < \frac{1}{2}$ *and* $1 < \mu \leq 2$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa^3}{4}, \frac{\mu}{20}\right\}\right)^2 = \frac{1}{2}\left(\max\left\{\kappa^3, \frac{\mu}{5}\right\}\right)^2 \leq \frac{1}{2}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{1}{2}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{1}{2}t. \end{aligned}$$

- *Consider* $\frac{1}{2} \leq \kappa \leq 1$ *and* $1 < \mu \leq 2$, *then*

$$\begin{aligned} 2s^q(\varpi(\Gamma\kappa, \Gamma\mu)) &= 8\left(\max\left\{\frac{\kappa^2}{6}, \frac{\mu}{20}\right\}\right)^2 = \frac{2}{9}\left(\max\left\{\kappa^2, \frac{3\mu}{10}\right\}\right)^2 \leq \frac{2}{9}\left(\max\left\{\kappa, \mu\right\}\right)^2 \\ &= \frac{2}{9}\varpi(\kappa, \mu) \leq \theta(N(\kappa, \mu)), \textit{ where } \quad \theta(t) = \frac{2}{9}t. \end{aligned}$$

Then, the mapping Γ *is an* η_s^q*-rational contractive. As in Example 5, we can find that the hypotheses (i), (ii) and (iv) of Theorem 3 are true and the fixed point here is* 0 *of* Γ.

4. Some Related Fixed Point on a Directed Graph

In this part, we shall consider the contractive condition (3) of Definition 6 without the function η to discuss some related fixed point in the framework of b-metric-like spaces endowed with a graph.

According to the results of Jachymski [30] in metric-like space, let (Ω, ω) be a metric-like space endowed with a directed graph G whose set of vertices $\Phi(G)$ coincides with Ω and the set of edges will be denoted by $\Xi(G)$.

The undirected graph obtained from G by ignoring the direction of edges denoted by the letter $\breve{G}$. Actually, it will be more convenient for us to treat $\breve{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$\Xi(\breve{G}) = \Xi(G) \cup \Xi(G^{-1}).$$

If κ and μ are vertices in a graph G, then a path in G from κ to μ of length $N(N \in \mathbb{N})$ is a sequence $\{\kappa_i\}_i^N = 0$ of $N+1$ vertices such that $\kappa_\circ = \kappa$, $\kappa_N = \mu$ and $(\kappa_{n-1}, \kappa_n) \in \Xi(G)$ for $i = 1, ..., N$. If there is a path between any two vertices, then a graph G is called connected and G is weakly connected if $\breve{G}$ is connected.

The first work in this direction was initiated by Jachymski [30] in which the author introduced the concept of a graph preserving mapping and G-contraction for a single valued mapping defined on a metric space endowed with a graph.

Definition 9 ([30]). *It is said that a nonlinear mapping $\Gamma : \Omega \to \Omega$ is G-contraction if Γ preserves edges of G, i.e.,*

$$\forall \kappa, \mu \in \Omega : (\kappa, \mu) \in \Xi(G) \Rightarrow (\Gamma(\kappa), \Gamma(\mu)) \in \Xi(G)$$

and Γ decreases weights of edges of G as for all $\kappa, \mu \in \Omega$, there exists $\beta \in (0,1]$, such that

$$(\kappa, \mu) \in \Xi(G) \Rightarrow \omega(\Gamma(\kappa), \Gamma(\mu)) \leq \beta\omega(\kappa, \mu).$$

Definition 10 ([31]). *If given $\kappa \in \Omega$ and any sequence $\{k_n\}$ of positive integers, a mapping $\Gamma : \Omega \to \Omega$ is called orbitally continuous, if*

$$\Gamma^{k_n}\kappa \to \mu \in \Omega \text{ as } n \to \infty \;\Rightarrow\; \Gamma(\Gamma^{k_n}\kappa) \to \Gamma\mu \text{ as } n \to \infty.$$

Definition 11 ([30]). *A mapping $\Gamma : \Omega \to \Omega$ is called G-continuous, if given $\kappa \in \Omega$ and sequence $\{\kappa_n\}$ for all $n \in \mathbb{N}$,*

$$\kappa_n \to \kappa \text{ as } n \to \infty \text{ and } (\kappa_n, \kappa_{n+1}) \in \Xi(G) \text{ imply } \Gamma\kappa_n \to \Gamma\kappa.$$

Definition 12 ([30]). *If given $\kappa, \mu \in \Omega$ and any sequence $\{k_n\}$ of positive integers for all $n \in \mathbb{N}$, a mapping $\Gamma : \Omega \to \Omega$ is called orbitally G-continuous if*

$$\Gamma^{k_n}\kappa \to \mu \in \Omega \text{ and } (\Gamma^{k_n}x, \Gamma^{k_n+1}x) \in \Xi(G) \;\Rightarrow\; \Gamma(\Gamma^{k_n}\kappa) \to \Gamma\mu \text{ as } n \to \infty.$$

Now, by generalizing the above results to a b-metric-like space, firstly, we introduce a cyclic (s, q)-graphic rational contractive mapping in such space as follows:

Definition 13. *Let (Ω, ϖ) be a b-metric-like space endowed with a graph G (with parameter $s \geq 1$). Let l be positive integer, $B_1, B_2, ...B_l$ be ϖ-closed subsets of Ω and $\Upsilon = \cup_{i=1}^{l} B_i$. the mapping Γ is called a cyclic (s, q)-graphic rational contractive if*

- $\Gamma(B_j) \subseteq B_{j+1}$, $j = 1, 2, ..., l$, *where* $B_{l+1} = B_1$;
- *for all* $\kappa, \mu \in \Omega : (\kappa, \mu) \in \Xi(G) \Rightarrow (\Gamma\kappa, \Gamma\mu) \in \Xi(G)$;
- *for any* $\kappa \in B_i, \mu \in B_{i+1}, i = 1, 2, ..., l$ *where* $B_{l+1} = B_1$ *and* $((\kappa, \Gamma\kappa), (\mu, \Gamma\mu)) \in \Xi(G)$, *we get*

$$2s^q\varpi(\Gamma\kappa, \Gamma\mu) \leq \theta(N(\kappa, \mu)),$$

where $\theta \in \Theta$, $q > 1$ is a constant, and

$$N(\kappa, \mu) = \max\left\{\varpi(\kappa, \mu), \varpi(\mu, \Gamma\kappa), \frac{\varpi(\kappa, \Gamma\kappa)\varpi(\mu, \Gamma\mu)}{\varpi(\kappa, \mu)}, \frac{\varpi(\mu, \Gamma\mu)[1 + \varpi(\kappa, \Gamma\kappa)]}{1 + \varpi(\kappa, \mu)}, \frac{\varpi(\kappa, \Gamma\mu) + \varpi(\mu, \Gamma\kappa)}{4s}\right\}.$$

Notice that, if we take $\Omega = B_i$, $i = 1, 2, ..., l$, in Definition 13, we say that Γ is (s, q)-graphic rational contraction.

Next, we state and prove our main theorem in this part.

Theorem 4. *Let (Ω, ϖ) be a complete b-metric-like space endowed with a graph G. Let l be a positive integer, $B_1, B_2, ...B_l$ be nonempty ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and the mapping $\Gamma : \Upsilon \to \Upsilon$ be a cyclic (s, q)-graphic rational contractive that fulfills the following hypotheses:*

(i) $(\kappa_\circ, \Gamma\kappa_\circ) \in \Xi(G)$, *if there exists* $\kappa_\circ \in \Omega$;

(ii) (a) *the nonlinear mapping* Γ *is orbitally* G*-continuous, or*

(b) *for any sequence* $\{\kappa_n\}$ *in* Ω, *if* $\lim_{n\to\infty} \kappa_n = \kappa$, *then* $(\kappa_n, \kappa_{n+1}) \in \Xi(G)$.

Then, there exists a unique fixed point $\kappa \in \cap_{i=1}^{l} B_i$ *of* Γ.

Proof. Consider that $\eta : \Upsilon \times \Upsilon \to [0, +\infty)$ is a mapping defined by

$$\eta(\kappa, \mu) = \begin{cases} 1, & \text{if } (\kappa, \mu) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

We shall prove that Γ is η-admissible. If $\eta(\kappa, \mu) \geq 1$, then $(\kappa, \mu) \in \Xi(G)$. As a mapping Γ is a cyclic (s, q)-graphic rational contractive, we get $(\Gamma\kappa, \Gamma\mu) \in \Xi(G)$. That is, $\eta(\Gamma\kappa, \Gamma\mu) \geq 1$. Thus, Γ is an η-admissible mapping. Suppose that Γ is G-continuous on Ω, which means that

$$\lim_{n\to\infty} \kappa_n = \kappa \text{ and } (\kappa_n, \kappa_{n+1}) \in \Xi(G) \text{ for all } n \in \mathbb{N} \text{ imply } \lim_{n\to\infty} \Gamma\kappa_n = \Gamma\kappa.$$

This yields

$$\lim_{n\to\infty} \kappa_n = \kappa \text{ and } \eta(\kappa_n, \kappa_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ imply } \lim_{n\to\infty} \Gamma\kappa_n = \Gamma\kappa,$$

which tells us that Γ is η-continuous on Ω. By condition (i), there exists $\kappa_\circ \in \Omega$ such that $(\kappa_\circ, \Gamma\kappa_\circ) \in \Xi(G)$. That is, $\eta(\kappa, \Gamma\kappa_\circ) \geq 1$.

Let $\kappa \in B_i$ and $\mu \in B_{i+1}$ where $\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1$. Then, $\kappa \in B_i$ and $\mu \in B_{i+1}$ where $(\kappa, \Gamma\kappa) \in \Xi(G)$ and $(\mu, \Gamma\mu) \in \Xi(G)$.

Now, since Γ is a cyclic (s, q)-graphic rational contractive, we get

$$2s^q\varpi(\Gamma\kappa, \Gamma\mu) \leq \theta(N(\kappa, \mu)).$$

Hence, Γ is a cyclic η_s^q-rational contractive mapping. Assume that $\{\kappa_n\} \subseteq \Omega$ is a sequence that satisfies $\lim_{n\to\infty} \kappa_n = \kappa$ and $\eta(\kappa_n, \kappa_{n+1}) \geq 1$. Therefore, $(\kappa_{n+1}, \kappa) \in \Xi(G)$ and then, from condition (ii), we have $(\kappa, \Gamma\kappa) \in \Xi(G)$. That is, $\eta(\kappa, \Gamma\kappa) \geq 1$. Furthermore, sine $\Xi(G) \supseteq \aleph$, we have $\eta(\kappa, \kappa) = 1$ for all $\kappa \in \Omega$. Thus, all hypotheses of Theorem 2 are verified and Γ has a unique fixed point in $\Upsilon = \cap_{i=1}^{l} B_i$. This ends the proof. $\square$

Finally, we state some consequences for Theorem 4: If in Theorem 4, we put $\theta(t) = \frac{t}{2}$, then we state the following consequence.

Corollary 4. *Suppose that (Ω, ϖ) is a complete b-metric-like space endowed with a graph G, l is a positive integer, $B_1, B_2, ...B_l$ be nonempty ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{q} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ is a nonlinear mapping. Consider that $\Gamma : \Upsilon \to \Upsilon$ is an operator satisfying the following hypotheses:*

(i) $\Gamma(B_j) \subseteq B_{j+1}$, $j = 1, 2, .., l$, *where* $B_{l+1} = B_1$;

(ii) $(\Gamma\kappa, \Gamma\mu) \in \Xi(G)$, *if* $(\kappa, \mu) \in \Xi(G)$, *for all* $\kappa, \mu \in \Upsilon$;

(iii) $(\kappa_\circ, \Gamma\kappa_\circ) \in \Xi(G)$, *if there exists* $\kappa_\circ \in \Omega$;

(iv) *(a) the nonlinear mapping Γ is orbitally G-continuous, or*

(b) for any sequence $\{\kappa_n\}$ in Ω, if $\lim_{n\to\infty}\kappa_n = \kappa$, and $(\kappa_n, \kappa_{n+1}) \in \Xi(G)$, then $(\kappa, \Gamma\kappa) \in \Xi(G)$;

(v) *there exists $h \in (0, \frac{1}{4}]$ such that $(\kappa, \Gamma\kappa)(\mu, \Gamma\mu) \in \Xi(G) \Longrightarrow$*

$$\varpi(\Gamma\kappa, \Gamma\mu) \leq h \max\left\{\varpi(\kappa,\mu), \varpi(\mu,\Gamma\kappa), \frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}, \frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s}\right\},$$

for any $\kappa \in B_i$, $\mu \in B_{i+1}$, $i = 1, 2, .., l$, where $B_{l+1} = B_1$.

Then, Γ has a unique fixed point $\kappa \in \cap_{i=1}^{l} B_i$.

If in Theorem 4, we take $B_i = X$, then, we have the following theorem.

Theorem 5. *Let (Ω, ϖ) be a complete b-metric-like space endowed with a graph G. Consider the mapping $\Gamma : \Upsilon \to \Upsilon$ is a cyclic (s, q)-graphic rational contractive that fulfills the following hypotheses:*

(i) *$(\Gamma\kappa, \Gamma\mu) \in \Xi(G)$, if $(\kappa, \mu) \in \Xi(G)$, for all $\kappa, \mu \in \Upsilon$;*

(ii) *$(\kappa_\circ, \Gamma\kappa_\circ) \in \Xi(G)$, if there exists $\kappa_\circ \in \Omega$;*

(iii) *(a) the nonlinear mapping Γ is orbitally G-continuous, or*

(b) or any sequence $\{\kappa_n\}$ in Ω, if $\lim_{n\to\infty}\kappa_n = \kappa$, then $(\kappa_n, \kappa_{n+1}) \in \Xi(G)$.

Then, Γ has a unique fixed point.

In the direction of integral type, we can state the following result:

Corollary 5. *Let (Ω, ϖ) be a complete b-metric-like space, l be a positive integer, $B_1, B_2, ...B_l$ be nonempty ϖ-closed subsets of Ω, $\Upsilon = \cup_{i=1}^{l} B_i$ and $\eta : \Upsilon \times \Upsilon \to [0, \infty)$ be a mapping. Let $\Gamma : \Upsilon \to \Upsilon$ be an operator satisfying the following hypotheses:*

(i) *$\Gamma(B_j) \subseteq B_{j+1}$, $j = 1, 2, .., l$, where $B_{l+1} = B_1$;*

(ii) *$(\Gamma\kappa, \Gamma\mu) \in \Xi(G)$, if $(\kappa, \mu) \in \Xi(G)$, for all $\kappa, \mu \in \Upsilon$;*

(iii) *there exists $\kappa_\circ \in \Upsilon$ such that $(\kappa_\circ, \Gamma\kappa_\circ) \geq 1$;*

(iv) *$(\triangle_1)$ either Γ is orbitally G-continuous, or*

$(\triangle_2)$ for any sequence $\{\kappa_n\}$ in Ω with $\eta(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \geq 0$ and $\lim_{n\to\infty}\kappa_n = \kappa$, then $(\kappa, \Gamma\kappa) \in \Xi(G)$.

(v) *there exists $h \in (0, \frac{1}{4}]$ such that*

$(\kappa, \Gamma\kappa)(\mu, \Gamma\mu) \in \Xi(G) \Longrightarrow$

$$\int_0^{s^q\varpi(\Gamma\kappa,\Gamma\mu)} \varphi(t)dt \leq h \int_0^{\max\left\{\varpi(\kappa,\mu),\varpi(\mu,\Gamma\kappa),\frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)},\frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)},\frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s}\right\}} \varphi(t)dt,$$

for any $\kappa \in B_i$ and $\mu \in B_{i+1}$, $i = 1, 2, .., l$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\epsilon} \varphi(t)dt > 0$ for $\epsilon > 0$.

Then, Γ has a unique fixed point $\kappa \in \cap_{i=1}^{l} B_i$.

5. The Analytical Solution of Fredholm Integral Equation of the Second Kind

This part shows the important applications of our theoretical results. Here, we shall find the existence of solution for the following Fredholm integral equation:

$$\kappa(\alpha) = \int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta, \ (\alpha, \beta) \in [0, A]^2, \tag{14}$$

where : $[0, A] \times [0, A] \to [0, \infty)$ and $\sigma : [0, A] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions.

Assume that $\Omega = C([0, A], \mathbb{R})$ is the set of real continuous functions on $[0, A]$ for $A > 0$ equipped with

$$\varpi(\kappa, \mu) = \max_{\alpha \in [0,A]} (|\kappa(\alpha)| + |\mu(\alpha)|)^m, all \kappa, \mu \in \Omega,$$

where $q > 1$ and $m > 1$. It is clear that, with a coefficient $s = 2^{m-1}$, (Ω, ϖ) is a complete b-metric-like space.

Suppose that $\Pi : \Omega \times \Omega \to \mathbb{R}$ is a function with the following assumptions:

- $\Pi(\kappa, \mu) \geq 0 \Longrightarrow \Pi(\Gamma\kappa, \Gamma\mu) \geq 0$,
- $\Pi(\kappa_\circ, \Gamma\kappa_\circ) \geq 0$, if there exists $\kappa_\circ \in \Omega$,
- for all $n \in \mathbb{N}$, if $\{\kappa_n\}$ is a sequence in Ω such that $\Pi(\kappa_n, \kappa_{n+1}) \geq 0$ and $\lim_{n\to\infty} \kappa_n = \kappa$, then $\Pi(\kappa, \Gamma\kappa) \geq 0$, where

$$\Gamma\kappa(\alpha) = \int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta, \text{ for all } \alpha \in [0, A].$$

Let $(a, c) \in \Omega \times \Omega$, $(a_\circ, c_\circ) \in \mathbb{R}^2$ such that

$$a_\circ \leq a(\alpha) \leq c(\alpha) \leq c_\circ, \text{ for all } \alpha \in [0, A]. \tag{15}$$

Assume that, for all $\alpha \in [0, A]$, we have

$$a(\alpha) \leq \int_0^A (\alpha, \beta)\sigma(\beta, c(\beta))d\beta \tag{16}$$

and

$$c(\alpha) \geq \int_0^A (\alpha, \beta)\sigma(\beta, a(\beta))d\beta. \tag{17}$$

Let, for all $\alpha \in [0, A]$, $\sigma(\alpha, .)$ be a decreasing function, that is,

$$\kappa, \mu \in \mathbb{R}, \ \kappa \geq \mu \Rightarrow \sigma(\alpha, \kappa) \leq \sigma(\alpha, \mu). \tag{18}$$

Consider

$$\sup_{\alpha \in [0,A]} \int_0^A (\alpha, \beta)d\beta \leq 1. \tag{19}$$

In addition, consider, $\beta \in [0, A]$, $\kappa, \mu \in \mathbb{R}$ with ($\kappa \leq c_\circ$ and $\mu \geq a_\circ$) or ($\kappa \geq a_\circ$ and $\mu \leq c_\circ$) and $\Pi(\mu, \Gamma\mu) \geq 1$, $\Pi(\kappa, \Gamma\kappa) \geq 1$, we have

$$|\sigma(\beta, \kappa) + \sigma(\beta, \mu)| \leq \left(\frac{1}{8s^3}\right)^{\frac{1}{m}} (|\kappa(\alpha)| + |\mu(\alpha)|), \tag{20}$$

with a parameter $s \geq 1$ and for positive integer $m \geq 2$. Now, our main theorem of this section becomes valid for viewing.

Theorem 6. *Problem (14) has a solution $\{\kappa \in \Omega : a \leq \kappa(\alpha) \leq c$ for all $\alpha \in [0, A]\}$, provided that the hypotheses (15)–(20) hold.*

Proof. Define closed subsets of Ω, $Q_1(\alpha)$ and $Q_2(\alpha)$ for all $\alpha \in [0, A]$ by

$$Q_1 = \{\kappa \in \Omega : \kappa \leq c\},$$

and

$$Q_2 = \{\kappa \in \Omega : \kappa \geq a\}.$$

Consider the operator $\Gamma : \Omega \to \Omega$ defined by

$$\Gamma\kappa(\alpha) = \int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta, \text{ for all } \alpha \in [0, A].$$

Let us show that Γ is a cyclic mapping, i.e.,

$$\Gamma(Q_1) \subseteq Q_2 \text{ and } \Gamma(Q_2) \subseteq Q_1. \tag{21}$$

Consider $\kappa \in Q_1$, that is,

$$\kappa(\alpha) \leq c(\alpha), \text{ for all } \alpha \in [0, A].$$

Applying the condition (18), since $(\alpha, \beta) \geq 0$ and $\sigma(\beta, \kappa(\beta))$ non-increasing for all $\alpha, \beta \in [0, A]$, one can write

$$(\alpha, \beta)\sigma(\beta, \kappa(\beta)) \geq (\alpha, \beta)\sigma(\beta, c(\beta)) \text{ for all } \alpha, \beta \in [0, A].$$

Integrating for β yields

$$\int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta \geq \int_0^A (\alpha, \beta)\sigma(\beta, c(\beta))d\beta.$$

Using (16), we can get

$$\int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta \geq \int_0^A (\alpha, \beta)\sigma(\beta, c(\beta))d\beta \geq a(\alpha),$$

for all $\alpha \in [0, A]$. Then, we have $\Gamma\kappa \in Q_2$.

Similarly, $\kappa \in Q_2$, that is

$$\kappa(\beta) \geq a(\beta), \text{ for all } \beta \in [0, A].$$

By condition (18), since $(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in [0, A]$, it follows that

$$(\alpha, \beta)\sigma(\beta, \kappa(\beta)) \leq (\alpha, \beta)\sigma(\beta, a(\beta)) \text{ for all } \alpha, \beta \in [0, A].$$

Integrating for β and applying hypothesis (17), we deduce that

$$\int_0^A (\alpha, \beta)\sigma(\beta, \kappa(\beta))d\beta \leq \int_0^A (\alpha, \beta)\sigma(\beta, a(\beta))d\beta \leq c(\alpha)$$

for all $\alpha \in [0, A]$. Then, we obtain $\Gamma\kappa \in Q_2$. Therefore, (21) holds.

Now, let $(\kappa, \mu) \in Q_1 \times Q_2$, that is, for all $\alpha \in [0, A]$,

$$\kappa(\alpha) \leq c(\alpha), \ \ \mu(\alpha) \geq a(\alpha).$$

This yields by condition (15) that for all $\alpha \in [0, A]$,

$$\kappa(\alpha) \leq c_\circ, \ \mu(\alpha) \geq a_\circ.$$

Let $\kappa \in Q_1$ and $\mu \in Q_2$, where $\Pi(\mu, \Gamma\mu) \geq 0$ and $\Pi(\kappa, \Gamma\kappa) \geq 0$. Applying (19), (20) and taking $q = 2$, we can write

$$\begin{aligned}
2s^2\varpi(\Gamma\kappa(\alpha), \Gamma\mu(\alpha)) &= 2s^2\left(|\Gamma\kappa(\alpha)| + |\Gamma\mu(\alpha)|\right)^m \\
&= 2s^2\left(\left|\int_0^A(\alpha,\beta)\sigma(\beta,\kappa(\beta))d\beta\right| + \left|\int_0^A(\alpha,\beta)\sigma(\beta,\mu(\beta))d\beta\right|\right)^m \\
&\leq 2s^2\left(\int_0^A(\alpha,\beta)\left(|\sigma(\beta,\kappa(\beta)) + \sigma(\beta,\mu(\beta))|\right)d\beta\right)^m \\
&\leq 2s^2\left(\int_0^A(\alpha,\beta)d\beta \times \left(\frac{1}{8s^3}\right)^{\frac{1}{m}}\left\{(|\kappa(\alpha)| + |\mu(\alpha)|)^m\right\}^{\frac{1}{m}}\right)^m \\
&\leq 2s^2 \times \frac{1}{8s^3}\left(\varpi^{\frac{1}{m}}(\kappa(\alpha),\mu(\alpha))\right)^m \\
&\leq \frac{1}{4s}\varpi(\kappa(\alpha),\mu(\alpha)) = \frac{1}{4s}\varpi(\kappa,\mu) \\
&\leq \theta(N(\kappa,\mu)).
\end{aligned}$$

By a similar manner, we can prove that the above inequality holds if $(\kappa, \mu) \in Q_2 \times Q_1$, where $\Pi(\mu, \Gamma\mu) \geq 0$ and $\Pi(\kappa, \Gamma\kappa) \geq 0$.

Now, if we define $\eta : \Omega \times \Omega \to [0, \infty)$ by

$$\eta(\kappa,\mu) = \begin{cases} 1, & \text{if } \Pi(\kappa,\mu) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\kappa \in B_i$, $\mu \in B_{i+1}$, $i = 1, 2$ and $\eta(\kappa, \Gamma\kappa)\eta(\mu, \Gamma\mu) \geq 1 \Rightarrow$

$$2s^q\varpi(\Gamma\kappa,\Gamma\mu) \leq \theta\left(\max\left\{\begin{array}{c} \varpi(\kappa,\mu), \varpi(\mu,\Gamma\kappa), \frac{\varpi(\kappa,\Gamma\kappa)\varpi(\mu,\Gamma\mu)}{\varpi(\kappa,\mu)}, \\ \frac{\varpi(\mu,\Gamma\mu)[1+\varpi(\kappa,\Gamma\kappa)]}{1+\varpi(\kappa,\mu)}, \frac{\varpi(\kappa,\Gamma\mu)+\varpi(\mu,\Gamma\kappa)}{4s} \end{array}\right\}\right)$$

with $\theta = \frac{1}{4s}$. That is, Γ is a cyclic η_s^q-rational contractive mapping. Hence, all conditions of Theorem 2 are satisfied and Γ has a fixed point ν in

$$Q_1 \cap Q_2 = \{\kappa \in C([0, A], \mathbb{R}) : a \leq \kappa(\alpha) \leq c, \text{ for all } \alpha \in [0, A]\}.$$

That is, $\nu \in Q_1 \cap Q_2$ is a unique solution to problem (14). $\square$

To demonstrate the requirements hypotheses of Theorem 6, we present the following example:

Example 7. *Under the same distance in this section, suppose that (Ω, ϖ) is a complete b-metric-like space and denote the set of real continuous functions on $[0,1]$ by $C(\Omega, [0,1])$. Assume that the following problem:*

$$\kappa(\alpha) = \frac{1}{4}\int_0^1 \alpha\beta\left(\frac{\kappa(\beta)}{64+\beta}\right)d\beta, \ \textit{for all} \ \beta \in [0,1]. \tag{22}$$

By comparing (22) and (14), we deduce that $A = 1$,

$$(\alpha, \beta) = \frac{1}{2}\alpha\beta \ \textit{and} \ \sigma(\beta, \kappa(\beta)) = \frac{\kappa(\beta)}{2(64+\beta)}.$$

Let $a(\alpha) = \frac{1}{8}\alpha$ and $c(\alpha) = 2\alpha$, for all $\alpha \in [0,1]$. It is clear that

$$a_\circ \le a(\alpha) = \frac{1}{8}\alpha \le 2\alpha = c(\alpha) \le c_\circ,$$

where $(a, c) \in \Omega \times \Omega$ and $(a_\circ, c_\circ) \in \mathbb{R}^2$.

In addition, for all $\alpha \in [0,1]$, we can write

$$a(\alpha) = \frac{1}{8}\alpha \le \frac{1}{4}\int_0^1 \alpha\beta\left(\frac{c(\beta)}{64+\beta}\right)d\beta = \int_0^1 (\alpha,\beta)\sigma(\beta, c(\beta))d\beta$$

and

$$c(\alpha) = 2\alpha \ge \frac{1}{4}\int_0^1 \alpha\beta\left(\frac{a(\beta)}{64+\beta}\right)d\beta = \int_0^1 (\alpha,\beta)\sigma(\beta, a(\beta))d\beta.$$

Again, for all $\beta \in [0,1]$, we observe that the function $\sigma(\beta, \kappa(\beta))$ is decreasing and

$$\sup_{t\in[0,1]} \int_0^1 (\alpha,\beta)d\beta = \frac{1}{2}\sup_{t\in[0,1]}\int_0^1 \alpha\beta d\beta \le 1.$$

In addition, suppose that, for each $\beta \in [0,1]$, $\kappa, \mu \in \mathbb{R}$ with ($\kappa \le c_\circ$ and $\mu \ge a_\circ$) or ($\mu \ge a_\circ$ and $\mu \le c_\circ$) and $\Pi(\mu, \Gamma\mu) \ge 1$, $\Pi(\kappa, \Gamma\kappa) \ge 1$, we have

$$\begin{aligned} |\sigma(\beta,\kappa) + \sigma(\beta,\mu)| &= \left|\frac{\kappa(\beta)}{2(64+\beta)} + \frac{\mu(\beta)}{2(64+\beta)}\right| = \frac{1}{2(64+\beta)}|\kappa(\beta)+\mu(\beta)| \\ &\le \frac{1}{64}|\kappa(\beta)+\mu(\beta)| \le \frac{1}{64}(|\kappa(\beta)| + |\mu(\beta)|) \\ &\le \left(\frac{1}{8\times 2^3}\right)^{\frac{1}{2}}(|\kappa(\alpha)|+|\mu(\alpha)|) = \left(\frac{1}{8s^3}\right)^{\frac{1}{m}}(|\kappa(\alpha)|+|\mu(\alpha)|), \end{aligned}$$

where $s = m = 2$. Therefore, all assumptions (15)–(20) of Theorem 6 are verified. Hence, the problem (22) has a unique solution $\{\kappa \in \Omega : \kappa(\alpha) \in [\frac{1}{8}\alpha, 2\alpha]\}$.

6. Conclusions

The analytical solution of nonlinear integral equations by using a cyclic contractive mapping is an important application in fixed point theory, where they have attracted the interest of a lot of authors in academic research. Continuing in this direction, this paper discusses some common fixed point theorems for cyclic η_s^q-rational contractive mappings in b-metric-like spaces. The obtained results here unify, improve the results of [24,28] and generalize various comparable known results. After that, we discuss some fixed point theorem in the framework of b-metric-like spaces endowed with a graph. Moreover, some pivotal examples are given to support our results as well as the existence of the solution of a nonlinear integral equation is presented as an application.

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