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Mei Symmetry and Invariants of Quasi-Fractional Dynamical Systems with Non-Standard Lagrangians

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Abstract: Non-standard Lagrangians play an important role in the systems of non-conservative dynamics or nonlinear differential equations, quantum field theories, etc. This paper deals with quasi-fractional dynamical systems from exponential non-standard Lagrangians and power-law non-standard Lagrangians. Firstly, the definition, criterion, and corresponding new conserved quantity of Mei symmetry in this system are presented and studied. Secondly, considering that a small disturbance is applied on the system, the differential equations of the disturbed motion are established, the definition of Mei symmetry and corresponding criterion are given, and the new adiabatic invariants led by Mei symmetry are proposed and proved. Examples also show the validity of the results.

Keywords: Mei symmetry; conserved quantity; adiabatic invariant; quasi-fractional dynamical system; non-standard Lagrangians

1. Introduction

The study of symmetry and invariants for non-conservative or nonlinear dynamics is of great significance. It is also a frontier research field of analytical mechanics. In a classical sense, the symmetries we refer to mainly include Noether symmetries [1] and Lie symmetries [2,3]. Noether symmetry and Lie symmetry are two different symmetries. After infinitesimal transformation, the former means the invariant property of the Hamilton action functional, and the latter means the invariant property of the differential equation. Unlike Noether symmetry or Lie symmetry, Mei proposed a new symmetry called form invariance in 2000 [4]. Form invariance, also known as Mei symmetry, refers to an invariant property, that is, the dynamical functions (such as Lagrangian, Hamiltonian, Birkhoffian, generalized force, etc.) that appear in the dynamical equations of the mechanical system still satisfy the original equations after the infinitesimal transformation. Under certain conditions, symmetry can lead to invariants, which are also called conserved quantities. Noether symmetry, Lie symmetry, and Mei symmetry of dynamical systems described by standard Lagrangian may lead to Noether conserved quantities or Mei conserved quantities [5], etc. Conserved quantities can also be independent of Lagrangian. For example, conserved quantities can be directly constructed from Lie symmetry neither utilizing Lagrangian nor Hamiltonian, or can be formulated for systems of differential equations by using symmetries and adjoint symmetries together regardless of the existence of a Lagrangian, see [5–8] and references therein. So far, much progress has been made in the study of the symmetries and corresponding invariants [9–22]. However, there are few reports on the symmetries and invariants of dynamical systems based on non-standard Lagrangians.

The concept of non-standard Lagrangian was first mentioned in Arnold's works in 1978 as non-natural Lagrangian [23], but it was ignored due to the lack of Hamilton form corresponding to it. Until 1984, when discussing the region adaptability of classical theories in Yang–Mills quantum field

theory [24], it was found that non-standard Lagrangians were directly related to the color constraint problem, which led to their renewed attention. The advantage of non-standard Lagrangians is that it can better describe nonlinear problems and plays an important role in non-conservative systems, dissipative systems, quantum field theory, etc. [25–34].

Fractional calculus can better describe natural phenomena and engineering problems [35–37]. Since Riewe [38,39] introduced fractional calculus into the modeling of non-conservative systems, fractional Lagrangian mechanics, fractional Hamiltonian mechanics and fractional Birkhoffian mechanics have been proposed and studied, and important progress has been made in fractional dynamics modeling, analysis, and calculation, see for example [40–47] and references therein. In 2005, El-Nabulsi proposed the fractional action-like variational approach to study non-conservative dynamical problem, in which the action is constructed by using the Riemann-Liouville definition of fractional integral [48,49], and extended it to the case of non-standard Lagrangians [28,50]. Considering the characteristic of fractional action-like variational approach, we call the non-conservative model obtained in this way as quasi-fractional order dynamical system. Here, we propose and study Mei symmetry and its invariants for the quasi-fractional order dynamical system with non-standard Lagrangians. New conserved quantities and new adiabatic invariants are derived from Mei symmetry of the quasi-fractional dynamical systems.

2. Mei Symmetry and Invariants of Quasi-Fractional Dynamical System Based on Exponential Lagrangians

For the quasi-fractional dynamical system whose action functional depends on exponential Lagrangian, the Euler–Lagrange equations that are derived in Appendix A can be expressed as

$$(t - \tau)^{\alpha-1} \exp L \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) = 0, \quad (s = 1, 2, \dots, n), \quad (1)$$

where $q_s (s = 1, 2, \dots, n)$ are the generalized coordinates, $L = L(\tau, q_s, \dot{q}_s)$ is the standard Lagrangian, $0 < \alpha \leq 1$, τ is the intrinsic time, t is the observer time, and τ is not equal to t .

Let us introduce the infinitesimal transformations as

$$\tau^* = \tau + \varepsilon \zeta_0(\tau, q_k, \dot{q}_k), \quad q_s^*(\tau^*) = q_s(\tau) + \varepsilon \xi_s(\tau, q_k, \dot{q}_k), \quad (s = 1, 2, \dots, n; k = 1, 2, \dots, n), \quad (2)$$

where ε is a small parameter, ζ_0 and ξ_s are the infinitesimals. After the transformation of Equation (2), $\exp L$ is transformed into the following form

$$\exp L^* = \exp L \left(\tau^*, q_s^*, \frac{dq_s^*}{d\tau^*} \right) = \exp L(\tau, q_s, \dot{q}_s) + \varepsilon X^{(1)}(\exp L) + O(\varepsilon^2), \quad (3)$$

where $X^{(1)}$ is the first extension of the infinitesimal generator X , that is [4]

$$X = \zeta_0 \frac{\partial}{\partial \tau} + \xi_s \frac{\partial}{\partial q_s}, \quad X^{(1)} = \zeta_0 \frac{\partial}{\partial \tau} + \xi_s \frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s \dot{\zeta}_0) \frac{\partial}{\partial \dot{q}_s}. \quad (4)$$

If L is replaced with L^* , Equation (1) still holds, namely

$$(t - \tau)^{\alpha-1} \exp L^* \left(\frac{\partial L^*}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial \dot{q}_s} \frac{dL^*}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L^*}{\partial \dot{q}_s} \right) = 0, \quad (s = 1, 2, \dots, n), \quad (5)$$

then this invariance is called Mei symmetry of quasi-fractional dynamical system (1). Substituting the formula (3) into Equation (5), and considering Equation (1), we have

$$(t-\tau)^{\alpha-1} \exp L \left(\frac{\partial X^{(1)}(L)}{\partial q_s} - \frac{d}{d\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \frac{dL}{d\tau} - \frac{\partial L}{\partial \dot{q}_s} \frac{dX^{(1)}(L)}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \right) = 0, \quad (s = 1, 2, \dots, n). \quad (6)$$

Equation (6) is the criterion for Mei symmetry of system (1).

Theorem 1. For the quasi-fractional dynamical system (1), if there is a gauge function $G = G(\tau, q_k, \dot{q}_k)$ such that the structural equation

$$\left(\frac{1-\alpha}{t-\tau} \varsigma_0 + \dot{\varsigma}_0 \right) X^{(1)}(\exp L) (t-\tau)^{\alpha-1} + X^{(1)}[X^{(1)}(\exp L)] (t-\tau)^{\alpha-1} + \dot{G} = 0 \quad (7)$$

holds, the Mei symmetry directly leads to the new conserved quantity

$$I_0 = (t-\tau)^{\alpha-1} X^{(1)}(\exp L) \varsigma_0 + (t-\tau)^{\alpha-1} \frac{\partial X^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \varsigma_0) + G = \text{const.} \quad (8)$$

Proof.

$$\begin{aligned} \frac{dI_0}{d\tau} &= \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} X^{(1)}(\exp L) \varsigma_0 + (t-\tau)^{\alpha-1} \frac{dX^{(1)}(\exp L)}{d\tau} \varsigma_0 + (t-\tau)^{\alpha-1} X^{(1)}(\exp L) \dot{\varsigma}_0 \\ &\quad + \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} \frac{\partial X^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \varsigma_0) + (t-\tau)^{\alpha-1} \frac{d}{d\tau} \frac{\partial X^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \varsigma_0) \\ &\quad + (t-\tau)^{\alpha-1} \frac{\partial X^{(1)}(\exp L)}{\partial \dot{q}_s} (\dot{\xi}_s - \dot{q}_s \dot{\varsigma}_0 - \ddot{q}_s \varsigma_0) - (t-\tau)^{\alpha-1} X^{(1)}(\exp L) \left(\frac{1-\alpha}{t-\tau} \varsigma_0 + \dot{\varsigma}_0 \right) \\ &\quad - (t-\tau)^{\alpha-1} X^{(1)}[X^{(1)}(\exp L)] \\ &= \left[-\frac{\partial X^{(1)}(L)}{\partial q_s} + \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\partial L}{\partial \dot{q}_s} \frac{dX^{(1)}(L)}{d\tau} + \frac{d}{d\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \frac{1-\alpha}{t-\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \right] \times \\ &\quad \times (\xi_s - \dot{q}_s \varsigma_0) (t-\tau)^{\alpha-1} \exp L + \left(\frac{1-\alpha}{t-\tau} \varsigma_0 + \dot{\varsigma}_0 \right) X^{(1)}(\exp L) (t-\tau)^{\alpha-1} \\ &\quad + X^{(1)}[X^{(1)}(\exp L)] (t-\tau)^{\alpha-1} + \left(-\frac{\partial L}{\partial q_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \times \\ &\quad \times (\xi_s - \dot{q}_s \varsigma_0) (t-\tau)^{\alpha-1} X^{(1)}(\exp L) + \dot{G}. \end{aligned} \quad (9)$$

Substituting Equations (1) and (6) into the formula (9), and using Equation (7), we obtain

$$\frac{dI_0}{d\tau} = \left\{ \left(\frac{1-\alpha}{t-\tau} \varsigma_0 + \dot{\varsigma}_0 \right) X^{(1)}(\exp L) + X^{(1)}[X^{(1)}(\exp L)] \right\} (t-\tau)^{\alpha-1} + \dot{G} = 0. \quad (10)$$

Thus, we get the desired result. \square

The new conserved quantity (8) is called Mei conserved quantity. Since the system is not disturbed, it is an exact invariant. However, in nature and engineering, it is often affected by disturbing forces. If the system is affected by small disturbance vQ_s , its Mei symmetry and the corresponding conserved quantity (8) will change correspondingly. The infinitesimals of transformations (2) without disturbance is denoted as ς_0^0, ξ_s^0 , while the infinitesimals are changed into ς_0, ξ_s when disturbed, and we have

$$\varsigma_0 = \varsigma_0^0 + v\varsigma_0^1 + v^2\varsigma_0^2 + \dots, \quad \xi_s = \xi_s^0 + v\xi_s^1 + v^2\xi_s^2 + \dots, \quad (s = 1, 2, \dots, n). \quad (11)$$

Meanwhile, we let G^0 represent the gauge function without disturbance, and G represent the gauge function of the disturbed system, which is the small perturbation on the basis of G^0 , i.e.,

$$G = G^0 + vG^1 + v^2G^2 + \dots. \quad (12)$$

If a small disturbance vQ_s is applied, Equation (1) is changed to

$$(t - \tau)^{\alpha-1} \exp L \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) = vQ_s. \quad (13)$$

Accordingly, Equation (6) is changed to

$$(t - \tau)^{\alpha-1} \exp L \left[\frac{\partial X^{(1)}(L)}{\partial q_s} - \frac{d}{d\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \frac{dL}{d\tau} - \frac{\partial L}{\partial \dot{q}_s} \frac{dX^{(1)}(L)}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} \right] = vX^{(1)}(Q_s), \quad (s = 1, 2, \dots, n). \quad (14)$$

Substituting the formulae (11) into Equation (14), we get

$$(t - \tau)^{\alpha-1} v^m \exp L \left[\frac{\partial X_m^{(1)}(L)}{\partial q_s} - \frac{d}{d\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} - \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} \frac{dL}{d\tau} - \frac{\partial L}{\partial \dot{q}_s} \frac{dX_m^{(1)}(L)}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} \right] = v^{m+1} X_m^{(1)}(Q_s), \quad (s = 1, 2, \dots, n). \quad (15)$$

where

$$X^{(1)} = v^m X_m^{(1)}, \quad X_m^{(1)} = \zeta_0^m \frac{\partial}{\partial \tau} + \xi_s^m \frac{\partial}{\partial q_s} + \left(\dot{\xi}_s^m - \dot{q}_s \zeta_0^m \right) \frac{\partial}{\partial \dot{q}_s}. \quad (16)$$

As a result, we have

Theorem 2. If the quasi-fractional dynamical system (1) is disturbed by a small disturbance vQ_s , and there is a gauge function $G = G(\tau, q_k, \dot{q}_k)$ such that the structural equation

$$\left(\frac{1-\alpha}{t-\tau} \zeta_0^m + \dot{\zeta}_0^m \right) X_m^{(1)}(\exp L)(t - \tau)^{\alpha-1} + X_m^{(1)} \left[X_m^{(1)}(\exp L) \right] (t - \tau)^{\alpha-1} + \dot{G}^m - \left[X_{m-1}^{(1)}(Q_s) + Q_s X_{m-1}^{(1)}(L) \right] (\xi_s^{m-1} - \dot{q}_s \zeta_0^{m-1}) = 0, \quad (s = 1, 2, \dots, n; m = 0, 1, 2, \dots), \quad (17)$$

holds, where $G = \sum_{m=0}^z v^m G^m$ and $\zeta_0^{-1} = \xi_s^{-1} = 0$, the Mei symmetry directly leads to the new adiabatic invariant

$$I_z = \sum_{m=0}^z v^m \left[(t - \tau)^{\alpha-1} X_m^{(1)}(\exp L) \zeta_0^m + (t - \tau)^{\alpha-1} \frac{\partial X_m^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s^m - \dot{q}_s \zeta_0^m) + G^m \right]. \quad (18)$$

Proof. By using Equations (13), (15), and (17), we have

$$\begin{aligned}
 \frac{dI_z}{d\tau} &= \sum_{m=0}^z v^m \left\{ \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} X_m^{(1)}(\exp L) \zeta_0^m + (t-\tau)^{\alpha-1} \frac{dX_m^{(1)}(\exp L)}{d\tau} \zeta_0^m \right. \\
 &\quad \left. + (t-\tau)^{\alpha-1} X_m^{(1)}(\exp L) \dot{\zeta}_0^m + \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} \frac{\partial X_m^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s^m - \dot{q}_s \zeta_0^m) \right. \\
 &\quad \left. + (t-\tau)^{\alpha-1} \frac{d}{d\tau} \frac{\partial X_m^{(1)}(\exp L)}{\partial \dot{q}_s} (\xi_s^m - \dot{q}_s \zeta_0^m) + (t-\tau)^{\alpha-1} \frac{\partial X_m^{(1)}(\exp L)}{\partial \dot{q}_s} (\dot{\xi}_s^m - \dot{q}_s \dot{\zeta}_0^m - \ddot{q}_s \zeta_0^m) + \dot{G}^m \right\} \\
 &= \sum_{m=0}^z v^m \left\{ \left[-\frac{\partial X_m^{(1)}(L)}{\partial q_s} + \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\partial L}{\partial \dot{q}_s} \frac{dX_m^{(1)}(L)}{d\tau} + \frac{d}{d\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} + \frac{1-\alpha}{t-\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} \right] \times \right. \\
 &\quad \times (\xi_s^m - \dot{q}_s \zeta_0^m) (t-\tau)^{\alpha-1} \exp L + \left(\frac{1-\alpha}{t-\tau} \zeta_0^m + \dot{\zeta}_0^m \right) X_m^{(1)}(\exp L) (t-\tau)^{\alpha-1} \\
 &\quad \left. + X_m^{(1)} \left[X_m^{(1)}(\exp L) \right] (t-\tau)^{\alpha-1} - \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} - \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \times \right. \\
 &\quad \left. \times (\xi_s^m - \dot{q}_s \zeta_0^m) (t-\tau)^{\alpha-1} X_m^{(1)}(L) \exp L + \dot{G}^m \right\} \\
 &= \sum_{m=0}^z v^m \left\{ -v X_m^{(1)}(Q_s) (\xi_s^m - \dot{q}_s \zeta_0^m) + X_{m-1}^{(1)}(Q_s) (\xi_s^{m-1} - \dot{q}_s \zeta_0^{m-1}) \right. \\
 &\quad \left. + Q_s X_{m-1}^{(1)}(L) (\xi_s^{m-1} - \dot{q}_s \zeta_0^{m-1}) - v Q_s X_m^{(1)}(L) (\xi_s^m - \dot{q}_s \zeta_0^m) \right\} \\
 &= -v^{z+1} \left[X_z^{(1)}(Q_s) + Q_s X_z^{(1)}(L) \right] (\xi_s^z - \dot{q}_s \zeta_0^z).
 \end{aligned} \tag{19}$$

According to the definition of adiabatic invariant [51], I_z is an adiabatic invariant of order z . This completes the proof. \square

Example 1. Considering the nonlinear dynamical system, its action functional based on exponential Lagrangian is

$$S = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \exp[L(\tau, q_s, \dot{q}_s)] (t-\tau)^{\alpha-1} d\tau, \tag{20}$$

where $L = \tau q \dot{q}$.

Equation (1) gives

$$(t-\tau)^{\alpha-1} \exp(\tau q \dot{q}) \left[\tau q \left(\frac{\alpha-1}{t-\tau} - q \dot{q} - \tau \dot{q}^2 - \tau q \ddot{q} \right) - q \right] = 0. \tag{21}$$

By calculation, we have

$$X_0^{(1)}(L) = q \dot{q} \zeta_0^0 + \tau \dot{q} \xi^0 + \tau q \left(\dot{\xi}^0 - \dot{q} \dot{\zeta}_0^0 \right), \tag{22}$$

$$X_0^{(1)}(\exp L) = \exp(\tau q \dot{q}) \left[q \dot{q} \zeta_0^0 + \tau \dot{q} \xi^0 + \tau q \left(\dot{\xi}^0 - \dot{q} \dot{\zeta}_0^0 \right) \right]. \tag{23}$$

If we take

$$\zeta_0^0 = \tau, \quad \xi^0 = \frac{1}{q}, \tag{24}$$

then we have

$$X_0^{(1)}(L) = 0, \quad X_0^{(1)}(\exp L) = 0. \tag{25}$$

According to the criterion (6), the infinitesimals (24) correspond to Mei symmetry. Substituting (24) into Equation (7), we get

$$G^0 = \tau \exp(\tau q \dot{q}) (t-\tau)^{\alpha-1}. \tag{26}$$

From Theorem 1, we have

$$I_0 = \tau \exp(\tau q \dot{q}) (t-\tau)^{\alpha-1} = \text{const.} \tag{27}$$

Let the small disturbance be

$$vQ = vq\dot{q}\exp(q^2/2). \quad (28)$$

The differential equation of the disturbed motion is

$$(t-\tau)^{\alpha-1}\exp(\tau q\dot{q})\left[\tau q\left(\frac{\alpha-1}{t-\tau}-q\dot{q}-\tau\ddot{q}^2-\tau q\ddot{q}\right)-q\right] = vq\dot{q}\exp(q^2/2). \quad (29)$$

Take

$$\varsigma_0^1 = \tau, \quad \xi^1 = \frac{1}{q}, \quad (30)$$

then we have

$$X_1^{(1)}(L) = 0, \quad X_1^{(1)}(\exp L) = 0, \quad X_0^{(1)}(Q) = X_1^{(1)}(Q) = 0. \quad (31)$$

According to the criterion (15), the infinitesimals (30) correspond to Mei symmetry. Substituting (30) into Equation (17), we have

$$G^1 = \tau \exp(\tau q\dot{q})(t-\tau)^{\alpha-1} + v \int \exp(q^2/2) dq. \quad (32)$$

By Theorem 2, we obtain

$$I_1 = \tau \exp(\tau q\dot{q})(t-\tau)^{\alpha-1} + v \left[\tau \exp(\tau q\dot{q})(t-\tau)^{\alpha-1} + v \int \exp(q^2/2) dq \right]. \quad (33)$$

The formula (33) is an adiabatic invariant led by Mei symmetry.

3. Mei Symmetry and Invariants of Quasi-Fractional Dynamical System Based on Power-Law Lagrangians

For the quasi-fractional dynamical system whose action functional depends on power-law Lagrangian, the Euler–Lagrange equations that are derived in Appendix B can be expressed as

$$(1+\gamma)(t-\tau)^{\alpha-1}L^\gamma\left(\frac{\partial L}{\partial q_s}-\frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right)=0, \quad (s=1,2,\dots,n), \quad (34)$$

where γ is not equal to -1 .

After the transformation of (2), $L^{1+\gamma}$ is transformed into the following form

$$L^{1+\gamma} = L^{1+\gamma}\left(\tau^*, q_s^*, \frac{dq_s^*}{d\tau^*}\right) = L^{1+\gamma}(\tau, q_s, \dot{q}_s) + \varepsilon X^{(1)}(L^{1+\gamma}) + O(\varepsilon^2). \quad (35)$$

If L is replaced with L^* , Equation (34) still holds, namely

$$(1+\gamma)(t-\tau)^{\alpha-1}L^{*\gamma}\left(\frac{\partial L^*}{\partial q_s}-\frac{d}{d\tau}\frac{\partial L^*}{\partial \dot{q}_s}-\frac{\gamma}{L^*}\frac{\partial L^*}{\partial \dot{q}_s}\frac{dL^*}{d\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L^*}{\partial \dot{q}_s}\right)=0, \quad (s=1,2,\dots,n). \quad (36)$$

then this invariance is called Mei symmetry of quasi-fractional dynamical system (34). Substituting the formula (35) into Equation (36), and considering Equation (34), we have

$$(1+\gamma)(t-\tau)^{\alpha-1}L^\gamma\left[\frac{\partial X^{(1)}(L)}{\partial q_s}-\frac{d}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{dL}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dX^{(1)}(L)}{d\tau}+\frac{\gamma}{L^2}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau}X^{(1)}(L)+\frac{\alpha-1}{t-\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right]=0, \quad (s=1,2,\dots,n). \quad (37)$$

Equation (37) is the criterion for Mei symmetry of system (34). Hence, we have

Theorem 3. For the quasi-fractional dynamical system (34), if there is a gauge function $G = G(\tau, q_k, \dot{q}_k)$ such that the structural equation

$$\left(\frac{1-\alpha}{t-\tau}\zeta_0 + \dot{\zeta}_0\right)X^{(1)}(L^{1+\gamma}) + X^{(1)}\{X^{(1)}(L^{1+\gamma})\} + (t-\tau)^{1-\alpha}\dot{G} = 0 \quad (38)$$

holds, the Mei symmetry directly leads to the new conserved quantity

$$I_0 = (t-\tau)^{\alpha-1}X^{(1)}(L^{1+\gamma})\zeta_0 + (t-\tau)^{\alpha-1}\frac{\partial X^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\zeta_0) + G = \text{const.} \quad (39)$$

Proof.

$$\begin{aligned} \frac{dI_0}{dt} &= \frac{1-\alpha}{t-\tau}(t-\tau)^{\alpha-1}X^{(1)}(L^{1+\gamma})\zeta_0 + (t-\tau)^{\alpha-1}\frac{dX^{(1)}(L^{1+\gamma})}{d\tau}\zeta_0 + (t-\tau)^{\alpha-1}X^{(1)}(L^{1+\gamma})\dot{\zeta}_0 \\ &\quad + \frac{1-\alpha}{t-\tau}(t-\tau)^{\alpha-1}\frac{\partial X^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\zeta_0) + (t-\tau)^{\alpha-1}\frac{d}{d\tau}\frac{\partial X^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\zeta_0) \\ &\quad + (t-\tau)^{\alpha-1}\frac{\partial X^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\zeta}_0 - \ddot{q}_s\zeta_0) + \dot{G} \\ &= \left[-\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \frac{d}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \frac{\gamma}{L}\frac{dL}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dX^{(1)}(L)}{d\tau} - \frac{\gamma}{L^2}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau}X^{(1)}(L) \right. \\ &\quad \left. + \frac{1-\alpha}{t-\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right](1+\gamma)(t-\tau)^{\alpha-1}L^\gamma(\xi_s - \dot{q}_s\zeta_0) + \left(\frac{1-\alpha}{t-\tau}\zeta_0 + \dot{\zeta}_0\right)X^{(1)}(L^{1+\gamma})(t-\tau)^{\alpha-1} \\ &\quad + \left(-\frac{\partial L}{\partial \dot{q}_s} + \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}_s} + \frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau} + \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right)(1+\gamma)(t-\tau)^{\alpha-1}X^{(1)}(L^\gamma)(\xi_s - \dot{q}_s\zeta_0) \\ &\quad + X^{(1)}(L)X^{(1)}(L)(1+\gamma)\gamma(t-\tau)^{\alpha-1}L^{\gamma-1} + X^{(1)}[X^{(1)}(L)](1+\gamma)(t-\tau)^{\alpha-1}L^\gamma + \dot{G}. \end{aligned} \quad (40)$$

Substituting Equations (34) and (37) into the formula (40), and using Equation (38), we obtain

$$\frac{dI_0}{dt} = \left(\frac{1-\alpha}{t-\tau}\zeta_0 + \dot{\zeta}_0\right)X^{(1)}(L^{1+\gamma})(t-\tau)^{\alpha-1} + X^{(1)}[X^{(1)}(L^{1+\gamma})](t-\tau)^{\alpha-1} + \dot{G} = 0. \quad (41)$$

Thus, we get the desired result. \square

Mei conserved quantity (39) is an exact invariant for the quasi-fractional dynamical systems (34).

If a small disturbance vQ_s is applied, Equation (34) is changed to

$$(1+\gamma)(t-\tau)^{\alpha-1}L^\gamma\left(\frac{\partial L}{\partial \dot{q}_s} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}_s} - \frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right) = vQ_s, \quad (s = 1, 2, \dots, n). \quad (42)$$

Accordingly, Equation (37) is changed to

$$\begin{aligned} (1+\gamma)(t-\tau)^{\alpha-1}L^\gamma\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \frac{d}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \frac{\gamma}{L}\frac{dL}{d\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - \frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dX^{(1)}(L)}{d\tau} \right. \\ \left. + \frac{\gamma}{L^2}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau}X^{(1)}(L) + \frac{\alpha-1}{t-\tau}\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right] = vX^{(1)}(Q_s), \quad (s = 1, 2, \dots, n). \end{aligned} \quad (43)$$

Substituting the formulae (11) into Equation (43), we get

$$\begin{aligned} (1+\gamma)(t-\tau)^{\alpha-1}v^mL^\gamma\left[\frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} - \frac{d}{d\tau}\frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} - \frac{\gamma}{L}\frac{dL}{d\tau}\frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} - \frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{dX_m^{(1)}(L)}{d\tau} \right. \\ \left. + \frac{\gamma}{L^2}\frac{\partial L}{\partial \dot{q}_s}\frac{dL}{d\tau}X_m^{(1)}(L) + \frac{\alpha-1}{t-\tau}\frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s}\right] = v^{m+1}X_m^{(1)}(Q_s), \quad (s = 1, 2, \dots, n). \end{aligned} \quad (44)$$

Therefore, we have

Theorem 4. If the quasi-fractional dynamical system (34) is disturbed by small disturbance vQ_s , and there is a gauge function $G = G(\tau, q_k, \dot{q}_k)$ such that the structural equation

$$\left(\frac{1-\alpha}{t-\tau}\zeta_0^m + \dot{\zeta}_0^m\right)X_m^{(1)}(L^{1+\gamma})(t-\tau)^{\alpha-1} + X_m^{(1)}\left[X_m^{(1)}(L^{1+\gamma})\right](t-\tau)^{\alpha-1} - X_{m-1}^{(1)}(Q_s)(\xi_s^{m-1} - \dot{q}_s\zeta_0^{m-1}) - \frac{\gamma}{L}Q_sX_{m-1}^{(1)}(L)(\xi_s^{m-1} - \dot{q}_s\zeta_0^{m-1}) + \dot{G}^m = 0, \quad (s = 1, 2, \dots, n; m = 0, 1, 2, \dots), \quad (45)$$

holds, where $G = \sum_{m=0}^z v^m G^m$ and $\zeta_0^{-1} = \xi_s^{-1} = 0$, the Mei symmetry directly leads to the new adiabatic invariant

$$I_z = \sum_{m=0}^z v^m \left[(t-\tau)^{\alpha-1} X_m^{(1)}(L^{1+\gamma}) \zeta_0^m + (t-\tau)^{\alpha-1} \frac{\partial}{\partial \dot{q}_s} X_m^{(1)}(L^{1+\gamma}) (\xi_s^m - \dot{q}_s \zeta_0^m) + G^m \right]. \quad (46)$$

Proof. By using Equations (42), (44), and (45), we have

$$\begin{aligned} \frac{dI_z}{d\tau} &= \sum_{m=0}^z v^m \left\{ \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} X_m^{(1)}(L^{1+\gamma}) \zeta_0^m + (t-\tau)^{\alpha-1} \frac{dX_m^{(1)}(L^{1+\gamma})}{d\tau} \zeta_0^m \right. \\ &\quad + (t-\tau)^{\alpha-1} X_m^{(1)}(L^{1+\gamma}) \dot{\zeta}_0^m + \frac{1-\alpha}{t-\tau} (t-\tau)^{\alpha-1} \frac{\partial X_m^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s} (\xi_s^m - \dot{q}_s \zeta_0^m) \\ &\quad \left. + (t-\tau)^{\alpha-1} \frac{d}{d\tau} \frac{\partial X_m^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s} (\xi_s^m - \dot{q}_s \zeta_0^m) + (t-\tau)^{\alpha-1} \frac{\partial X_m^{(1)}(L^{1+\gamma})}{\partial \dot{q}_s} (\dot{\xi}_s^m - \dot{q}_s \dot{\zeta}_0^m - \ddot{q}_s \zeta_0^m) + \dot{G}^m \right\} \\ &= \sum_{m=0}^z v^m \left\{ \left[-\frac{\partial X_m^{(1)}(L)}{\partial q_s} + \frac{d}{d\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} + \frac{\gamma}{L} \frac{dL}{d\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} + \frac{\gamma}{L} \frac{\partial L}{\partial \dot{q}_s} \frac{dX_m^{(1)}(L)}{d\tau} \right. \right. \\ &\quad \left. \left. - \frac{\gamma}{L^2} \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} X_m^{(1)}(L) + \frac{1-\alpha}{t-\tau} \frac{\partial X_m^{(1)}(L)}{\partial \dot{q}_s} \right] (1+\gamma) (t-\tau)^{\alpha-1} L^\gamma (\xi_s^m - \dot{q}_s \zeta_0^m) \right. \\ &\quad + \left(-\frac{\partial L}{\partial q_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{\gamma}{L} \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) (1+\gamma) (t-\tau)^{\alpha-1} X_m^{(1)}(L^\gamma) (\xi_s^m - \dot{q}_s \zeta_0^m) \\ &\quad + \left(\frac{1-\alpha}{t-\tau} \zeta_0^m + \dot{\zeta}_0^m \right) X_m^{(1)}(L^{1+\gamma}) (t-\tau)^{\alpha-1} + X_m^{(1)}(L) X_m^{(1)}(L) (1+\gamma) \gamma (t-\tau)^{\alpha-1} L^{\gamma-1} \\ &\quad \left. + X_m^{(1)} \left[X_m^{(1)}(L) \right] (1+\gamma) (t-\tau)^{\alpha-1} L^\gamma + \dot{G}^m \right\} \\ &= \sum_{m=0}^z v^m \left\{ -v X_m^{(1)}(Q_s) (\xi_s^m - \dot{q}_s \zeta_0^m) - v \frac{\gamma}{L} Q_s X_m^{(1)}(L) (\xi_s^m - \dot{q}_s \zeta_0^m) \right. \\ &\quad \left. + X_{m-1}^{(1)}(Q_s) (\xi_s^{m-1} - \dot{q}_s \zeta_0^{m-1}) + \frac{\gamma}{L} Q_s X_{m-1}^{(1)}(L) (\xi_s^{m-1} - \dot{q}_s \zeta_0^{m-1}) \right\} \\ &= -v^{z+1} \left[X_z^{(1)}(Q_s) + \frac{\gamma}{L} Q_s X_z^{(1)}(L) \right] (\xi_s^z - \dot{q}_s \zeta_0^z). \end{aligned} \quad (47)$$

According to the definition of adiabatic invariant [51], I_z is an adiabatic invariant of order z . So that ends the proof. \square

Example 2. Considering the nonconservative dynamical system, its action functional based on power-law Lagrangian is [50]

$$A = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left[L^{1+\gamma}(\tau, q_s, \dot{q}_s) \right] (t-\tau)^{\alpha-1} d\tau, \quad (48)$$

where $L = \dot{q} - q(\tau - t)$, $\gamma = 1$.

Equation (34) gives

$$2(t-\tau)^{\alpha-1} \left[-\ddot{q} + \frac{\alpha-1}{t-\tau} \dot{q} + ((\tau-t)^2 + \alpha)q \right] = 0. \quad (49)$$

By calculation, we have

$$X_0^{(1)}(L) = -q\zeta_0^0 - (\tau-t)\xi_0^0 + \dot{\xi}_0^0 - \dot{q}\zeta_0^0, \quad (50)$$

$$X_0^{(1)}(L^2) = 2L \left[-q\varsigma_0^0 - (\tau - t)\xi_0^0 + \xi_0^0 - \dot{q}\dot{\varsigma}_0^0 \right]. \quad (51)$$

Let

$$\varsigma_0^0 = 0, \quad \xi_0^0 = \exp[(t + \tau)^2/2], \quad (52)$$

then

$$X_0^{(1)}(L) = 0, \quad X_0^{(1)}(L^2) = 0. \quad (53)$$

According to the criterion (37), the infinitesimals (52) correspond to Mei symmetry. Substituting (52) into Equation (38), we get

$$G^0 = 2[\dot{q} - q(\tau - t)](t - \tau)^{\alpha-1} \exp[(\tau - t)^2/2]. \quad (54)$$

From Theorem 3, we have

$$I_0 = 2[\dot{q} - q(\tau - t)](t - \tau)^{\alpha-1} \exp[(\tau - t)^2/2] = \text{const}. \quad (55)$$

Let the small disturbance be

$$\nu Q = \nu \sin \tau \exp[-(\tau - t)^2/2]. \quad (56)$$

The differential equation of the disturbed motion is

$$2(t - \tau)^{\alpha-1} \left[-\ddot{q} + \frac{\alpha - 1}{t - \tau} \dot{q} + ((\tau - t)^2 + \alpha)q \right] = \nu \sin \tau \exp[-(\tau - t)^2/2]. \quad (57)$$

Take

$$\varsigma_0^1 = 0, \quad \xi_0^1 = \exp[(t + \tau)^2/2], \quad (58)$$

then it is easy to verify

$$X_1^{(1)}(L) = 0, \quad X_1^{(1)}(L^2) = 0, \quad X_0^{(1)}(Q) = X_1^{(1)}(Q) = 0. \quad (59)$$

According to the criterion (44), the infinitesimals (58) correspond to Mei symmetry. Substituting (58) into Equation (45), we have

$$G^1 = 2[\dot{q} - q(\tau - t)](t - \tau)^{\alpha-1} \exp[(\tau - t)^2/2] - \nu \cos \tau. \quad (60)$$

By Theorem 4, we obtain

$$I_1 = 2[\dot{q} - q(\tau - t)](t - \tau)^{\alpha-1} \exp[(\tau - t)^2/2] + \nu \{ 2[\dot{q} - q(\tau - t)](t - \tau)^{\alpha-1} \exp[(\tau - t)^2/2] - \nu \cos \tau \}. \quad (61)$$

Formula (61) is an adiabatic invariant led by Mei symmetry.

4. Conclusions

Symmetry is closely related to invariants, and it is of great significance to find the invariants of complex system dynamics. First, even if the equations of motion are difficult to solve, the existence of some conserved quantity makes it possible to understand the local physical state or dynamical behavior of the system. Secondly, we can reduce the differential equations of motion by using conserved quantities. Thirdly, we can study the motion stability of complex dynamical systems by using conserved quantities. Based on the quasi-fractional dynamical model proposed by El-Nabulsi according to the Riemann–Liouville definition of fractional integral, we studied Mei symmetry and its corresponding invariants of quasi-fractional dynamics system whose action functional is composed of non-standard Lagrangians. The main results of this paper are its four theorems. In this paper, we provided a method

to study nonlinear or non-conservative dynamics and obtained new conserved quantities and new adiabatic invariants, and the results are expected to be generalized or applied to the dynamics of constrained systems, such as those of nonholonomic systems.

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Appendix A. Derivation of the Euler–Lagrange Equations for Quasi-Fractional Dynamical System with Exponential Lagrangians

Consider a nonlinear dynamical system whose configuration is determined by n generalized coordinates $q_s (s = 1, 2, \dots, n)$, its action functional based on exponential Lagrangian is

$$S = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \exp[L(\tau, q_s, \dot{q}_s)] (t - \tau)^{\alpha-1} d\tau. \quad (A1)$$

where $L = L(\tau, q_s, \dot{q}_s)$ is the standard Lagrangian, $0 < \alpha \leq 1$, τ is the intrinsic time, t is the observer time, and τ is not equal to t .

The isochronous variational principle

$$\delta S = 0, \quad (A2)$$

which satisfies the following commutation relation

$$d\delta q_s = \delta dq_s, \quad (s = 1, 2, \dots, n), \quad (A3)$$

and given boundary condition

$$\delta q_s|_{t=t_1} = \delta q_s|_{t=t_2} = 0, \quad (s = 1, 2, \dots, n) \quad (A4)$$

can be called the Hamilton principle of the quasi-fractional dynamical system with exponential Lagrangians.

Expanding the Hamilton principle (A2), we have

$$\begin{aligned} 0 = \delta S &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \delta [\exp L (t - \tau)^{\alpha-1}] d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} \exp L \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right) d\tau \end{aligned} \quad (A5)$$

Due to

$$\begin{aligned} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} \exp L \frac{\partial L}{\partial q_s} \delta q_s d\tau &= \left[(t - \tau)^{\alpha-1} \exp L \frac{\partial L}{\partial q_s} \delta q_s \right]_{t_1}^{t_2} \\ &\quad - \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} \exp L \left(-\frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau \\ &= - \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} \exp L \left(-\frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau. \end{aligned} \quad (A6)$$

Substituting the formula (A6) into Equation (A5), we have

$$\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} \exp L \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau = 0. \quad (A7)$$

Because of the arbitrariness of the interval $[t_1, t_2]$ and the independence of δq_s ($s = 1, 2, \dots, n$), using the fundamental lemma [23] of the calculus of variations, we get

$$(t - \tau)^{\alpha-1} \exp L \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) = 0, \quad (s = 1, 2, \dots, n). \quad (A8)$$

Equation (A8) can be called the Euler–Lagrange equations for quasi-fractional dynamical system with exponential Lagrangians.

Appendix B. Derivation of the Euler–Lagrange Equations for Quasi-Fractional Dynamical System with Power-Law Lagrangians

Consider a nonlinear dynamical system whose configuration is determined by n generalized coordinates q_s ($s = 1, 2, \dots, n$), its action functional based on power-law Lagrangian is

$$A = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} [L^{1+\gamma}(\tau, q_s, \dot{q}_s)] (t - \tau)^{\alpha-1} d\tau \quad (A9)$$

where $L = L(\tau, q_s, \dot{q}_s)$ is the standard Lagrangian, γ is not equal to -1 , $0 < \alpha \leq 1$, τ is the intrinsic time, t is the observer time, and τ is not equal to t .

The isochronous variational principle

$$\delta A = 0, \quad (A10)$$

which satisfies the following commutation relation

$$d\delta q_s = \delta dq_s, \quad (s = 1, 2, \dots, n), \quad (A11)$$

and given boundary condition

$$\delta q_s|_{t=t_1} = \delta q_s|_{t=t_2} = 0, \quad (s = 1, 2, \dots, n) \quad (A12)$$

can be called the Hamilton principle of the quasi-fractional dynamical system with power-law Lagrangians.

Expanding the Hamilton principle (A10), we have

$$\begin{aligned} 0 = \delta A &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \delta [L^{1+\gamma} (t - \tau)^{\alpha-1}] d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (1 + \gamma) (t - \tau)^{\alpha-1} L^\gamma \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right) d\tau \end{aligned} \quad (A13)$$

Due to

$$\begin{aligned} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} L^\gamma \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s d\tau &= \left[(t - \tau)^{\alpha-1} L^\gamma \frac{\partial L}{\partial \dot{q}_s} \delta q_s \right] \Big|_{t_1}^{t_2} \\ &- \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} L^\gamma \left(-\frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{\gamma}{L} \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau \\ &= - \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} L^\gamma \left(-\frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{\gamma}{L} \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau. \end{aligned} \quad (A14)$$

Substituting the formula (A14) into Equation (A13), we have

$$\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (1 + \gamma) (t - \tau)^{\alpha-1} L^\gamma \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\gamma}{L} \frac{dL}{d\tau} \frac{\partial L}{\partial \dot{q}_s} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) \delta q_s d\tau = 0. \quad (A15)$$

Because of the arbitrariness of the interval $[t_1, t_2]$ and the independence of δq_s ($s = 1, 2, \dots, n$), using the fundamental lemma [23] of the calculus of variations, we get

$$(1 + \gamma)(t - \tau)^{\alpha-1} L^\gamma \left(\frac{\partial L}{\partial q_s} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_s} - \frac{\gamma}{L} \frac{\partial L}{\partial \dot{q}_s} \frac{dL}{d\tau} + \frac{\alpha-1}{t-\tau} \frac{\partial L}{\partial \dot{q}_s} \right) = 0, \quad (s = 1, 2, \dots, n). \quad (A16)$$

Equation (A16) can be called the Euler–Lagrange equations for quasi-fractional dynamical system with power-law Lagrangians. Equation (A16) is consistent with the results given in [50].

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