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The Meir–Keeler Fixed Point Theorem for Quasi-Metric Spaces and Some Consequences

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Abstract: We obtain quasi-metric versions of the famous Meir–Keeler fixed point theorem from which we deduce quasi-metric generalizations of Boyd–Wong's fixed point theorem. In fact, one of these generalizations provides a solution for a question recently raised in the paper "On the fixed point theory in bicomplete quasi-metric spaces", *J. Nonlinear Sci. Appl.* **2016**, *9*, 5245–5251. We also give an application to the study of existence of solution for a type of recurrence equations associated to certain nonlinear difference equations.

Keywords: fixed point; quasi-metric space; Meir–Keeler; Boyd–Wong

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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integer numbers. Moreover, if T is a self-map of a set X and $x \in X$, we will write Tx instead of $T(x)$ if no confusion arises.

In a recent paper [1], several distinguished fixed point theorems, formulated in terms of φ-contractions, were extended from the setting of complete metric spaces to the realm of bicomplete quasi-metric spaces. The only exception to this approach was the famous Boyd and Wong fixed point theorem [2] (Theorem 1). In fact, it was given in [1] (Example 2.14) an easy example of a self-map T of a bicomplete quasi-metric space (X, d) for which there exists a right upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(t) < t$ for all $t > 0$, and $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in X$, but T has no fixed points.

Motivated by this example, it was raised in [1] (p. 5250) the question of adding some extra reasonable condition in order to obtain a suitable extension of Boyd–Wong's theorem to bicomplete quasi-metric spaces. Here, we obtain, among other results, a solution to this question which will be easily deduced from a quasi-metric version of the celebrated Meir–Keeler fixed point theorem.

Let us recall that the Meir–Keeler fixed point theorem [3] provides a nice and real improvement (see [3] (Example)) of the generalization of the Banach contraction principle obtained by Boyd and Wong in their paper [2]. Meir–Keeler's theorem has been extended and generalized in several directions (see e.g., [4–19]). In this paper, we will discuss conditions of Meir–Keeler's type in the realm of quasi-metric spaces and some fixed point theorems are obtained. Quasi-metric generalizations of Boyd–Wong's fixed point theorem are deduced and various illustrative examples are given. Finally, an application to the study of existence and uniqueness of solution for a type of recurrence equations associated to certain nonlinear difference equations is discussed.

We finish this section by recalling some concepts and properties on quasi-metric spaces which will be useful later on, Ref. [20] being our basic reference.

By a quasi-metric on a set X, we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow d(x,y) = d(y,x) = 0$, and

(ii) $d(x,z) \leq d(x,y) + d(y,z)$.

A quasi-metric space is a pair (X,d) such that X is a set and d is a quasi-metric on X.

Given a quasi-metric d on a set X, the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If τ_d is a T_1 topology on X, we say that (X,d) is a T_1 quasi-metric space, and, if τ_d is a T_2 topology on X, we say that (X,d) is a Hausdorff quasi-metric space.

It is well known, and easy to check, that every quasi-metric d on a set X induces a partial order $\leq_d$ on X defined as $x \leq_d y$ if and only if $d(x,y) = 0$.

Hence, a self-map T of a quasi-metric space (X,d) is $\leq_d$-increasing if and only if $d(Tx,Ty) = 0$ whenever $d(x,y) = 0$. Similarly, T is $\leq_d$-decreasing if and only if $d(Ty,Tx) = 0$ whenever $d(x,y) = 0$. Note that, if (X,d) is a T_1 quasi-metric space, then every self-map of X is $\leq_d$-increasing and $\leq_d$-decreasing.

A quasi-metric space (X,d) is called bicomplete if (X,d^s) is a complete metric space, and it is called sequentially complete if every Cauchy sequence in the metric space (X,d^s) converges for τ_d.

Obviously, every bicomplete quasi-metric space is sequentially complete, and it is well known that the converse is not true in general not even for compact Hausdorff quasi-metric spaces. We illustrate this fact with the following easy example.

Example 1. *Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and let d be the quasi-metric on X given by $d(x,x) = 0$ for all $x \in X$, $d(x,0) = 1$ for all $x \in X \backslash \{0\}$, and $d(x,y) = |x-y|$ otherwise. Clearly, (X,d) is a Hausdorff compact quasi-metric space (note that the sequence $(1/n)_{n\in\mathbb{N}}$ converges to 0 for τ_d), and hence it is sequentially complete). However, $(1/n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X,d^s) which does not converge for the (metric) topology τ_{d^s}.*

2. Results

A self-map T of a metric space (X,d) is called a Meir–Keeler map on (X,d) if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x, y \in X$,

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Ty) < \varepsilon.$$

Meir and Keeler proved in [3] their celebrated theorem that every Meir–Keeler map on a complete metric space has a unique fixed point.

In contrast to Meir–Keeler's theorem, consider the bicomplete quasi-metric space (X,d) where $X = \{0,1\}$, and $d(0,0) = d(1,1) = d(0,1) = 0$ and $d(1,0) = 1$. Now (compare [1], Example 2.14) define $T : X \to X$ as $T0 = 1$ and $T1 = 0$. Given $\varepsilon > 0$, choose $\delta = \varepsilon$ and let $x, y \in X$ such that $\varepsilon \leq d(x,y) < 2\varepsilon$. Then, $x = 1$, $y = 0$, and consequently $d(Tx,Ty) = d(T1,T0) = d(0,1) = 0 < \varepsilon$.

Note that the self-map T of the preceding simple example is not $\leq_d$-increasing. This fact motivates the following notion.

Definition 1. *Let (X,d) be a quasi-metric space. A self-map T of X is called a d–Meir–Keeler map on (X,d) if it is $\leq_d$-increasing and for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x, y \in X$,*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Ty) < \varepsilon. \qquad \text{(MK)}$$

Remark 1. *Note that, if T is a d–Meir–Keeler map on a quasi-metric space (X,d), then $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in X$. Indeed, if $d(x,y) = 0$, then $d(Tx,Ty) = 0$ because T is $\leq_d$-increasing, and if $d(x,y) > 0$, then $d(Tx,Ty) < d(x,y)$ by condition (MK) in Definition 1.*

Remark 2. *As in the classical metric case, it is clear that every (Banach) contraction on a quasi-metric space* (X, d) *is a* d*–Meir Keeler map on* (X, d)*. Indeed, let* $T : X \to X$ *such that there exists* $c \in (0, 1)$ *with* $d(Tx, Ty) \leq cd(x, y)$ *for all* $x, y \in X$*. Then, given* $\varepsilon > 0$ *take* $\delta = \varepsilon(1-c)/c$*, and thus* $d(Tx, Ty) < \varepsilon$ *whenever* $d(x, y) < \varepsilon + \delta$*.*

Proposition 1. *Let* (X, d) *be a quasi-metric space. Then,*

(1) Every d*–Meir–Keeler map on* (X, d) *is a Meir–Keeler map on the metric space* (X, d^s)*.*

(2) If T *is a* d*–Meir–Keeler map on* (X, d)*, the sequence* $(T^n x)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* (X, d^s) *for every* $x \in X$*.*

Proof of Proposition 1. (1) Let T be a d–Meir–Keeler map on (X, d). Choose an $\varepsilon > 0$. Then, there exists $\delta > 0$ such that condition (MK) in Definition 1 is satisfied.

Let $x, y \in X$ such that $\varepsilon \leq d^s(x, y) < \varepsilon + \delta$. We shall show that $d^s(Tx, Ty) < \varepsilon$.

To this end, assume, without loss of generality, that $d^s(Tx, Ty) = d(Tx, Ty)$. Then, $d^s(Tx, Ty) \leq d(x, y)$ by Remark 1.

We distinguish three cases:

Case 1. $\varepsilon \leq d(x, y) < \varepsilon + \delta$. Then, by condition (MK) in Definition 1, we have $d(Tx, Ty) < \varepsilon$, i.e., $d^s(Tx, Ty) < \varepsilon$.

Case 2. $d(x, y) < \varepsilon$. Then, $d^s(Tx, Ty) < \varepsilon$ because $d^s(Tx, Ty) \leq d(x, y)$.

Case 3. $d(x, y) \geq \varepsilon + \delta$. Then, $d^s(x, y) \geq \varepsilon + \delta$, a contradiction.

We have shown that T is a Meir–Keeler map on the metric space (X, d^s).

(2) Let T be a d–Meir–Keeler map on (X, d). By the statement (1), it is a Meir–Keeler map on the metric space (X, d^s). Then, the proof of Meir–Keeler's fixed point theorem shows that $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s) for every $x \in X$. □

Theorem 1. *Every* d*–Meir–Keeler map on a bicomplete quasi-metric space* (X, d) *has a unique fixed point.*

Proof of Theorem 1. Let T be a d–Meir–Keeler map on a bicomplete quasi-metric space (X, d). By Proposition 1(1), T is a Meir–Keeler map on the complete metric space (X, d^s), so it has a unique fixed point. □

Theorem 2. *Every* d*–Meir–Keeler map on a Hausdorff sequentially complete quasi-metric space* (X, d) *has a unique fixed point.*

Proof of Theorem 2. Let T be a d–Meir–Keeler map on a sequentially complete Hausdorff quasi-metric space (X, d). Fix an $x \in X$. By Proposition 1(2), $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Hence, there exists $y \in X$ such that $d(y, x_n) \to 0$ as $n \to \infty$. By Remark 1, $d(Ty, Tx_n) \to 0$ as $n \to \infty$. Since (X, d) is Hausdorff, we deduce that $y = Ty$. Finally, suppose that $z \in X$ is a fixed point of T. Then, $d(y, z) = d(Ty, Tz)$. If $y \neq z$, it follows from Remark 1 that $d(Ty; Tz) < d(y, z)$, i.e., $d(y, z) < d(y, z)$, a contradiction. □

As easy consequences of Theorems 1 and 2, respectively, we have the following quasi-metric versions of the Boyd and Wong fixed point theorem, respectively. In particular, Theorem 3 below answers the question raised in [1] and mentioned in Section 1.

Theorem 3. *Let* (X, d) *be a bicomplete quasi-metric space. If* T *is a* $\leq_d$*-increasing self-map of* X *for which there exists a right upper semicontinuous function* $\varphi : [0, \infty) \to [0, \infty)$ *satisfying* $\varphi(t) < t$ *for all* $t > 0$*, and* $d(Tx, Ty) \leq \varphi(d(x, y))$ *for all* $x, y \in X$*, then* T *has a unique fixed point.*

Theorem 4. *Let (X,d) be a sequentially complete Hausdorff quasi-metric space. If T is a self-map of X for which there exists a right upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(t) < t$ for all $t > 0$, and $d(Tx,Ty) \leq \varphi(d(x,y))$ for all $x, y \in X$, then T has a unique fixed point.*

Furthermore, from Theorem 3, we immediately deduce the following well-known quasi-metric version of the Banach contraction principle.

Corollary 1. *Let (X,d) be a bicomplete quasi-metric space. If there exists a constant $c \in [0,1)$ such that $d(Tx,Ty) \leq cd(x,y)$ for all $x, y \in X$, then T has a unique fixed point.*

Since every complete metric space is a Hausdorff bicomplete quasi-metric space and, hence, a sequentially complete Hausdorff quasi-metric space, we deduce that the example of Meir and Keeler mentioned in Section 1, shows that Theorems 1 and 2 are real generalizations of Theorems 3 and 4, respectively.

Next, we present pertinent examples illustrating the differences between the four theorems obtained above. The first one, which constitutes a quasi-metric variant of Meir–Keeler's example mentioned above, provides an example where we can apply Theorem 1 but not Theorems 2, 3 and 4.

Example 2. *Let $X = A \cup B$, where $A = [0,1]$ and $B = \{n \in \mathbb{N} : n \geq 2\}$, and let d be the quasi-metric on X given by $d(x,y) = \max\{x-y, 0\}$ if $x, y \in A$, and $d(x,y) = |x-y|$ otherwise. Then, (X,d) is bicomplete because $d^s(x,y) = |x-y|$ for all $x, y \in X$, and thus (X,d^s) is clearly a complete metric space.*

Now define a self-map T of X as follows: $Tx = x/2$ if $x \in A$, $Tx = 0$ if $x \in B$ with x even, and $Tx = x/(x+1)$ if $x \in B$ with x odd.

We show that T is a d–Meir–Keeler map on (X,d). Indeed, we first note that T is $\leq_d$-increasing because, if $d(x,y) = 0$ with $x \neq y$, it follows that $x, y \in A$ with $x < y$, and hence $Tx, Ty \in A$ with $Tx < Ty$, so $d(Tx,Ty) = 0$.

Now take an $\varepsilon > 0$.

If $\varepsilon < 1$, choose $\delta > 0$ such that $\delta < \varepsilon$ and $\varepsilon + \delta < 1$. Now let $x, y \in X$ satisfying $\varepsilon \leq d(x,y) < \varepsilon + \delta$. Then, $x, y \in A$ and, consequently, $d(Tx,Ty) = d(x,y)/2 < (\varepsilon+\delta)/2 < \varepsilon$.

If $\varepsilon \geq 1$, we deduce that $d(Tx,Ty) < \varepsilon$ for all $x, y \in X$ because $d(Tx,Ty) < 1$ for all $x, y \in X$.

We conclude that all conditions of Theorem 1 are satisfied. In fact, 0 is the unique fixed point of T.

Observe that (X,d) is not a T_1 quasi-metric space, and thus not Hausdorff, so we cannot apply Theorem 2, and hence not Theorem 4, to this instance.

Finally, we can also not apply Theorem 3. Indeed, assume the contrary. Then, there exists a function φ for which the conditions of Theorem 3 are satisfied. In particular, for each $n \in \mathbb{N}$, we would have

$$d(T(2n+1), T(2n)) = d(\frac{2n+1}{2n+2}, 0) = \frac{2n+1}{2n+2} \leq \varphi(d(2n+1, 2n)) = \varphi(1),$$

which implies $\varphi(1) \geq 1$, contradicting in this way the assumption that $\varphi(t) < t$ for all $t > 0$.

In our next example, we can apply Theorem 2 but not Theorems 1, 3 and 4.

Example 3. *Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\backslash\{1\}\} \cup \{n \in \mathbb{N} : n \geq 2\}$, and let d be the function defined on $X \times X$ as $d(1/n, 0) = n/(n+1)$ for all $n \in \mathbb{N}\backslash\{1\}$, and $d(x,y) = |x-y|$ otherwise. It is routine to verify that d is a quasi-metric on X (in fact, note that the only check where we should have more care is showing the triangle inequality $d(1/n, 1/m) \leq d(1/n, 0) + d(0, 1/m)$, when $n < m$; recalling, in this case, that $n > 1$).*

Clearly, the quasi-metric space (X,d) is Hausdorff. Moreover, it is sequentially complete but not bicomplete: indeed, the sequence $(1/n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s) that converges to 0 for τ_d but does not converge for τ_{d^s} (note that the only non-eventually constant Cauchy sequences in (X,d^s) are the subsequences of $(1/n)_{n\in\mathbb{N}}$, which obviously converges to 0 for τ_d).

Now define a self-map T of X as follows: $T0 = 0$, $T(1/n) = 0$ for all $n \in \mathbb{N}\backslash\{1\}$, $T(2n) = 0$ for all $n \in \mathbb{N}$, and $T(2n+1) = 1/(2n+1)$ for all $n \in \mathbb{N}$. We show that T is a d–Meir–Keeler map on (X, d). First, note that T is $\leq_d$-increasing because (X, d) is Hausdorff and, hence, T_1. Now take an $\varepsilon > 0$.

If $\varepsilon < 1$, choose $\delta > 0$ such that $\delta < \varepsilon$ and $\varepsilon + \delta < 1$. Now let $x, y \in X$ satisfying $\varepsilon \leq d(x, y) < \varepsilon + \delta$. Then, $d(x, y) < 1$, so $Tx = Ty = 0$, and thus $d(Tx, Ty) = 0 < \varepsilon$.

If $\varepsilon \geq 1$, we deduce that $d(Tx, Ty) < \varepsilon$ for all $x, y \in X$ because $d(Tx, Ty) < 1$ for all $x, y \in X$.

We conclude that all conditions of Theorem 2 are satisfied. In fct, 0 is the unique fixed point of T.

Since (X, d) is not bicomplete, we cannot apply Theorem 1, and hence not Theorem 3, to this instance.

Finally, an argument similar to the one give in Example 2 shows that we cannot apply Theorem 4. Indeed, assume the contrary. Then, there exists a function φ for which the conditions of Theorem 4 are satisfied. In particular, for each $n \in \mathbb{N}$, we would have

$$d(T(2n+1), T(2n)) = d(\frac{1}{2n+1}, 0) = \frac{2n+1}{2n+2} \leq \varphi(d(2n+1, 2n)) = \varphi(1),$$

which implies $\varphi(1) \geq 1$, contradicting in this way the assumption that $\varphi(t) < t$ for all $t > 0$.

We now present an example where we can apply Theorem 3, and hence Theorem 1, but not Theorems 2 and 4, and Corollary 1.

Example 4. *Let $X = [0, \infty)$ and let d be the quasi-metric on X given by $d(x, y) = 0$ if $x \leq y$, and $d(x, y) = x$ otherwise. Since the non-eventually constant Cauchy sequences in (X, d^s) are sequences that converge to 0 for τ_{d^s}, we deduce that (X, d) is a bicomplete quasi-metric space. Let T be the self-map of X defined by $Tx = x/(1+x)$ for all $x \in [0, \infty]$, and let $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = t/(1+t)$ for all $t \geq 0$.*

Obviously, φ is a continuous function on $[0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$.

On the other hand, T is $\leq_d$-increasing. Indeed, let $d(x, y) = 0$ with $x \neq y$. Then, $x < y$ and hence $Tx < Ty$,so $d(Tx, Ty) = 0$.

Next, we show that $d(Tx, Ty) \leq \varphi(d(x, y))$, for all $x, y \in X$. Indeed, if $d(Tx, Ty) = 0$, the conclusion is obvious. Suppose that $d(Tx, Ty) > 0$. Then, $Tx > Ty$ and, thus, $x > y$. Therefore,

$$d(Tx, Ty) = Tx = \frac{x}{1+x} = \varphi(x) = \varphi(d(x, y)).$$

Consequently, all conditions of Theorem 3, and hence all conditions of Theorem 1, are satisfied. However, it is not possible to apply Theorems 2 and 4. (X, d) is not a T_1 quasi-metric space.

We also show that it is not possible to apply Corollary 1 to this example. Indeed, let $c \in (0, 1)$. Choose an $\varepsilon \in (0, 1)$ such that $c(1+\varepsilon) < 1$. Take $x = \varepsilon$ and $y = 0$. Then,

$$d(Tx, Ty) = T\varepsilon = \frac{\varepsilon}{1+\varepsilon} > c\varepsilon = cd(x, y).$$

The following well-known example shows that "Hausdorff" cannot be replaced with "T_1" in Theorem 4, and hence not in Theorem 2.

Example 5. *Let d be the quasi-metric on $\mathbb{N}$ given by $d(n, n) = 0$, for all $n \in \mathbb{N}$ and $d(n, m) = 1/m$ whenever $n \neq m$. It is well known, and easy to check, that $(\mathbb{N}, d)$ is a sequentially complete T_1 quasi-metric which is neither bicomplete nor Hausdorff. The self-map T of $\mathbb{N}$ defined as $Tn = 2n$ for all $n \in \mathbb{N}$, satisfies $d(Tn, Tm) = d(2n, 2m) = 1/2m = d(n, m)/2$ for all $n, m \in \mathbb{N}$. Therefore, all conditions of Theorem 4 are satisfied with the exception of the Hausdorffness of the quasi-metric space $(\mathbb{N}, d)$, and T has no fixed point.*

The preceding example motivates the following variant of the notion of a d–Meir–Keeler map as given in Definition 1.

Definition 2. *Let (X, d) be a quasi-metric space. A self-map T of X is called a d^*–Meir–Keeler map on (X, d) if it is $\leq_d$-decreasing, and, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$,*

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow d(Ty, Tx) < \varepsilon. \qquad (\mathrm{MK}^*)$$

Remark 3. *Note that, similarly to Remark 1, if T is a d^*–Meir–Keeler map on a quasi-metric space (X, d), then $d(Ty, Tx) \leq d(x, y)$ for all $x, y \in X$. In particular, $d(Ty, Tx) < d(x, y)$ whenever $d(x, y) > 0$.*

Now, slight modifications of the proofs of Proposition 1 and Theorems 1 and 2 yield the following results (we only will give the proof of Theorem 6 because it presents some interesting aspects).

Proposition 2. *Let (X, d) be a quasi-metric space. Then,*

(1) Every d^*–Meir–Keeler map on (X, d) is a Meir–Keeler map on the metric space (X, d^s).

(2) If T is a d^*–Meir–Keeler map on (X, d), the sequence $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s) for every $x \in X$.

Theorem 5. *Every d^*-Meir–Keeler map on a bicomplete quasi-metric space (X, d) has a unique fixed point.*

Theorem 6. *Every d^*-Meir–Keeler map on a sequentially complete T_1 quasi-metric space (X, d) has a unique fixed point.*

Proof of Theorem 6. Let T be a d^*-Meir–Keeler map on a sequentially complete T_1 quasi-metric space (X, d). Fix an $x \in X$. By Proposition 2(2), $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Hence, there exists $y \in X$ such that $d(y, x_n) \to 0$ as $n \to \infty$. Therefore, $d(Tx_n, Ty) \to 0$ as $n \to \infty$. Since (X, d) is T_1, we deduce that $y = Ty$. Finally, suppose that $z \in X$ is a fixed point of T. Then, $d(y, z) = d(Ty, Tz)$ and $d(z, y) = d(Tz, Ty)$. If $y \neq z$, it follows from Remark 3 that $d(Ty, Tz) < d(z, y)$, and $d(Tz, Ty) < d(y, z)$, i.e., $d(y, z) < d(z, y)$ and $d(z, y) < d(y, z)$, a contradiction. □

Theorem 7. *Let (X, d) be a bicomplete quasi-metric space. If T is a $\leq_d$-decreasing self-map of X for which there exists a right upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(t) < t$ for all $t > 0$, and $d(Ty, Tx) \leq \varphi(d(x, y))$ for all $x, y \in X$, then T has a unique fixed point.*

Theorem 8. *Let (X, d) be a sequentially complete T_1 quasi-metric space. If T is a self-map of X for which there exists a right upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(t) < t$ for all $t > 0$, and $d(Ty, Tx) \leq \varphi(d(x, y))$ for all $x, y \in X$, then T has a unique fixed point.*

In this context, we immediately deduced from Theorem 7 the following quasi-metric version of the Banach contraction principle (compare Corollary 1).

Corollary 2. *Let (X, d) be a bicomplete quasi-metric space. If there exists a constant $c \in [0, 1)$ such that $d(Ty, Tx) \leq cd(x, y)$ for all $x, y \in X$, then T has a unique fixed point.*

Remark 4. *Note that Examples 2 and 3 provide examples where we can apply Theorems 1 and 2, respectively, but not Theorems 5, 6, 7 and 8.*

We finish this section with an example where we can apply Theorem 7, and hence Theorem 5, but not the rest of theorems obtained in this section as well as Corollary 2.

Example 6. *Let $X = [-1, 1]$ and let d be the quasi-metric on X given by $d(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. Since, for each $x, y \in X$, $d^s(x, y) = |x - y|$, we deduce that (X, d) is a bicomplete quasi-metric space. Let T be the self-map of X defined by $Tx = -x^2/2$ for all $x \in (0, 1]$, and $Tx = 0$ for all $x \in [-1, 0]$; and let $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = 2t/(2 + t)$ for all $t \geq 0$.*

Obviously φ is a continuous function on $[0,\infty)$ such that $\varphi(t) < t$ for all $t > 0$.

On the other hand T is $\leq_d$-decreasing. Indeed, let $d(x,y) = 0$ with $x \neq y$. Then $x < y$. If $x, y \leq 0$, we have $d(Ty, Tx) = d(0,0) = 0$. If $x \leq 0 < y$, we have $d(Ty, Tx) = d(-y^2/2, 0) = 0$. Finally, if $x, y > 0$, we have $d(Ty, Tx) = d(-y^2/2, -x^2/2) = 0$, because $x < y$, and hence, $x^2 < y^2$.

Next we show that $d(Ty, Tx) \leq \varphi(d(x,y))$, for all $x, y \in X$. Indeed, suppose that $d(Ty, Tx) > 0$. Then $Ty - Tx > 0$, so $Ty > Tx$. Therefore, we can consider, without loss of generality, the following two cases:

a) $x, y > 0$. b) $y \leq 0 < x$.

In case a), from $Ty > Tx$ we deduce that $x > y$, and obtain

$$d(Ty, Tx) = d(-y^2/2, -x^2/2) = \frac{x^2 - y^2}{2} = \frac{(x-y)(x+y)}{2} \leq \frac{2(x-y)}{1+2(x-y)} = \varphi(d(x,y)).$$

In case b), we obtain

$$d(Ty, Tx) = d(0, -x^2/2) = \frac{x^2}{2} \leq \frac{x}{2} \leq \frac{x-y}{2} \leq \frac{2(x-y)}{1+2(x-y)} = \varphi(d(x,y)).$$

Consequently, all conditions of Theorem 7, and hence all conditions of Theorem 5, are satisfied. However, it is not possible to apply any of Theorems 1, 2, 3, 4, because T is not $\leq_d$-increasing. Moreover, we cannot apply Theorems 6 and 8 because (X, d) is not a T_1 quasi-metric space.

To conclude we also show that it is not possible to apply Corollary 2 to this example. Indeed, let $c \in (0,1)$. Choose an $\varepsilon \in (0,1)$ such that $2c + \varepsilon < 2$. Now take $x = 1$ and $y = 1 - \varepsilon$. Then

$$d(Ty, Tx) = \frac{1 - (1-\varepsilon)^2}{2} = \frac{(2-\varepsilon)\varepsilon}{2} > \frac{2c\varepsilon}{2} = cd(x,y).$$

3. An Application

Quasi-metric spaces (for instance, the so-called complexity quasi-metric space [21,22] and other related ones) provide an efficient setting to the analysis of complexity of programs and algorithms. In particular, to prove the existence of solution for many recurrence equations typically associated to well-known algorithms as Quicksort (worst case), Hanoi, Largetwo (average cae), Divide and Conquer, etc., via the fixed point theory in quasi-metric spaces (see e.g., [21–26]).

In many cases it suffices to apply the quasi-metric version of the Banach contraction principle given in Corollary 1 to obtain the corresponding solution. Here we present a novel application where, with the help of our version of Boyd-Wong's fixed point theorem obtained in Theorem 3, we shall show the existence (and uniqueness) of solution for a kind of recurrence equations naturally associated to certain difference equations. We shall also observe that Corollary 1 can not be applied to this case.

Denote by ω the set $\mathbb{N} \cup \{0\}$ and let $R : \omega \to (0, \infty)$ be the recurrence equation defined on ω by $R(0) = a$, $a \in (0,1]$, and

$$R(n) = \frac{R(n-1)}{1 + R(n-1)},$$

for $n \in \mathbb{N}$.

This equation arises in the study of the nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}x_n}, \qquad n \in \omega.$$

(see e.g., [27] (Equation (5) on p. 81), and also [28,29]).

We shall apply Theorem 3 to show that the recurrence equation R has a (unique) solution.

To this end, we denote by $[0,1]^\omega$ the set of all functions from ω to $[0,1]$, and then we associate to R the self-map $\mathcal{F}$ of $[0,1]^\omega$ defined as follows:

$\mathcal{F}(f)(0) = a$, and

$$\mathcal{F}(f)(n) = \frac{f(n-1)}{1+f(n-1)}, \qquad n \in \mathbb{N},$$

for all $f \in [0,1]^{\omega}$.

Obviously, if $\mathcal{F}$ has a fixed point f, then f is solution for the recurrence equation R.

We denote by Q the quasi-metric on $[0,1]^{\omega}$ given, for each $f, g \in [0,1]^{\omega}$ by

$$Q(f,g) = \sup_{n\in\omega} \max(f(n) - g(n), 0).$$

Since Q^s is the well-known supremum metric on $[0,1]^{\omega}$, then $([0,1]^{\omega}, Q^s)$ is a complete metric space, so the quasi-metric space $([0,1]^{\omega}, Q)$ is bicomplete. Note also that is $\leq_Q$-increasing: In fact, if $Q(f,g) = 0$, it follows that $f \leq g$, so $\mathcal{F}(f) \leq \mathcal{F}(g)$, and consequently, $Q(\mathcal{F}(f), \mathcal{F}(g)) = 0$.

Next we show that we cannot apply the quasi-metric version of the Banach contraction principle (Corollary 1), to the self-map $\mathcal{F}$ of $([0,1]^{\omega}, Q)$.

Indeed, let $c \in (0,1)$. We distinguish two cases:

Case 1. $c < 1/2$. Take $f, g \in [0,1]^{\omega}$ defined by $f(0) = 1$, $f(n) = 0$ for all $n \in$, and $g(n) = 0$ for all $n \in \omega$. Then $Q(f,g) = 1$. Moreover, $\mathcal{F}(f)(0) = \mathcal{F}(g)(0) = a$, $\mathcal{F}(f)(1) = f(0)/(1+f(0)) = 1/2$, $\mathcal{F}(f)(n) = 0$ for all $n > 1$, and $\mathcal{F}(g)(n) = 0$ for all $n \in \mathbb{N}$. Therefore

$$Q(\mathcal{F}(f), \mathcal{F}(g)) = \frac{1}{2} > c = cQ(f,g).$$

Case 2. $c \geq 1/2$. Choose $b \in (0, (1-c)/c)$, and take $f, g \in [0,1]^{\omega}$ defined by $f(0) = b$ and $f(n) = 0$ for all $n \in \mathbb{N}$, and $g(n) = 0$ for all $n \in \omega$. Then $Q(f,g) = b$, and, similarly, to Case 1, we deduce that

$$Q(\mathcal{F}(f), \mathcal{F}(g)) = \frac{f(0)}{1+f(0)} = \frac{b}{1+b} > cb = cQ(f,g).$$

Finally, we shall prove that for each $f, g \in [0,1]^{\omega}$, one has $Q(\mathcal{F}(f), \mathcal{F}(g)) \leq \varphi(Q(f,g))$, where $\varphi(t) = t/(1+t)$ for all $t \geq 0$, and thus we can apply our quasi-metric version of Boyd-Wong's fixed point theorem, obtained in Theorem 3 above.

Indeed, let $f, g \in [0,1]^{\omega}$. If $f \leq g$, then $\mathcal{F}(f) \leq \mathcal{F}(g)$, so $Q(\mathcal{F}(f), \mathcal{F}(g)) = 0$.

Hence, we shall suppose that $f(m) > g(m)$ for some $m \in \omega$. By the definition of Q and the fact that $\mathcal{F}(f)(0) = \mathcal{F}(g)(0)$, it suffices to show that for any $n \in \mathbb{N}$ satisfying $\mathcal{F}(f)(n) > \mathcal{F}(g)(n)$ it follows that

$$\mathcal{F}(f)(n) - \mathcal{F}(g)(n) \leq \frac{Q(f,g)}{1+Q(f,g)}.$$

To this end, let $n \in \mathbb{N}$ be such that $\mathcal{F}(f)(n) > \mathcal{F}(g)(n)$. Then

$$0 < \mathcal{F}(f)(n) - \mathcal{F}(g)(n) = \frac{f(n-1) - g(n-1)}{(1+f(n-1))(1+g(n-1))} \leq \frac{f(n-1) - g(n-1)}{1+f(n-1)} \tag{1}$$

Since $f(n-1) - g(n-1) \leq Q(f,g)$ and $f(n-1) - g(n-1) \leq f(n-1)$, we deduce that

$$(f(n-1) - g(n-1))(1+Q(f,g)) \leq Q(f,g) + f(n-1)Q(f,g),$$

so

$$f(n-1) - g(n-1) \leq \frac{(1+f(n-1))Q(f,g)}{1+Q(f,g)} \tag{2}$$

From inequalities (1) and (2) it follows that

$$\mathcal{F}(f)(n) - \mathcal{F}(g)(n) \leq \frac{Q(f,g)}{1 + Q(f,g)}.$$

We conclude that

$$Q(\mathcal{F}(f), \mathcal{F}(g)) \leq \frac{Q(f,g)}{1 + Q(f,g)} = \varphi(Q(f,g)).$$

By Theorem 3, $\mathcal{F}$ has a unique fixed point, which is the unique solution of recurrence equation R.

4. Conclusions

We have obtained four different extensions of the famous Meir–Keeler fixed point to the realm of bicomplete and sequentially complete quasi-metric spaces, respectively, from which we have derived four different extensions of the celebrated Boyd–Wong fixed point theorem. Our study is motivated, in part, by an example presented in [1] showing that Boyd–Wong's fixed point theorem does not admit a direct generalization to bicomplete quasi-metric spaces; and, then, by the natural question derived from this example and also raised in [1] of adding a reasonable condition for which such a generalization holds. We show (Theorems 1 and 3) that the condition that the self map be $\leq_d$-increasing provides the appropriate ingredient to obtain a solution to this question and also a satisfactory generalization both of Meir–Keeler's and of Boyd–Wong's theorem to bicomplete quasi-metric spaces (recall that every self map on a T_1 quasi-metric space, and hence on a metric space, is $\leq_d$-increasing). Our study is completed by obtaining the corresponding extensions and generalizations under the assumption that the self map is $\leq_d$-decreasing instead of $\leq_d$-increasing and giving an application to the study of existence of solution for a recurrence equation associated to certain nonlinear difference equations. Several examples showing the differences between the obtained results are also discussed.

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