



# Article **Do Fractals Confirm the General Theory of Relativity?**

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**Abstract:** The relatively high abundance of fractal properties of complex systems on Earth and in space is considered an argument in support of the general relativity of the geometric theory of gravity. The fractality may be called the fractal symmetry of physical interactions providing self-similarities of complex systems. Fractal symmetry is discrete. A class of geometric solutions of the general relativity equations for a complex scalar field is offered. This class allows analogy to spatial fractals in large-scale structures of the universe due to its invariance with respect to the discrete scale transformation of the interval  $ds \leftrightarrow qd\tilde{s}$ . The method of constructing such solutions is described. As an application, the treatment of spatial variations of the Hubble constant  $H_0^{HST}$  (Riess et al., 2016) is considered. It is noted that the values  $H_0^{HST}$  form an almost fractal set. It has been shown that: a) the variation  $H_0^{HST}$  may be connected with the local gravitational perturbations of the space-time metrics in the vicinity of the galaxies containing Cepheids and supernovae selected for measurements; b) the value of the variation  $H_0^{HST}$  can be a consequence of variations in the space-time metric on the outskirts of the local supercluster, and their self-similarity indicates the fractal distribution of matter in this region.

**Keywords:** fractals; self-similarity; fractal symmetry; general relativity; geometric solutions; Hubble constant; gravitational perturbations

### 1. Introduction

For more than a hundred years, on 29 May 1919, the Eddington expedition made observations of a solar eclipse. The purpose of the observation was to test one of the three effects proposed by Einstein to confirm his general relativity (GR)—the curvature of light lines from a distant star in the gravitational field of the Sun. The results of Eddington's observations confirmed the predictions of GR [1]. It was the first experiment to test the main postulate of this theory as a geometrical model of gravitational interaction. Over the past century, many direct experiments have been made. The absolute majority of these experiments confirmed Einstein's theory [2].

However, at present, criticism of GR has renewed, in particular, because of the appearance of, the so-called "dark sector" in cosmology. Astrophysicists and cosmologists introduced the dark matter, and then the dark energy, to explain the characteristics of the rotation curves of galaxies, the stability of galactic disks, the growth of large-scale structures in the early universe, the Hubble parameter (accelerated expansion of space), and the anisotropy of the CMB radiation. Here these problems will not be discussed; but some relevant publications are given in [3]. So far, the "dark sector" methodology could not provide a satisfactory explanation of these problems because of the large variety of dark energy and dark matter models.

Modifications of the GR appeared in response to the "paradoxes of the dark sector". The main disadvantage of these modifications is that they introduce additional fundamental constants of gravitational interaction, which do not allow the use of the effects of the "dark sector". However, these parameters are defined from the same astrophysical observations of galaxies which need to be explained. It does not cause satisfaction. In GR, the fundamental constants measured in terrestrial experiments are used.

Modern information technologies allow the numerical solution of the non-linear equations of GR or its modifications. They are also used in observations, in processing observational data, and in the analysis of these data within a chosen cosmological model. The authors of various cosmological simulations fit their results to observations with good accuracy, which becomes better with an increase in the number of free parameters for baryonic matter (for example, different types of viscosity), for dark matter (from cold to hot), and for dark energy (different variants of scalar field potentials).

In work [3], facts indicating that the dark matter hypothesis is groundless from the point of view of the interpretation of the observations are given. Galaxies and clusters of galaxies are located inside the cosmic web. They are open systems that interact with the circumgalactic medium and the intergalactic medium. The description of the dynamics of such open systems within the framework of GR has not yet been done. Therefore, there is no indisputable evidence that would justify the rejection of GR.

In this article, the relatively high abundance of fractal properties of complex systems on Earth and in space is considered as an argument in support of the GR as a geometric theory of gravity. Geometrically, fractals are self-similar structures. Not always this self-similarity can be observed directly, especially in astrophysical objects and cosmological structures. Meanwhile, a sign of self-similar discrete sets is power dependencies between the physical characteristics of complex systems. For example, let us consider the Tully-Fisher relation  $L \sim v^3$  in spirals for the luminosity L and the rotation rate of spiral galaxies v. Suppose that the values of the luminosities of galaxies and the rotation rates form geometric progressions (the most obvious example of self-similarity in the state space of the system):  $L_{i+1} = aL_i$ ,

 $v_{i+1} = bv_i$ , where *a* and *b* are constant parameters, i = 1, 2, ... Number *i* can be written as  $i = \frac{\ln \frac{v_i}{v_1}}{\ln b}$ . Then for luminosity values we find  $L_i = (v_i)^{\frac{\ln a}{\ln b}} \left(\frac{L_1}{(v_1)^{\frac{\ln a}{\ln b}}}\right) \sim (v_i)^{\frac{\ln a}{\ln b}}$ . The Tully-Fisher relation follows

from this formula if we assume  $\frac{\ln a}{\ln b} = 3$ . Spiral galaxies are similar physical objects. The power of the Tully-Fisher relation indicates that these objects form a fractal discrete set in the state space of these objects; we observe the projection of this set onto the L - v plane as the Tully-Fisher relation.

The power of the relationship is often met in very different astrophysics objects. Let's give several examples:

- Main sequence stars obey a mass-luminosity relation,  $L \sim m^{\alpha}$ ,  $\alpha \approx 3.52$ ;
- Period-luminosity relation for pulsating variable stars,  $L \sim P^{\alpha}$ , for classical Cepheids  $\alpha \approx 1.15$ ;
- Luminosity functions for stars, galaxies, and quasars,  $\Phi \sim L^{\alpha}$ ,  $-2.6 < \alpha < -0.6$ ;
- Faber-Jackson relation for luminosity and their central stellar velocity dispersion of stars of elliptical galaxies, L ~ σ<sup>4</sup>;
- Tully-Fisher relation for the luminosity and rotation rate of spiral galaxies,  $L \sim v^3$ ;
- Brightness distribution in the images of bright elliptical galaxies,  $I \sim r^{-\alpha} \left(1 + \left(\frac{r}{r_0}\right)^{\beta}\right)^{\gamma}$ ,  $-1.37 < \alpha < 0.85$ ,  $0.28 < \beta < 1$ ,  $-7.57 < \gamma < -0.18$ ;
- Power-law spectra of the radio emission of the jets from active galactic nuclei, *F* ~ ν<sup>−α</sup>, 0.6 < α < 0.7 (jets are composed of plasma clouds with a magnetic field);</li>
- Baldwin's ratio in the active galactic nucleus for the equivalent width of the emission lines and the luminosity of its galaxy,  $EW(CaII) \sim L^{-\alpha}$ ,  $\alpha > 1$ ;
- Red shift distribution of absorption lines in quasar spectra,  $\frac{dN}{dz} \sim (1+z)^{\alpha}$ , 1.67 <  $\alpha$  < 2.09;
- Spatial correlation function of galaxies in the clusters of galaxies and clusters in superclusters,  $\xi(r) \sim r^{-\alpha}$ ,  $1.6 < \alpha < 2.2$ .

The given power relationships are of course correlations. However, new observations only specify them.

The relative high abundance of fractals indicates that perhaps fractality is a fundamental property of physical interactions. The fractality may be called the fractal symmetry of physical interactions providing self-similarity of complex structures. Fractal symmetry is discrete, since a complex system with fractal symmetry consists of self-similar subsystems and the physical parameters of these subsystems form geometric progressions. Fractality can be compared with quantization (discreteness) of the states of the systems in the microcosm.

Can fractal symmetry be a consequence of the geometric model of gravity in GR? This question is logical if we take into account the geometric properties of the phase spaces of the systems, which we call fractals. It is also logical from a historical point of view in the development of science. For example, in the classical work of Minkowski (1909, [4]) it was proposed to back out of the hypothesis of ether, but to combine space and time into a four-dimensional homogeneous and isotropic set of points (the Minkowski world). Minkowski proved his offer relying on postulates of the special theory of relativity and the Lorentz invariance of the Maxwell's equations. According to Minkowski, we discover a homogeneous and isotropic space-time through the invariance of Maxwell's equations with respect to the Lorentz transformations. From the modern point of view, Minkowski considered the Lorentz's symmetry for an electromagnetic field as a consequence of the properties of four-dimensional space-time: the vector potential of the electromagnetic field is also transformed like any vector in four-dimensional space-time. Theoretical cosmology as a science about the structure and evolution of space-time of the universe began with this research by Minkowski. It should be noted that the fundamental hypothesis by Weyl, where he proposed (1918, 1929 [5,6]) to use a scalar field  $\Phi(x^k)$ (now it is called dilation) with transformation  $\Phi \leftrightarrow \sigma^{1/2} \widetilde{\Phi}$  and the continuous conformal symmetry of space-time  $ds^2 \leftrightarrow \sigma(x^k) d\bar{s}^2$  for the electron model. This field should change the Newtonian gravitational interaction of all systems and its consequences would be self-similar properties of these systems. However, the existence of the dilaton field is not detected.

Now symmetries play an important role in the development of our ideas about space-time and the physics of matter. The description of the observed properties of matter by means of dynamic equations and symmetries of the interaction fields allows us to understand the nature of the universe at the micro and macro levels. This paradigm inspires cosmologists.

In this paper, a class of geometric solutions of GR is proposed for a complex scalar field. The solutions admit the possibility of spatial fractals in the large-scale structure of the universe [7–9]. Quasar groups are an example of such fractals. It was found in [8], that the number of groups  $N_i$  with size  $\vartheta_i$  depends on the size  $N_i \sim (\vartheta_i)^{-2.02}$ . Let's consider the quasar group with number *i* as the so-called "island system", that has a space-time with an interval  $ds_i$ . If the intervals of different island systems form a geometric progression  $ds_{i+1} = qds_i$  (*q* is a number, discrete scale transformation), then a distant observer using the metric of the Minkowski's space will observe the geometric progression  $\vartheta_{i+1} = q\vartheta_i$  for the angular sizes of the quasar groups. The introduction of a geometric progression for interval means, that we consider the space-time, consists of an ensemble of geometrically self-similar regions.

Below we consider an example of the solution of the GR equations, which allows for the discrete scale transformation of the space-time interval. The complex scalar field is used as a source in the GR equations. It is assumed that the field has a unitary symmetry U (1), which is given in the algebraic form:

$$(\psi_1)^2 + (\psi_2)^2 = \Psi^2 = const.$$
 (1)

The complex field is used in the scalar electrodynamics to introduce charges, that are conserved due to symmetry (1). In the Standard Model of the physics of elementary particles SU(n) symmetries are introduced as a generalization of the circle (1) for multidimensional fields.

In the problem considered here, the discrete transform of the field phase  $\varphi \leftrightarrow \tilde{\varphi} + \alpha$  ( $\alpha$  is a constant) without changing its amplitude  $\Psi$  (unitary symmetry) corresponds to a field transform  $\tilde{\psi} \leftrightarrow e^{-i\alpha}\psi$  and a discrete transform of metric tensor  $g_{mn}(\varphi) \leftrightarrow \frac{\tilde{U}_0}{U_0}\tilde{g}_{mn}(\tilde{\varphi})$ , where  $U_0$  and  $\tilde{U}_0$  are the parameters of the field potentials  $U = U_0\psi\psi^*$ ,  $\tilde{U} = \tilde{U}_0\tilde{\psi}\tilde{\psi}^*$ , an asterisk means complex conjugation. In this case, the two regions of space-time are geometrically similar, their intervals are connected

by a scale transformation  $ds^2 \leftrightarrow \frac{\widetilde{U}_0}{U_0} d\widetilde{s}^2$  (an analogue of the conformal discrete transformation). An ensemble of such regions can be called a fractal, if  $(ds^2)_{i+1} = (\frac{\widetilde{U}_0}{U_0})(d\widetilde{s}^2)_{i'}$ , i = 1, 2, ...

Let's define a fractal cosmological model as a model in which fractal properties are characteristic of the spatial distribution of matter. The space-time of this model consists of a set of regions in space-time, the metric tensors of which are connected by discrete scale transformations.

Note, that in the fractal cosmological model, the simulation of the large-scale structure of the galaxy distribution should require less computational time, because our need is to model the self-similar parts of this structure and not their ensembles as a whole.

The rest of the article is organized as follows. Section 2 presents the method of constructing a geometrical solution of the Lagrange and Einstein equations for a complex scalar field. Section 3 discusses the contradiction in the measurements of the Hubble constant and its interpretation within the framework of the fractal cosmological model.

# 2. Exact Geometrical Solution of the Lagrange and Einstein Equations for a Complex Scalar Field with Symmetry U (1)

In the preprint [10], a method is presented for constructing the class of exact solutions discussed here for the GR equations of a complex scalar field. The following describes the revised and expanded version of this method. Let's consider a system with the Hilbert—Einstein's action

$$S = -\frac{c^3}{16\pi G} \int \left( R - \frac{8\pi G}{c^4} L \right) \sqrt{-g} d^4 x, \tag{2}$$

where the curvature scalar is R, g < 0 is the determinant of the metric tensor  $g_{mn}$ , the space-time interval is  $ds^2 = g_{mn}dx^m dx^n$ , the indices run through the values 0, 1, 2, 3, the metric signature is (+ - -). Here, the physical system of units is maintained to simplify the comparison with the observations.

The Lagrangian of a complex scalar field has the form:

$$L = \frac{1}{hc} \left( g^{mn} \frac{\partial \psi}{\partial x^m} \frac{\partial \psi^*}{\partial x^n} - U(\psi\psi^*) \right), \tag{3}$$

where *h* is the Planck constant,  $U(\psi\psi^*)$  is the field potential, which we choose in a simple form  $U(\psi\psi^*) = U_0\psi\psi^* = U_0\Psi^2$  for a field  $\psi = \Psi e^{i\varphi}$  with symmetry (1).

From the Lagrange equation

$$\left(\frac{\partial^2 \psi}{\partial x^m \partial x^n} - \Gamma^l_{mn} \frac{\partial \psi}{\partial x^l}\right) g^{mn} = -\frac{\partial U}{\partial \psi^*},\tag{4}$$

we derive an equation for the field phase  $\varphi(x^m)$ :

$$\left(-\psi\frac{\partial\varphi}{\partial x^m}\frac{\partial\varphi}{\partial x^n}+i\psi\frac{\partial^2\varphi}{\partial x^m\partial x^n}-i\psi\Gamma_{mn}^l\frac{\partial\varphi}{\partial x^l}\right)g^{mn}=-U_0\psi.$$
(5)

The goal is to choose the dependence of the Christoffel's symbols  $\Gamma_{mn}^l$  and the metric tensor  $g_{mn}$  on the derivatives  $\frac{\partial \varphi}{\partial x^m}$  in such away so that Equation (5) for the phase  $\varphi(x^m)$  is identically satisfied, and the Einstein equation for the metric tensor  $g_{mn}$  can be reduced to an algebraic form for the vector  $e_n = \frac{\partial \varphi}{\partial x^n}$ . In this case, the Einstein's equation is invariant relative to the discrete phase transformation  $\varphi \leftrightarrow \widetilde{\varphi} + \alpha$ . Achieving this goal is carried out in two stages.

#### 2.1. Selection of the Christoffel's Symbols and Metric Tensor

By direct substitution, we can see that Equation (5) becomes an identity if the following formulas are used:

$$e_m e_n g^{mn} = U_0, (6)$$

$$\Gamma_{mn}^{l} = \frac{1}{U_0} \frac{\partial^2 \varphi}{\partial x^m \partial x^n} \left( e^l + b^l \right),\tag{7}$$

$$e_l b^l = 0, (8)$$

where  $b^l$  is a vector.

The left-hand sides of formulas (6) and (8) are the tensors, therefore the formulas are covariance invariant. For the covariant conservation of formula (7), it is necessary that the covariant derivatives be equal to zero,  $e_{n,m} = 0$ ,  $b_{m}^{l} = 0$  and

$$\left(\frac{\partial^2 \varphi}{\partial x^m \partial x^n}\right)_l = \left(\frac{\partial e_n}{\partial x^m}\right)_l = 0.$$
(9)

We define the metric tensor by the following formula:

$$g_{mn} = \frac{4}{U_0} e_m e_n + a_m e_n + a_n e_m,$$
 (10)

where there  $a_m$  is a vector. From the condition  $g_{mn}g^{mn} = 4$ , the equation follows,  $a_m e^m = 0$ .

The equation for the covariant derivative  $g_{mn,k} = 0$  is identically satisfied, if  $e_{n,m} = 0$  and  $a_{m,n} = 0$ , then

$$\frac{\partial a_m}{\partial x^l} = -\frac{3}{U_0} \frac{\partial e_m}{\partial x^l},\tag{11}$$

$$\frac{\partial e_m}{\partial x^n} = \frac{\partial e_n}{\partial x^m}, \frac{\partial a_m}{\partial x^n} = \frac{\partial a_n}{\partial x^m}, \frac{\partial b_m}{\partial x^n} = \frac{\partial b_n}{\partial x^m}.$$
(12)

Calculation of the Cristoffel's symbols  $\Gamma_{ml}^n = \frac{1}{2}g^{kn}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{lk}}{\partial x^m} - \frac{\partial g_{ml}}{\partial x^k}\right)$  and comparison with the definition (7) lead to the following equation:

$$\frac{1}{U_0}\frac{\partial e_m}{\partial x^l}(e^n+b^n) = \frac{4}{U_0}e^n\frac{\partial e_m}{\partial x^l} + a^n\frac{\partial e_m}{\partial x^l} + e^n\frac{\partial a_m}{\partial x^l},$$
(13)

if the calculations take into account the equalities (12). Equation (13) turns into an identity, if we take into account Equation (12) and

$$b^n = U_0 a^n. (14)$$

Formulas (6)–(14) determine the relationships between the vector  $e_m$  and  $g_{mn}$ ,  $\Gamma_{mn}^l$ ,  $a_m$ , and  $b^l$ .

#### 2.2. Algebraic form of the Einstein's Equations

To determine the vector  $e_n$  and metric tensor (10), one must use the Einstein's equation  $R_{km} = \frac{8\pi G}{c^4} (T_{km} - \frac{1}{2}g_{km}T)$  (here we use the notation of the classic textbook [11]). The energy-momentum tensor of the complex field is  $T_{km} = \frac{2\Psi^2}{hc} U_0 e_k e_m$ . The calculation of the Ricci tensor  $R_{mn}$  through the symbols  $\Gamma_{mn}^l$  and Equation (12) allows us to obtain the Einstein's equation in the following form:

$$\left(\frac{\partial^2 e_m}{\partial x^l \partial x^n} - \frac{\partial^2 e_m}{\partial x^n \partial x^l}\right) \left(e^l + U_0 a^l\right) = -\zeta U_0 (2e_m e_n + U_0 a_m e_n + U_0 a_n e_m),\tag{15}$$

where  $\zeta = \frac{8\pi G}{hc^5} \Psi^2$ . The derivatives of the left side of Equation (15) are determined from Equation (9):

$$\frac{\partial^2 e_m}{\partial x^n \partial x^l} = \Gamma^k_{mn} \frac{\partial e_k}{\partial x^l} + \Gamma^k_{ln} \frac{\partial e_k}{\partial x^m} = \frac{1}{U_0} \left( e^k + U_0 a^k \right) \left( \frac{\partial e_m}{\partial x^n} \frac{\partial e_k}{\partial x^l} + \frac{\partial e_l}{\partial x^n} \frac{\partial e_k}{\partial x^m} e_n \right). \tag{16}$$

In this case, Equation (15) is rewritten as follows:

$$\left(\frac{\partial e_m}{\partial x^k}\frac{\partial e_n}{\partial x^l} - \frac{\partial e_m}{\partial x^n}\frac{\partial e_k}{\partial x^l}\right)\!\left(e^k + U_0a^k\right)\!\left(e^l + U_0a^l\right) = -\zeta(U_0)^2(2e_me_n + U_0a_me_n + U_0a_ne_m). \tag{17}$$

The Equation (17) has an algebraic equation for a vector  $e_m$  if we introduce another definition for the tensor  $\frac{\partial e_m}{\partial x^n}$ :

$$\frac{\partial e_m}{\partial x^n} = e_m \gamma_n + e_n \gamma_m,\tag{18}$$

where  $\gamma_m$  is a vector. We also use the general form of the solution of Equation (11):

$$U_0 a_m = -3e_m + \xi_m,\tag{19}$$

where the vector  $\xi_m$  does not depend on coordinates  $x^k$ , i.e.,  $\frac{\partial \xi_m}{\partial x^n}$  and  $\xi_m e^m = 3U_0$ . As a result, Equation (17) is written in algebraic form:

$$e_m e_n (y - 2z)^2 - U_0 (y - 2z) (e_n \gamma_m + e_m \gamma_n) + (U_0)^2 \gamma_m \gamma_n = \zeta (U_0)^2 (4e_m e_n - \xi_m e_n - \xi_n e_m),$$
(20)

where  $y = \gamma_k \xi^k$  and  $z = \gamma_k e^k$ . The vector  $e_m$  can be found by convolving Equation (20) with the vector  $e^n$ :

$$e_m = U_0 \frac{(y-3z)\gamma_m - \zeta(U_0)^2 \xi_m}{(y-2z)(y-3z) - \zeta(U_0)^2}.$$
(21)

Equations (18) and (21) define the class of geometrical solutions for problem (1)–(3) for the metric tensor

$$g_{mn} = \frac{1}{U_0} (-2e_m e_n + e_n \xi_m + e_m \xi_n),$$
(22)

and the Cristoffel's symbols

$$\Gamma_{mn}^{l} = \frac{1}{U_0} (e_n \gamma_m + e_m \gamma_n) \Big( -2e^l + \xi^l \Big), \tag{23}$$

if the vectors  $\gamma_k$  and  $\xi_k$  are chosen. Options, that lead to contradictions, should be excluded from this choice: y = 3z, y = 0, z = 0, y = 2z, and  $\gamma_m \gamma^m = 0$ .

So, the Einstein's equation is reduced to the algebraic form (20). Using it and Equation (18), we find the function  $e_m(x^n)$ .

Convolution of Equation (20) with  $e^m e^n$ ,  $\gamma^m \gamma^n$ , and  $\xi^m \xi^n$  gives three corresponding equations:

$$(y-3z)^2 = -2\zeta(U_0)^2,$$
(24)

$$(z(y-2z) - U_0 \gamma_m \gamma^m)^2 = -2\zeta (U_0)^2 z(y-2z),$$
<sup>(25)</sup>

$$2(y-z)^2 = 3\zeta U_0(6U_0 - \xi_n \xi^n).$$
<sup>(26)</sup>

Equation (24) defines the relationship between *y* and *z*, Equations (25) and (26) define the functions  $\gamma_m \gamma^m$  and  $\xi_n \xi^n$ .

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Using formula (24) in formula (21) we get,  $e_m = U_0 \frac{\gamma_m + \beta U_0 \xi_m}{z + 3\beta U_0}$ , where  $\beta = \left(-\frac{\zeta}{2}\right)^{1/2}$  is an imaginary number. Then the Equation (18) is reduced to the form (the condition  $\frac{\partial \xi_m}{\partial x^n} = 0$  is used):

$$\frac{\partial \gamma_m}{\partial x^n} - \frac{\gamma_m + \beta U_0 \xi_m}{z + 3\beta U_0} \frac{\partial z}{\partial x^n} = 2\gamma_m \gamma_n + \beta U_0 (\xi_m \gamma_n + \xi_n \gamma_m).$$
(27)

The Equation (27) includes three unknowns—two functions  $\gamma_m(x^k)$ ,  $z(x^k)$ , and a constant vector  $\xi_m$ . Their choice determines the vector  $e_m(x^n)$ . As an example, we present the following choice of equations for functions  $\gamma_m(x^k)$ ,  $z(x^k)$ :

$$\frac{\partial \gamma_m}{\partial x^n} = 2\gamma_m \gamma_n + \beta U_0 (\gamma_n + \beta \xi_n) \xi_m, \tag{28}$$

$$\frac{\partial z}{\partial x^n} = \beta(z + 3\beta U_0)\xi_n.$$
(29)

They turn Equation (27) into an identity. The complex solution of Equation (29) has the form:

$$z = e^{\beta\xi_n x^n} - 3\beta U_0 = \cos\left(\sqrt{\frac{\zeta}{2}}\xi_n x^n\right) + i\left(\sin\left(\sqrt{\frac{\zeta}{2}}\xi_n x^n\right) - 3U_0\sqrt{\frac{\zeta}{2}}\right).$$
(30)

The solution of Equation (28) is more bulky:

$$\gamma_m = \beta U_0 \xi_m e^{2\int \frac{y}{\xi_m \xi^m} d\eta} \int \frac{\frac{y}{(\xi_m \xi^m)} + \beta}{e^{2\int \frac{y}{\xi_m \xi^m} d\eta}} d\eta,$$
(31)

where  $d\eta = \xi_n dx^n$ , the functions  $\xi_n \xi^n$  and y are determined from Equations (25) and (26) using solution (30). To determine the vector  $e_m$  and metric tensor (22), it is necessary to use the real parts of the functions (30) and (31). The given solution of problem (1)–(3) has a pulsating space-time.

Despite the relative bulkiness of the described method of constructing a solution to problem (1)–(3), it is simple and easily verified by analytical calculations, and also allows numerical modeling of all solutions.

Note that for the problem in question, Lagrangian (3) is zero, but the field energy density is not zero:

$$E = \frac{1}{hc} \left( g^{mn} \frac{\partial \psi}{\partial x^m} \frac{\partial \psi^*}{\partial x^n} + U(\psi\psi^*) \right) = \frac{2}{hc} U_0 \Psi^2.$$
(32)

The interval of the phase trajectory for the circle (1) is equal to

$$dF^{2} = (d\psi_{1})^{2} + (d\psi_{2})^{2} = d\psi d\psi^{*} = \Psi^{2} e_{m} e_{n} dx^{m} dx^{n}.$$
(33)

Using formulas (22), one can transform formula (33) to the following form:

$$dF^{2} = \frac{1}{2}\Psi^{2}U_{0}\left(-ds^{2} + \frac{1}{U_{0}}(e_{m}\xi_{n} + e_{n}\xi_{m})dx^{m}dx^{n}\right).$$
(34)

Formula (34) shows how the local geometry of the phase space (1) is related to the local space-time geometry.

When the phase transformation is  $\varphi \leftrightarrow \tilde{\varphi} + \alpha$  and the field amplitude is  $\Psi = \tilde{\Psi}$ , but the potential parameter is changed  $U_0 \leftrightarrow \tilde{U}_0$ , then the Lagrange's equation, the Einstein's equations, covariant vectors  $e_m$ ,  $\xi_m$ ,  $\gamma_m$ , the Christoffel's symbols do not change. However, the metric tensor is multiplied by a constant coefficient:

$$g_{mn}(\varphi) \leftrightarrow \frac{\widetilde{U}_0}{U_0} \widetilde{g}_{mn}(\widetilde{\varphi}).$$
 (35)

As a result, the transformation of the phase of the complex field leads to the compression or stretching of the space-time interval  $ds^2 \leftrightarrow \frac{\widetilde{U}_0}{U_0} d\widetilde{s}^2$ . This means that the volumes of space-time with fields  $\psi$  and  $\widetilde{\psi}$  are similar geometrically, therefore, the directly observed Newtonian law of law does not change. On the contrary, continuous conformal transformations distort the geometry and are accompanied by a change in the Newtonian law of gravitation.

Thus, if there are two regions of space-time in which complex scalar fields are connected by a phase transformation  $\varphi \leftrightarrow \widetilde{\varphi} + \alpha$ , then the spatial scales in these regions are connected by a transformation  $dl \leftrightarrow \left(\frac{\widetilde{u}_0}{U_0}\right)^{1/2} d\widetilde{l}$ . Let there be a set of regions in which the phases of the fields and the spatial scales are connected by the equations  $\varphi_i = \varphi + \alpha_i$  and  $(dl)_{i+1} \leftrightarrow \left(\frac{\widetilde{u}_0}{U_0}\right)^{1/2} (dl)_i$ , respectively. Then a sample of physically similar structures from these regions will form a fractal.

#### 3. Spatial Variations of the Hubble Constant and Local Gravitational Perturbations

Hubble's article on "relation between distance and radial velocity among extra-galactic nebular" was published 90 years ago [12]. This discovery was the first law of the observational cosmology. The Hubble constant is the most important cosmological parameter. It is used to estimate cosmological distances to galaxies and their clusters. It is used in the theoretical cosmology through the Hubble parameter,  $H = \frac{\dot{a}}{a}$ , where *a* is the scale factor of the cosmological model and a dot above the letter means the differentiation according to the cosmological time of the model *t*. The Hubble parameter characterizes the speed of the expansion of the universe space. In a homogeneous and isotropic cosmological model, the Hubble parameter depends only on cosmological time *t* (at each moment of time, the expansion rate at all points of the 3D-space is identical). Then in the modern era, the Hubble constant  $H_0 = H(t_0)$  has to be identical in all directions.

The results of the latest local measurements  $H_0$  were very carefully performed in [13,14] using data on Cepheids in 23 galaxies, supernovae SN Ia were observed in 19 galaxies (redshifts 0.01 < z < 0.15, observations by the Hubble Space Telescope). The average value is equal to  $H_0^{HST} = (73.48 \pm 1.66) km/s/Mpc$  for the standard cosmological model.

Global estimates  $H_0$  are obtained by fitting cosmological parameters of the model to observations of the CMB anisotropy and galaxy clustering. According to the latest data from the Planck collaboration for a spatially flat model, we have  $H_0^{Pl} = (67.4 \pm 0.5) km/s/Mpc$  [15].

In recent publications, an unexpectedly large difference between  $H_0^{HST}$  and  $H_0^{Pl}$  is explained by the possibility of the presence of still unclear errors of measurements or hypotheses about the dark energy model (for example, [16]) or about the inhomogeneity of dark matter distribution (for example, [17,18]).

Here we propose a simpler hypothesis to explain the inequality  $H_0^{HST} \neq H_0^{pl}$  remaining within the GR framework and without the use of exotic forms of matter. The hypothesis itself appeared due to a careful look at the measurement data  $H_0^{HST}$  in various galaxies [13]. A sample of these data is shown in Table 1. It appears that the measured values  $H_0^{HST}$  form a manifold that is approximately described by the power of low characterizing fractals. It has the following form  $H_0^{HST} \approx q_i H^*$ , and q = 1.023. Here  $H^* = 65km/s/Mpc$  and the values of the exponent *i* are shown in Table 1. The proximity of value q = 1.023 to 1 may indicate a very small deviation from the isotropic expansion of space.

Let's suppose that this form reflects the real local properties of the space expansion in the vicinity of the supernovae (or galaxies) that were used for the measurement. Variations of the expansion speed relative to the value  $H_0^{Pl}$  indicate the presence of local gravitational perturbations in the vicinity of each source (local changes of the space-time metric relative to the metric of the background reference system). The fractality indicates that the processes generating these gravitational perturbations have an identical nature. Such processes can be gravitational waves generated by the asymmetric collapse of a supernova core or cosmological gravitational perturbations. The fractality of the Hubble constant should also be observed in the fractal cosmological model.

Galaxy	$H_0^{HST}$ (km/s/Mpc)	i
M101	68.39	2
N1015	80.09	9
N1309	70.24	3
N1365	68.39	2
N1448	77.77	8
N2442	73.42	5
N3021	63.94	-1
N3370	76.00	7
N3447	74.37	6
N3972	77.98	8
N3982	64.80	-1
N4038	79.69	9
N4258	72.25	5
N4424	63.97	-1
N4536	71.48	4
N4639	77.98	8
N5584	78.67	9
N5917	72.75	5
N7250	74.75	6
U9391	66.53	1

Table 1. Hubble constant (data [13]).

Let's consider local gravitational perturbations in the vicinity of the galaxy which hosts Cepheids and supernovae. These sources are used to measure the Hubble constant in [13]. We use the synchronous space-time metric

$$ds^{2} = g_{ik}dx^{i}dx^{k} = a^{2}d\eta^{2} + \left(a^{2}\delta_{\alpha\beta} + h_{\alpha\beta}\right)dx^{\alpha}dx^{\beta},$$
(36)

where Latin indices run through values 0, 1, 2, 3, and Greek 1, 2, 3. The coordinates of the background reference system are  $\eta$  and  $x^{\alpha}$ , the cosmological time *t* is related to the conformal time  $\eta$  by the Equation  $ad\eta = cdt$ , and  $h_{\alpha\beta}$  is a local gravitational perturbation.

The commoving distance of galaxy  $r_g$  is measured along the isotropic geodesic of the space-time with metric (1). For the isotropic geodesic ds = 0, then  $c^2 dt^2 = -(g_{\alpha\beta} + h_{\alpha\beta})k^{\alpha}k^{\beta}dr^2$ . Here the isotropic wave vector is  $k^i = \frac{dx^i}{dr}$ ,  $k_ik^i = 0$ , and  $k^{\alpha} = \frac{q^{\alpha}}{a^2}$ ,  $k_{\alpha} = q_{\alpha}$ ,  $q^{\alpha}$  and  $q_{\alpha}$  don't depend on t,  $k_{\alpha}k^{\alpha} = \frac{q^2}{a^2}$ . The commoving distance of galaxy  $r_g$  is equal to

$$r_g = \int_0^{z_g} k_\alpha dx^\alpha = c \int_0^{z_g} \frac{dt}{dz} \frac{k_\alpha k^\alpha}{\sqrt{-(g_{\alpha\beta} + h_{\alpha\beta})k^\alpha k^\beta}} dz = cq^2 \int_0^{z_g} \frac{dt}{dz} \frac{1}{a} \frac{dz}{\sqrt{(\delta_{\alpha\beta} - \frac{h_{\alpha\beta}}{a^2})q^\alpha q^\beta}}$$
(37)

Let's consider that  $\frac{dt}{dz} = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}}$ , where the cosmological parameters for the flat  $\Lambda CDM$  model are  $\Omega_m = 0.28$ ,  $\Omega_\Lambda = 0.72$ . The luminosity distance of galaxy is  $d_L = (1 + z_g)r_g$ . It is used in the distance modulus equation of source,  $(m - M)_{PL} = 5 \log \frac{d_L}{Mpc} + 25$ , where  $(m - M)_{PL}$  is a distance module for Cepheids. The authors of [11] found values  $H_0^{HST}$  for dozens of Cepheids in each galaxy. The most probable value  $H_0^{HST}$  (included in Table 1) was determined from the condition of equality of the distance modules for Cepheids and a supernova in the same galaxy.

Using Equation (37), we found the expression for the Hubble constant taking into account the gravitational perturbation:

$$H_0 = cq(1+z_g)10^{5-\frac{(m-M)_{PL}}{5}} \int_0^{z_g} \frac{1+z}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \frac{dz}{\sqrt{\left(1 - \frac{h_{\alpha\beta}q^\alpha q^\beta}{a^2q^2}\right)}}.$$
(38)

The Hubble constant for the background frame  $H_0^{Pl}$  is obtained from Equation (38) with  $\frac{h_{\alpha\beta}q^{\alpha}q^{\beta}}{a^2a^2} = 0$ . Then the relative variation of the Hubble constant is:

$$\frac{H_0 - H_0^{Pl}}{H_0^{Pl}} \approx \frac{\frac{1}{2} \int_0^{z_g} \frac{1+z}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} \frac{h_{\alpha\beta} q^{\alpha} q^{\beta}}{a^2 q^2} dz}{\int_0^{z_g} \frac{1+z}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}} dz},$$
(39)

where  $\frac{h_{\alpha\beta}q^{\alpha}q^{\beta}}{a^2q^2} \ll 1$ . For a local perturbation of the metric we can assume that the main contribution to the integral of the term of a fraction (38) is a perturbation in the vicinity of the galaxy at an epoch  $z = z_g$ . Then, to estimate the variation of the value of the Hubble constant, we used two conditions  $\frac{h_{\alpha\beta}q^{\alpha}q^{\beta}}{a^2q^2} \approx \frac{q^{\alpha}q^{\beta}}{q^2} \left(\frac{h_{\alpha\beta}}{a^2}\right)_{z_1}$ , and  $h_{\alpha\beta}(z < z_g) \approx 0$ . In this case we obtained:

$$\frac{H_0 - H_0^{Pl}}{H_0^{Pl}} \approx \frac{1}{2} \frac{q^\alpha q^\beta}{q^2} \left(\frac{h_{\alpha\beta}}{a^2}\right)_{z_g}.$$
(40)

Let's consider the case, when the galaxies used in [13] belong to the outskirts of the local supercluster. The medium density contrast in the superclusters is of the order  $\frac{\delta \varepsilon}{\varepsilon} \approx 0.1$  (see, for example, [19]). This density contrast corresponds to the metric perturbation  $h_{\alpha\beta}$  and  $\frac{\delta\varepsilon}{\varepsilon} \approx \frac{q^{\alpha}q^{\beta}}{q^2} \left(\frac{h_{\alpha\beta}}{a^2}\right)_{z_g}$ . As a result,

the variation of the Hubble constant is  $\frac{H_0 - H_0^{Pl}}{H_0^{Pl}} \approx \frac{1}{2} \frac{\delta \varepsilon}{\varepsilon} \approx 0.05$ . This value is comparable to the average variation of the Hubble constant in measurements [11]. The above example shows that variations of the Hubble constant can be used to study local gravitational perturbations.

In the fractal cosmological model, the space-time is considered as a set of self-similar regions of space-time. This self-similarity is due, in particular, to the fact that the transition from one region to another is reduced to stretching or shrinking the interval. Metric tensors of two regions are related by the relation:  $\tilde{g}_{\alpha\beta} = \frac{U_0}{\tilde{U}_0} g_{\alpha\beta}$ , where  $\frac{U_0}{\tilde{U}_0}$  is a constant number, it can be different for different pairs of regions. Consider the metric  $\tilde{g}_{\alpha\beta}$  in the vicinity of the galaxy as weakly perturbed relative to the metric of a homogeneous and isotropic space-time  $g_{\alpha\beta}$  for the region in which the observer is located. Then  $\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$ , and  $h_{\alpha\beta} = \left(\frac{U_0}{\overline{U}_0} - 1\right)g_{\alpha\beta}$ . The direction of the isotropic geodetic upon transition from one region to another does not change. Using Equation (39), we found that in such a model, the observer will find a variation of the Hubble constant,  $\frac{H_0 - H_0^{Pl}}{H_0^{Pl}} \approx \frac{1}{2} \left( \frac{\frac{U_0}{\overline{U}_0} - 1}{a^2} \right)_{T_0}$ . The values of the Hubble

constant for a set of self-similar regions will form a fractal set,  $H_{0n} \approx \delta^i H_0^{pl}$ , where  $\delta = 1 + \frac{1}{2} \left( \frac{\frac{-\omega}{U_0} - 1}{a^2} \right)$ .

#### 4. Discussion

Above, it is proposed to consider the puzzle of fractals as a consequence of the properties of space-time and the symmetry of the interaction fields of matter. In this case, it is necessary to use the geometric theory of gravity, which is similar to GR. We have described the method of constructing the problems solution (1)–(3) and shown that it had simple and easily verified analytical solutions with geometrical properties. It is important that these solutions for complex scalar fields be generalized to multidimensional fields.

This class of geometric solutions of the general relativity equation allows analogy to spatial fractals in large-scale structures of the universe due to its invariance with respect to discrete scale transformation of interval,  $ds \leftrightarrow qd\bar{s}$ . As an application, the treatment of spatial variations of the Hubble constant  $H_0^{HST}$  [13] is considered. It is noted that the values  $H_0^{HST}$  form almost fractal sets. It has been shown that the variations of  $H_0^{HST}$  may be connected with local gravitational perturbations of the space-time metrics in the vicinity of the galaxies containing Cepheids and supernovae selected for measurement. The value of variational perturbations) on the outskirts of the local supercluster, and their self-similarity indicates the fractal distribution of matter in this region. This example shows that the variations of to study local gravitational perturbations.

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