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Optimal Control of Nonsmooth Production Systems with Deteriorating Items, Stock-Dependent Demand, with or without Backorders

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Received: 9 January 2019; Accepted: 28 January 2019; Published: 4 February 2019



Abstract: We propose a nonsmooth dynamic system integrating production and inventory where the items may deteriorate and the demand is stock-dependent. We aim to derive the optimal production rate. In our first model, backorders are not allowed, while in the second model they are. Using optimal control, necessary optimality conditions are obtained for general forms of the cost, demand, and deterioration rates and closed form solutions are derived for specific forms of these rates. Numerical simulations are presented and sensitivity of the solutions are examined.

Keywords: nonsmooth production inventory systems; item deterioration; backorder; optimal control; maximum principle

MSC: 90B05

1. Introduction

To take in consideration the nature of the dynamic behavior of inventory–production systems, many authors have successfully used control theory techniques. However, the literature on the subject is rather meager. The earliest reference is that of Simon, in [1]. In the absence of deterioration, some dynamic models have been studied using an optimal control approach. For example, [2] treated the optimal control of a manufacturing system subject to failure under the assumption of restarting costs. Riddalls and Bennett, in [3], used a similar technique to cater for batch production costs which, often, are not included in cumulative production problems. Zhang et al. [4] studied the scheduling problem of a marketing production system where the demand depended on the status of the market. Khmelnitsky and Gerchak, in [5], were interested in the solution, using optimal control theory of a production system with state-dependent demand. Kiesmüller [6] was concerned with the optimal control of recovery systems and took into account the remanufacturing of used products to reduce waste. Dobos, in [7], considered reverse logistic systems with customer returns. Finally, Hedjar et al. [8] investigated, using predictive control, the case of a periodic-review policy with deteriorating items.

In this work, we are interested in some complex, dynamic inventory–production system featuring many of the characteristics available in the literature. We list these characteristics below:

- System dependent parameters: Taking into account the effect of the system parameters on an inventory system can lead to an improvement of its performance. The dependence of the demand

rate on the stock is, without doubt, the dependence that received the most attention, and the literature on this topic is abundant. Among the most recent references we cite [9–13].

- Item deterioration: The deterioration of stocked items plays an important role in inventory management. The literature on this subject, again, is immense and an excellent survey, in which deteriorating inventory systems are thoroughly classified, has been done in [14] and recently in [13].
- Nonlinear costs: The traditional approach regarding cost parameters is to assume that they are linear. This assumption is somewhat unrealistic. Nonlinear holding costs have been introduced by Naddor [15]. Among recent references we suggest [16–19].
- Items backorders: In many real-life situations, demand is not met on time and shortages occur, the condition that exists when the inventory on hand is not sufficient to cover needs. Shortages are undesirable because they are quite expensive. However, in certain situations, management may find it desirable from a cost point of view not only to allow shortages but to plan for them. This specific shortage is called a backorder. After the exhaustion of inventory, we allow a period of time over which backorders accumulate to some level. When allowing backorders, we have, in addition to the usual costs, the additional cost of backordering. For more works on item backorders, we refer the reader to [18,20,21].

Note that in the case where backorders are allowed, the objective function is nonsmooth (nondifferentiable). As far as we know, the paper [5] is the only one with such a mathematical complexity. Our intent here is to extend the state equation in that paper by incorporating item deterioration. Further, [5] maximizes a linear profit function while we will be minimizing a nonlinear cost function. Therefore, we propose two models. In the first one, the demand has a general form of a function of time and of the inventory level. We analysed a deteriorating production inventory system with holding cost (resp. production cost) taken to be nonlinear functionals of the inventory level (resp. production rate). Also, the deterioration rate depends, in a general way, on time and inventory level. In the second model, we extend this first one to a more general model where backorders are allowed and the shortage is given in terms of on-hand stock. For the proposed models, we use optimal control techniques to establish an optimal policy. The policy minimizes the total cost of the inventory and production for the first model. For the second model, we incorporate the shortage cost into the previous total cost.

The mathematical complexity observed in [5] is due to the state variable which can be either positive or negative. When it is positive, a holding cost is incurred and when it is negative, a shortage cost is incurred, rendering the objective function nonsmooth (nondifferentiable). However, they use linear terms in their objective function. It is well known that when the objective function is linear, the optimal control is given by a bang function, see [22].

In Section 2, we describe the first inventory model and we attack the solution of the optimal control problem. In Section 3, we conduct the second model similarly. Various numerical examples along with sensitivity analysis on the system parameters are given. Section 4 summarizes the paper.

2. Model without Backorders

We propose a firm producing a single item. We assumed that the decision was made on a compact interval $[0, T]$. We considered a finite planning horizon because many firms are concerned with short and/or intermediate term market activities.

For a given unit time t in the interval $[0, T]$, we denoted by $I(t)$ the inventory level, $D(t, I(t))$ the demand rate, $\theta(t, I(t))$ the deterioration rate, and $h(I(t))$ the holding cost rate. We also let $K(P(t))$ stand for the cost rate corresponding to a production rate $P(t)$. Let $\rho \geq 0$ be a constant discount rate. All functions are assumed to be non-negative.

For a given $T > 0$, we considered the optimal control problem to minimize the total inventory–production costs:

$$(\mathcal{P}) \quad \begin{cases} \min_{P(t) \geq D(t, I(t)) + \theta(t, I(t))} J(P, I) = \int_0^T e^{-\rho t} \{h(I(t)) + c[P(t) - D(t, I(t))] + K(P(t))\} dt \\ \frac{d}{dt} I(t) = P(t) - D(t, I(t)) - \theta(t, I(t)), \quad I(0) = I_0, \quad I(T) = I_T, \end{cases}$$

where $c > 0$ is the unit cost. The model can be seen as an optimal control problem with one state variable ($I(t)$) and one control variable ($P(t)$). Observe that the demand, at rate D , decreases the inventory level and the production, at rate P , increases it. Therefore the inventory level $I(t)$ evolves according to the above state equation. Taking into account the constraint $P(t) \geq D(t, I(t)) + \theta(t, I(t))$, we clearly obtained, using the state equation, that I was nondecreasing and $I(t) \geq I_0$. Therefore, shortages are not allowed in this model.

Using the Pontryagin maximum principle [23], the necessary conditions for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ are that there should exist a constant β , a continuous and piecewise continuously differentiable function λ and a piecewise continuous function μ (called the adjoint) and Lagrange multiplier functions, respectively, such that

$$H(t, I^*(t), P^*(t), \lambda(t)) \geq H(t, I^*(t), P(t), \lambda(t)), \text{ for all } P(t) \geq D(t, I^*(t)), \quad (1)$$

$$-\frac{d}{dt} \lambda(t) = \frac{\partial}{\partial I} L(t, I(t), P(t), \lambda(t), \mu(t)), \quad (2)$$

$$I(0) = I_0, \quad I(T) = I_T, \quad \lambda(T) = \beta, \quad (3)$$

$$\frac{\partial}{\partial P} L(t, I(t), P(t), \lambda(t), \mu(t)) = 0, \quad (4)$$

$$P(t) - D(t, I(t)) - \theta(t, I(t)) \geq 0, \quad \mu(t) \geq 0, \quad \mu(t) [P(t) - D(t, I(t)) - \theta(t, I(t))] = 0, \quad (5)$$

where

$$H(t, I(t), P(t), \lambda(t)) = -e^{-\rho t} \{h(I(t)) + c[P(t) - D(t, I(t))] + K(P(t))\} + \lambda(t) \{P(t) - D(t, I(t)) - \theta(t, I(t))\}, \quad (6)$$

is the Hamiltonian function and

$$L(t, I(t), P(t), \lambda(t), \mu(t)) = H(t, I(t), P(t), \lambda(t)) + \mu(t) \{P(t) - D(t, I(t)) - \theta(t, I(t))\}, \quad (7)$$

is the Lagrangian function. Rewriting the Equation (2) we obtained

$$\frac{d}{dt} \lambda(t) = e^{-\rho t} \left[\frac{d}{dI} h(I(t)) - c \frac{d}{dI} D(t, I(t)) \right] + [\lambda(t) + \mu(t)] \left[\frac{\partial}{\partial I} D(t, I(t)) + \frac{\partial}{\partial I} \theta(t, I(t)) \right]. \quad (8)$$

Equation (4) is equivalent to

$$\lambda(t) + \mu(t) = e^{-\rho t} \left[\frac{d}{dP} K(P(t)) + c \right]. \quad (9)$$

Now, consider Equation (5). Then for any t , we distinguished two cases, either we had $P(t) - D(t, I(t)) - \theta(t, I(t)) = 0$ or $P(t) - D(t, I(t)) - \theta(t, I(t)) > 0$.

Case 1: $P(t) - D(t, I(t)) - \theta(t, I(t)) = 0$ on some subset S of $[0, T]$. This means that the firm has to produce the exact total amount corresponding to the amount consumed plus the amount lost due to deterioration. In this case $\frac{d}{dt}I(t) = 0$ on S and I^* is obviously constant on S and

$$P^*(t) = D(t, I^*(t)) + \theta(t, I^*(t)), \quad \text{for all } t \in S. \quad (10)$$

Substituting the Equation (9) into Equation (8), we obtained

$$\frac{d}{dt}\lambda(t) = e^{-\rho t} \left[\frac{d}{dI}h(I(t)) + c \frac{\partial}{\partial I}\theta(t, I(t)) + \frac{d}{dP}K(P(t)) \left(\frac{\partial}{\partial I}D(t, I(t)) + \frac{\partial}{\partial I}\theta(t, I(t)) \right) \right].$$

To get an explicit form of λ and β , we integrated the previous differential equation. Then, we used Equation (9) to derive an explicit form of the Lagrange multiplier function μ . We pointed out that if the obtained function μ was not nonnegative, then we did not accept the solutions stated in Equation (10).

Case 2: $P(t) - D(t, I(t)) - \theta(t, I(t)) > 0$ for $t \in [0, T] \setminus S$. The firm should produce more than the total amount corresponding to the amount consumed plus the amount lost due to deterioration, in order to avoid a shortage situation. In this case, $\mu(t) = 0$ on $[0, T] \setminus S$, and so the necessary conditions in Equations (3), (8) and (9) become

$$\frac{d}{dt}\lambda(t) = e^{-\rho t} \left[\frac{d}{dI}h(I(t)) - c \frac{d}{dI}D(t, I(t)) \right] + \lambda(t) \left[\frac{\partial}{\partial I}D(t, I(t)) + \frac{\partial}{\partial I}\theta(t, I(t)) \right],$$

and

$$I(0) = I_0, \quad I(T) = I_T, \quad \lambda(T) = \beta, \quad \lambda(t) = e^{-\rho t} \left[\frac{d}{dP}K(P(t)) + c \right].$$

Combining the state equation with these equations yields the following second order differential equation:

$$\frac{d}{dt}P(t) \frac{d^2}{dP^2}K(P(t)) - \left[\rho + \frac{\partial}{\partial I}D(t, I(t)) + \frac{\partial}{\partial I}\theta(t, I(t)) \right] \left[\frac{d}{dP}K(P(t)) + c \right] = \frac{d}{dI}h(I(t)) - c \frac{\partial}{\partial I}D(t, I(t)), \quad (11)$$

and

$$I(0) = I_0, \quad I(T) = I_T, \quad c + \frac{d}{dP}K(P(T)) = \beta e^{\rho T}.$$

These equations are enough to determine the optimal solution of problem $(\mathcal{P})$. To be able to push the derivations any further, one needs to have an explicit form for the functions involved. For illustration purposes, let us assume the following forms for the cost rates

$$K(P) = \frac{KP^2}{2}, \quad h(I) = \frac{hI^2}{2},$$

and for the exogenous functions

$$D(t, I(t)) = d_1(t) + d_2I(t), \quad \theta(t, I(t)) = \theta_1(t) + \theta_2I(t).$$

Here K, h, d_2 , and θ_2 are positive constants. For these functions the necessary conditions for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ become

$$\frac{d^2}{dt^2}I(t) - \rho \frac{d}{dt}I(t) - \left[\frac{h}{K} + (d_2 + \theta_2)(\rho + d_2 + \theta_2) \right] I(t) = \alpha(t), \quad (12)$$

with

$$\alpha(t) = (\rho + d_2 + \theta_2)(d_1(t) + \theta_1(t)) - \frac{d}{dt}d_1(t) - \frac{d}{dt}\theta_1(t) + \frac{c\theta_2}{K},$$

and

$$I(0) = I_0, \quad I(T) = I_T. \quad (13)$$

This is a two-point boundary value problem ($\mathcal{P}_{TPBV}$) that we solved in the next proposition.

Lemma 1. The solution I^* of ($\mathcal{P}_{TPBV}$) is given by

$$I^*(t) = a_1 e^{m_1 t} + a_2 e^{m_2 t} + Q(t), \quad (14)$$

and its corresponding P^* is given by

$$P^*(t) = a_1(m_1 + d_2 + \theta_2)e^{m_1 t} + a_2(m_2 + d_2 + \theta_2)e^{m_2 t} + \frac{d}{dt}Q(t) + (d_2 + \theta_2)Q(t) + d_1(t) + \theta_1(t), \quad (15)$$

where the constants a_1 , a_2 , m_1 , and m_2 are unique and given in the proof below, and $Q(t)$ is a particular solution of Equation (12).

Proof. We used the standard method to solve Equation (12). The characteristic equation is

$$m^2 - \rho m - \left[\frac{h}{K} + (\rho + d_2 + \theta_2)(d_2 + \theta_2) \right] = 0.$$

It has two real roots of opposite signs, given by

$$\begin{aligned} m_1 &= \frac{1}{2} \left(\rho - \sqrt{\rho^2 + 4 \left[\frac{h}{K} + (\rho + d_2 + \theta_2)(d_2 + \theta_2) \right]} \right) < 0, \\ m_2 &= \frac{1}{2} \left(\rho + \sqrt{\rho^2 + 4 \left[\frac{h}{K} + (\rho + d_2 + \theta_2)(d_2 + \theta_2) \right]} \right) > 0, \end{aligned}$$

and therefore $I^*(t)$ is given by Equation (14). The initial and terminal conditions in Equation (13) are used to determine the constants a_1 and a_2 as follows. From the initial condition we had

$$a_1 + a_2 + Q(0) = I_0,$$

and from the terminal condition we had

$$a_1 e^{m_1 T} + a_2 e^{m_2 T} + Q(T) = I_T.$$

By putting

$$\begin{aligned} b_1 &= I_0 - Q(0), \\ b_2 &= I_T - Q(T), \end{aligned}$$

we obtained a system of two linear equations in two unknowns which had the following unique solution

$$\begin{aligned} a_1 &= \frac{b_2 - e^{m_2 T} b_1}{e^{m_1 T} - e^{m_2 T}}, \\ a_2 &= \frac{b_1 e^{m_1 T} - b_2}{e^{m_1 T} - e^{m_2 T}}. \end{aligned}$$

The expression of P^* is deduced using the optimal expression of I^* along with the state equation. $\square$

From the above analysis we had the following theorem characterizing the optimal solution of $(\mathcal{P})$.

Theorem 1. The optimal solution (P^*, I^*) of $(\mathcal{P})$ has the form given in Equation (10) on S , and the form in Equations (14) and (15) on $[0, T] \setminus S$.

Example 1.

1. We illustrated the results obtained by considering a production system with the following characteristics: planning horizon of length $T = 5$, initial and terminal inventory levels, $I_0 = 0$, $I(T) = 10$, unit costs and discount factor $c = h = 0.1$, $K = 5$ and $\rho = 0$, respectively. The demand rate is such that $d_1(t) = \sin(t) + 1$, $d_2 = 0.1$ and the deterioration rate is such that $\theta_1(t) = e^{-t}$, $\theta_2 = 0.1$. The optimal control and state are displayed in Figure 1. The optimal objective function value is $J = 216.67$.
2. To assess the effect of the deterioration rate on the value of the optimal objective function, we set $\theta_1 \equiv 0$ and varied the value of θ_2 from 0.0005 to 0.2560, and we kept all the other parameters as in Example (1). As shown by Table 1, the resulting optimal cost increases as θ_2 increases.
3. Next, we studied the effect of the discount factor on the value of the optimal objective function, we set $\theta_1 \equiv 0$ and varied the value of ρ from 0 to 0.1, and we keep all the other parameters as in Example (1). As shown by Table 2, the resulting optimal cost increases as ρ increases.

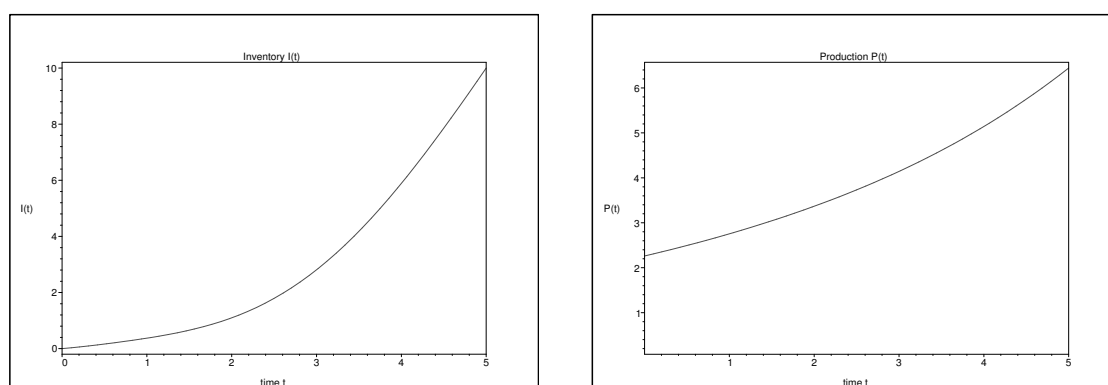


Figure 1. Variations of I^* and P^* as functions of time t .

Table 1. Sensitivity of J with respect to θ_2 .

θ_2	0.0005	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.256
J	490.49	491.19	492.61	495.45	501.15	512.64	536.02	584.22	685.60	902.52

Table 2. Sensitivity of J with respect to ρ .

ρ	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
J	216.67	664.06	664.36	664.86	665.55	666.42	667.48	668.71	670.11	671.68	673.41

3. Model with Backorders Allowed

We mentioned in the Introduction that a firm may be better off if it plans for backorders. Therefore, in this section, we extended the previous model by assuming that the firm allows for backorders. In this case, a holding cost is incurred when the inventory is positive and a shortage cost is incurred when the inventory is negative. Deterioration, and hence a deterioration cost, happens only when the inventory is positive. The optimal control problem is restated as follows:

$$(\mathcal{P}_b) \quad \begin{cases} \min_{P \geq 0} J(P, I) = \int_0^T e^{-\rho t} \{h^+(I(t)) + g^-(I(t)) + c^+(I) [P(t) - D(t, I(t))] + K(P(t))\} dt \\ \frac{d}{dt} I(t) = P(t) - D(t, I(t)) - \theta^+(t, I(t)), \quad I(0) = I_0, \quad I(T) = I_T, \end{cases}$$

where the holding, shortage, and deterioration costs are, respectively:

$$h^+(I) = \begin{cases} h(I), & I > 0, \\ 0, & I \leq 0, \end{cases} \quad g^-(I) = \begin{cases} 0, & I \geq 0, \\ -g(I), & I < 0, \end{cases} \quad c^+(I) = \begin{cases} c, & I > 0, \\ 0, & I \leq 0, \end{cases}$$

and the deterioration rate is

$$\theta^+(t, I(t)) = \begin{cases} \theta(t, I(t)), & I > 0, \\ 0, & I \leq 0, \end{cases}$$

with $g(0) = h(0) = 0$ and $\theta(t, 0) = 0$, $D(t, 0) = P(t)$, $\forall t$. The constraint $P(t) \geq D(t, I(t)) + \theta(t, I(t))$ that we had in the previous model is no longer necessary since we allowed negative values of the inventory level $I(t)$. Since in case of backorders the objective function is nonsmooth (nondifferentiable), an extension of Pontryagin maximum principle (see for example [24]) is used to derive the following necessary conditions for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P}_b)$; there exists a constant β and a continuous and piecewise continuously differentiable function λ such that

$$\frac{\partial}{\partial P} H(t, I(t), P(t), \lambda(t)) = 0, \quad (16)$$

$$-\frac{d}{dt} \lambda(t) = \frac{\partial}{\partial I} H(t, I(t), P(t), \lambda(t)), \quad (17)$$

$$I(0) = I_0, \quad I(T) = I_T, \quad \lambda(T) = \beta, \quad (18)$$

where

$$H(t, I(t), P(t), \lambda(t)) = -e^{-\rho t} \{h^+(I(t)) + g^-(I(t)) + c^+(I) [P(t) - D(t, I(t))] + K(P(t))\} + \lambda(t) \{P(t) - D(t, I(t)) - \theta^+(t, I(t))\}. \quad (19)$$

Equation (16) is equivalent to

$$\lambda(t) = \begin{cases} e^{-\rho t} \left[\frac{d}{dP} K(P(t)) + c \right], & I(t) > 0, \\ e^{-\rho t} \frac{d}{dP} K(P(t)), & I(t) \leq 0. \end{cases} \quad (20)$$

Equation (17) is equivalent to

$$-\frac{d}{dt} \lambda(t) \in \begin{cases} \{a(t, I(t))\}, & I(t) > 0, \\ [b(t, I(t)), a(t, I(t))], & I(t) = 0, \\ \{b(t, I(t))\}, & I(t) < 0, \end{cases} \quad (21)$$

where

$$a(t, I(t)) = -\lambda(t) \left[\frac{\partial}{\partial I} D(t, I(t)) + \frac{\partial}{\partial I} \theta(t, I(t)) \right] + e^{-\rho t} \left[c \frac{\partial}{\partial I} D(t, I(t)) - \frac{d}{dI} h(I(t)) \right],$$

and

$$b(t, I(t)) = e^{-\rho t} \frac{d}{dI} g(I(t)) - \lambda(t) \frac{\partial}{\partial I} D(t, I(t)).$$

We distinguished the following three cases:

Regime 1 : $I(t) > 0$.

Regime 2 : $I(t) = 0$.

Regime 3 : $I(t) < 0$.

As we did in the previous section, we show how computations can be carried out when specific functions are available. Take, for example, the following cost rates

$$K(P) = \frac{KP^2}{2}, \quad h(I) = \frac{hI^2}{2}, \quad g(I) = gI,$$

and the following demand and deterioration functions

$$D(t, I(t)) = d_1(t) + d_2 I(t), \quad \theta(t, I(t)) = \theta_1(t) + \theta_2 I(t),$$

where K , h , g , d_2 , and θ_2 are positive constants and $\theta_1 \equiv 0$ on those intervals where $I \equiv 0$. Combining Equations (20) and (21) we got the following conditions which were necessary for the regimes to happen.

When $I(t) > 0$, we had from the state equation $P(t) = \frac{d}{dt} I(t) + D(t, I(t)) + \theta(t, I(t))$ and therefore regime 1 happened only over the time intervals where

$$\frac{d}{dt} I(t) + D(t, I(t)) + \theta(t, I(t)) \geq 0 \quad \text{and} \quad I(t) > 0.$$

The optimal control is $P(t) = \frac{d}{dt} I(t) + D(t, I(t)) + \theta(t, I(t))$, where $I(t)$ is the solution of the following differential equation:

$$\frac{d^2}{dt^2} I(t) - \rho \frac{d}{dt} I(t) - \left[\frac{h}{K} + (d_2 + \theta_2)(\rho + d_2 + \theta_2) \right] I(t) = \alpha(t), \quad (22)$$

with

$$\alpha(t) = (\rho + d_2 + \theta_2)(d_1(t) + \theta_1(t)) - \frac{d}{dt} d_1(t) - \frac{d}{dt} \theta_1(t) + \frac{c\theta_2}{K}.$$

Regime 2 occurs only over those time intervals where

$$0 \in [a(t), b(t)], \quad (23)$$

where

$$a(t) = -\frac{d}{dt} d_1(t) + \rho d_1(t) - d_2 d_1(t) + \frac{g}{K},$$

and

$$b(t) = -\frac{d}{dt} d_1(t) + \rho d_1(t) - (\theta_2 + d_2) \left[(\theta_1(t) + d_1(t)) + \frac{c}{K} \right].$$

The optimal control is $P(t) = d_1(t)$.

Regime 3 happens only over the time intervals where

$$\frac{d}{dt} I(t) + D(t, I(t)) \geq 0 \quad \text{and} \quad I(t) < 0.$$

The optimal control is $P(t) = \frac{d}{dt}I(t) + D(t, I(t))$, where $I(t)$ is the solution of the following differential equation:

$$\frac{d^2}{dt^2}I(t) - \rho \frac{d}{dt}I(t) - [d_2(\rho + d_2)] I(t) = d_1(t)(\rho + d_2) - \frac{d}{dt}d_1(t) + \frac{g}{K}. \quad (24)$$

Solution Approach:

Our approach to determine I^* depended on the initial and the terminal inventory levels and on the time intervals where regime 2 arose, if it did.

Scenario 1. Regime 2 arises on intervals: For simplicity we assumed that there was only one subinterval $[t_0, t_1] \subset [0, T]$ on which $0 \in [a(t), b(t)]$. In this scenario, there are four cases; $I_0 < 0 < I_T$, $I_T < 0 < I_0$, $I_0, I_T > 0$, and $I_0, I_T < 0$.

Case 1. $I_0 < 0 < I_T$: In this case we proceed as follows:

- * Solve Equation (24) with the boundary conditions $I_0 < 0$ and $I(t_0) = 0$.
- * Solve Equation (22) with the boundary conditions $I(t_1) = 0$ and $I_T > 0$.
- * The optimal level I^* is the function given by the solution of Equation (24) over the interval $[0, t_0]$ and by the solution of Equation (22) on the interval $[t_1, T]$, and $I^* \equiv 0$ on $[t_0, t_1]$.

Case 2. $I_T < 0 < I_0$. This case is unlikely to happen in practice, but if it does, then we proceed as follows:

- * Solve Equation (22) with the boundary conditions $I_0 > 0$ and $I(t_0) = 0$.
- * Solve Equation (24) with the boundary conditions $I(t_1) = 0$ and $I_T < 0$.
- * The optimal level I^* is the function given by the solution of Equation (22) over the interval $[0, t_0]$ and by the solution of Equation (24) on the interval $[t_1, T]$, and $I^* \equiv 0$ on $[t_0, t_1]$.

Case 3. $I_0, I_T < 0$. In this case we solved Equation (24) twice, once with the boundary conditions $I_0 < 0$ and $I(t_0) = 0$ and once with the boundary conditions $I(t_1) = 0$ and $I_T < 0$. The optimal level I^* is the function given by the solution of Equation (24) over the interval $[0, t_0]$ and by the solution of Equation (24) on the interval $[t_1, T]$, and $I^* \equiv 0$ on $[t_0, t_1]$.

Case 4. $I_0, I_T > 0$. In this case we proceed as in case 3, using Equation (22) instead of Equation (24).

Scenario 2. Regime 2 does not arise on intervals: As in the previous scenario, we again had to consider the four cases; $I_0 < 0 < I_T$, $I_T < 0 < I_0$, $I_0, I_T > 0$, and $I_0, I_T < 0$.

Case 1. $I_0 < 0 < I_T$: For simplicity we assumed that there is only one point $t_0 \in [0, T]$ with $I(t_0) = 0$. In this case we proceed as follows:

Step 1.

- Solve Equation (24) with the boundary conditions I_0 and $I(T)$, and determine the value t_0 for which $I(t_0) = 0$.
- Solve Equation (22) with the boundary conditions $I(t_0) = 0$ and I_T .
- Denote by $\hat{I}$ the function given by the solution of Equation (24) over the interval $[0, t_0]$ and by the solution of Equation (22) on the interval $[t_0, T]$ and compute $\hat{J} := J(\hat{I})$.

Step 2.

- Solve Equation (22) with the boundary conditions I_0 and $I(T)$, and determine the value t_0 for which $I(t_0) = 0$.
- Solve Equation (24) with the boundary conditions $I(0) = I_0$ and $I(t_0) = 0$.

- Denote by $\tilde{I}$ the function given by the solution of Equation (22) over the interval $[0, t_0]$ and by the solution of Equation (24) on the interval $[t_0, T]$ and compute $\tilde{J} := J(\tilde{I})$.

Step 3. The optimal level I^* is the one with the smallest objective function value.

Case 2. $I_T < 0 < I_0$. This case is unlikely to happen in practice, but if it does, then we proceed as in case 1.

Case 3. $I_0, I_T < 0$. In this case we solved the differential Equation (24) with the boundary conditions $I_0 < 0$ and $I_T < 0$.

Case 4. $I_0, I_T > 0$. In this case we solved the differential Equation (22) with the boundary conditions $I_0 > 0$ and $I_T > 0$.

Remark 1. For simplicity we had assumed that there was only one subinterval $[t_0, t_1] \subset [0, T]$ on which $0 \in [a(t), b(t)]$. This assumption is solely for simplicity of exposition, i.e., the extension to multiple subintervals for which $0 \in [a(t), b(t)]$ is straightforward, except for notation.

Example 2. Take $\rho = 0, g = c = h = 0.1, K = 5, \theta_2 = 0.1, d_2 = 0.1, T = 5, I_0 = -10, I_T = 5, d_1(t) = e^t$, and $\theta_1 \equiv 0$. We can check that $0 \notin [a(t), b(t)]$, for all $t \in [0, T]$, so that regime 2 does not exist. This is case 1 of scenario 2. Using Maple, we found $\hat{J} = 20,632.26$ and $\tilde{J} = 20,626.63$ and so the optimal solution is the one obtained in step 2. The graphs of the optimal inventory level I^* and the optimal production rate P^* are shown in Figure 2.

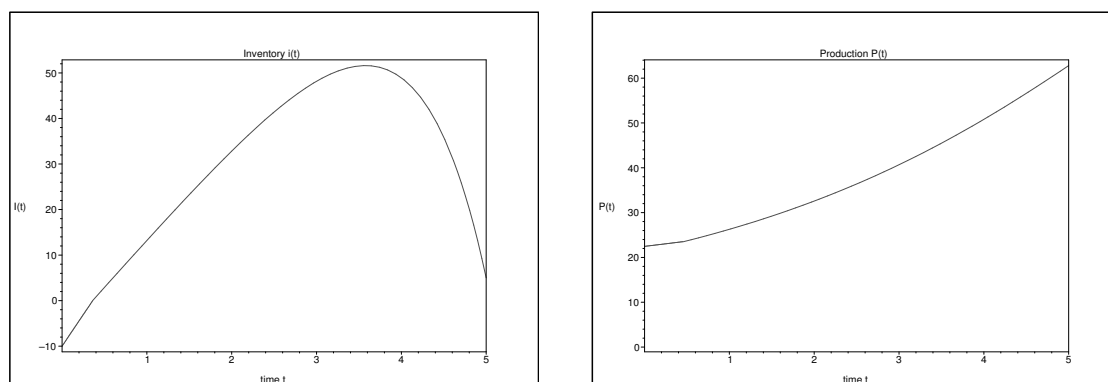


Figure 2. Variations of I^* and P^* as functions of time t .

4. Conclusions

We have considered in this paper two inventory–production systems with deteriorating items and stock-dependent demand. The first model does not allow backorders while the second does. The optimal control approach is very effectively used to determine the optimal production rate in both models. Using Pontryagin maximum principle, we derived in a general framework the necessary optimality conditions for optimal production rate P^* and optimal inventory level I^* . Explicit expressions of P^* and I^* are obtained under assumption of explicit forms of the functionals involved. Different scenarios and regimes describe the solution approach in the second model. Numerical examples illustrate the efficiency of the proposed solutions.

This work can be extended in various ways. For example, instead of minimizing the total cost, one may want to maximize the total profit. Usually these two problems are equivalent and to avoid this, one may take a unit revenue rate, that is both function of time and of the inventory level, in the maximization model. Another extension would be to consider an infinite planning horizon. Also, one may consider the case of multi-item production systems and/or systems with stochastic inventory levels or stochastic demand rates.

Author Contributions: All authors contributed equally to this article.

Funding: The authors extend their appreciations to the Deanship of Scientific Research at King Saud University for funding the work through the research group Project no. RGP-024.

Acknowledgments: The authors would like to thank the referees for their valuable remarks and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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