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Group Decision Making Based on Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operators

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Abstract: In this paper, a new concept of the triangular neutrosophic cubic fuzzy numbers (TNCFNs), their score and accuracy functions are introduced. Based on TNCFNs, some new Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy Einstein weighted averaging (TNCFEWA), triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging (TNCFEOWA) and triangular neutrosophic cubic fuzzy Einstein hybrid weighted averaging (TNCFEHWA) operators are developed. Furthermore, their application to multiple-attribute decision-making with triangular neutrosophic cubic fuzzy (TNCF) information is discussed. Finally, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: triangular neutrosophic cubic fuzzy number; Einstein t-norm; arithmetic averaging operator; Multi-attribute decision making; numerical application

1. Introduction

Atanassov [1] introduced the IFS, which is a generalization of FS. Atanassov [2] introduced operations and relations over IFSs taking as a point of departure respective definitions of relations and operations over fuzzy sets. Bustince et al. [3] introduced the characterization of certain structures of intuitionistic relations according to the structures of two concrete fuzzy relations. Deschrijver et al. [4] established the relationships between intuitionistic fuzzy sets (Atanassov, VII ITKR's Session, Sofia, June 1983 (Deposited in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian)), L-fuzzy sets. Deschrijver et al. [5] defined the mathematical relationship between intuitionistic fuzzy sets and other models of imprecision. Jun et al. [6] introduced the cubic set. Mohiuddin et al. [7] showed that the union of two internal cubic soft sets might not be internal. Turksen [8] showed that the proposed representation (1) exists for certain families of the conjugate pairs of t-norms and t-norms, and (2) resolves some of the difficulties associated with particular interpretations of conjunction, disjunction, and implication in fuzzy set theories.

Xu [9] developed some aggregation operators, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid aggregation operator, to aggregate intuitionistic fuzzy values. Xu et al. [10] developed some new geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator and the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator. Xu et al. [11] provided a survey of the aggregation techniques of intuitionistic fuzzy information and their applications in various fields, such as decision making, cluster analysis, medical diagnosis, forecasting, and

manufacturing grid. Liu et al. [12] introduced and discussed the concept of intuitionistic fuzzy point operators. Zeng et al. [13] defined the situation with intuitionistic fuzzy information and developed an intuitionistic fuzzy ordered weighted distance (IFOWD) operator. The fuzzy set was introduced by Zadeh [14]. Zadeh [15] introduced the interval-valued fuzzy set Li et al. [16] proposed group decision-making methods of the interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm and Choquet integral. Zhao et al. [17] developed some hesitant triangular fuzzy aggregation operators based on the Einstein operation: the hesitant triangular fuzzy Einstein weighted averaging (HTFEWA) operator. Xu et al. [18] introduced two new aggregation operators: dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator.

The Neutrosophic Set (NS) was projected by Smarandache [19,20]. Neutrosophic sets are characterized by fact participation, an indeterminacy-enrollment work and misrepresentation participation, which are inside the ordinary or nonstandard unit interim $]^{-0}, 1^{+}[$ in order to apply NS to genuine applications. In order to apply NS to real-world applications, Aliya et al. [21] introduced the concept of the triangular cubic fuzzy number. Aliya et al. [22] introduced the triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator.

Beg et al. [23] introduced a computational means to manage situations in which experts assess alternatives in possible membership and non-membership values. Przemyslaw et al. [24] introduced a simple test that sometimes might be helpful in detecting non-separability at a glance.

The differences between Reference 21, 22 and the current paper are as Table 1:

Table 1. Difference between references 21, 22 and current paper.

Reference 21	Reference 22	Current Paper
Defines a new extension of the triangular cubic fuzzy number by using a cubic set.	Defines a new extension of the triangular cubic hesitant fuzzy number by using a cubic set.	Defines a new extension of the triangular neutrosophic cubic fuzzy number by using a neutrosophic set.
Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions.	Introduced the triangular cubic hesitant fuzzy number, operational laws, and their score, accuracy functions, membership uncertainty index and hesitation index.	Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions, membership uncertainty index and hesitation index.
Introduced the triangular cubic fuzzy hybrid aggregation operator.	Introduced three Einstein aggregation operators, such as the triangular cubic fuzzy hybrid aggregation operator, and the TCHFEWA, TCHFOWA and TCHFEHWA operators	Introduced three Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy hybrid aggregation operator, and the TNCFEWA, TNCFEOWA and TNCFEHWA operators

Based on the above analysis, in this paper we develop TNCFNs, which is the generalization of the triangular neutrosophic intuitionistic fuzzy number and triangular neutrosophic interval fuzzy number. We perform some operations based on Einstein T-norm and Einstein T-conorm for TNCFNs. We also develop score and accuracy functions to compare two TNCFNs. Due to the developed operation, we propose the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, to aggregate a collection of TNCFNs.

This paper is organized as follows. In Section 2, we define some concepts of FS, CS, and TNCFNs. In Section 3, we discuss some Einstein operations on TNCFNs and their properties. In Section 4, we first develop some novel arithmetic averaging operators, such as the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, for aggregating a group of TNCFNs. In Section 5, we apply the

TNCFEHW operator to MADM with TNCFNs material. In Section 6, we offer a numerical example consistent with our approach. In Section 7, we discuss comparison analysis. In Section 8, we present a conclusion.

2. Preliminaries

Definition 1. [15]. Let H be a fixed set, a FS F in H is defined as: $F = \{(h, \Gamma_F(h)) | h \in H\}$ where $\Gamma_F(h)$ is a mapping from h to the closed interval $[0, 1]$ and for each $h \in H$, $\Gamma_F(h)$ is called the degree of membership of h in H .

Definition 2. Let H is a fixed set and an interval-valued fuzzy set I in H is defined as $I = \{h, R_I^-(h), R_I^+(h) | h \in H\}$, where $R_I^- : H \rightarrow [0, 1]$ and $R_I^+ : H \rightarrow [0, 1]$. The $R_I^-(h)$ is lower membership and $R_I^+(h)$ is upper membership such that $0 \leq R_I^-(h) \leq R_I^+(h) \leq 1$.

Definition 3. [1]. An IFS $\mathcal{D}$ in H is given by $\mathcal{D} = \{(h, R_{\mathcal{D}}(h), \Omega_{\mathcal{D}}(h)) | h \in H\}$, where $R_{\mathcal{D}} : H \rightarrow [0, 1]$ and $\Omega_{\mathcal{D}} : H \rightarrow [0, 1]$, with the condition $0 \leq R_{\mathcal{D}}(h) + \Omega_{\mathcal{D}}(h) \leq 1$.

The numbers $R_{\mathcal{D}}(h)$ and $\Omega_{\mathcal{D}}(h)$ represent, respectively, the membership degree and non-membership degree of the element h to the set $\mathcal{D}$.

Triangular Neutrosophic Cubic Fuzzy Number

Definition 4. Let $A_1 = \left\{ \begin{array}{c} [p_1(h), q_1(h), \\ r_1(h)], \\ \langle [Y_1^-(h), \\ R_1^-(h), \delta_1^-(h)], \\ [Y_1^+(h), \\ R_1^+(h), \delta_1^+(h)], \\ [Y_1(h), \\ R_1(h), \delta_1(h)] \rangle \\ | h \in H \end{array} \right\}$ and $A_2 = \left\{ \begin{array}{c} [p_2(h), q_2(h), \\ r_2(h)], \\ \langle [Y_2^-(h), \\ R_2^-(h), \delta_2^-(h)], \\ [Y_2^+(h), \\ R_2^+(h), \delta_2^+(h)], \\ [Y_2(h), \\ R_2(h), \delta_2(h)] \rangle \\ | h \in H \end{array} \right\}$ are two TNCFNs, some

operations on TNCFNs are defined as follows:

(a) $A_1 \subseteq A_2$ iff $\forall h \in H, p_1(h) \geq p_2(h), q_1(h) \geq q_2(h), r_1(h) \geq r_2(h), Y_1^-(h) \geq Y_2^-(h), R_1^-(h) \geq R_2^-(h), \delta_1^-(h) \geq \delta_2^-(h), Y_1^+(h) \geq Y_2^+(h), R_1^+(h) \geq R_2^+(h), \delta_1^+(h) \geq \delta_2^+(h)$ and $Y_1(h) \geq Y_2(h), \delta_1(h) \geq \delta_2(h)$.

(b) $A_1 \cap_{T,S} A_2 = T[p_1(h), p_2(h)], T[q_1(h), q_2(h)], T[r_1(h), r_2(h)], \langle T[Y_1^-(h), Y_2^-(h)], T[R_1^-(h), R_2^-(h)], T[\delta_1^-(h), \delta_2^-(h)], T[Y_1^+(h), Y_2^+(h)], T[R_1^+(h), R_2^+(h)], T[\delta_1^+(h), \delta_2^+(h)], S[Y_1(h), Y_2(h)], S[R_1(h), R_2(h)], S[\delta_1(h), \delta_2(h)] \rangle$

Example 1. Let $\ddot{A}_1 = \left\{ \begin{array}{c} \langle [0.1, 0.2, 0.3], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \rangle \end{array} \right\}$ and $\ddot{A}_2 = \left\{ \begin{array}{c} \langle [0.103, 0.104, 0.105], \\ [0.100, 0.102, 0.104], \\ [0.102, 0.104, 0.106], \\ [0.101, 0.103, 0.105] \rangle \end{array} \right\}$ be two TNCFNs

(a) $\ddot{A}_1 \subseteq \ddot{A}_2$, if $\forall \ddot{z} \in Z, 0.1 \geq 0.103, 0.2 \geq 0.104, 0.3 \geq 0.105, 0.2 \geq 0.100, 0.4 \geq 0.102, 0.6 \geq 0.104, 0.4 \geq 0.102, 0.6 \geq 0.104, 0.8 \geq 0.106$ and $0.3 \leq 0.101, 0.5 \leq 0.103, 0.7 \leq 0.105$.

(b) $\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = T[0.1, 0.103], T[0.2, 0.104], T[0.3, 0.105], T[0.2, 0.10], T[0.4, 0.12], T[0.6, 0.14], T[0.4, 0.12], T[0.6, 0.14], T[0.8, 0.16]$ and $S[0.3, 0.11], S[0.7, 0.15]$.

Definition 5. Let $C = \left\{ \begin{array}{l} [p_C(h), q_C(h), r_C(h)] \\ \langle [A_C^-(h), R_C^-(h), \tilde{U}_C^-(h)], \\ [A_C^+(h), R_C^+(h), \tilde{U}_C^+(h)], \\ [A_C(h), R_C(h), \tilde{U}_C(h)] \rangle \end{array} \middle| h \in H \right\}$ be a TNCFN and then the score function

$S(C)$, accuracy function $N(C)$, membership uncertainty index $T(C)$ and hesitation uncertainty index $G(C)$ of a TNCFN C are defined by

$$\begin{aligned} S(C) &= \frac{\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + R_C^-(h) + \tilde{U}_C^-(h)] \\ &\quad + [A_C^+(h) + R_C^+(h) + \tilde{U}_C^+(h)] \\ &\quad - [A_C(h) + R_C(h) + \tilde{U}_C(h)] \rangle}{27}, \\ N(C) &= \frac{\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + R_C^-(h) + \tilde{U}_C^-(h)] \\ &\quad + [A_C^+(h) + R_C^+(h) + \tilde{U}_C^+(h)] + [A_C(h) + R_C(h) + \tilde{U}_C(h)] \rangle}{27}, \\ T(C) &= \langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + R_C^+(h) + \tilde{U}_C^+(h)] + \\ &\quad [A_C(h) + R_C(h) + \tilde{U}_C(h)] - [A_C^-(h) + R_C^-(h) + \tilde{U}_C^-(h)] \rangle, \\ G(C) &= \langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + R_C^+(h) + \tilde{U}_C^+(h)] \\ &\quad + [A_C^-(h) + R_C^-(h) + \tilde{U}_C^-(h)] - [A_C(h) + R_C(h) + \tilde{U}_C(h)] \rangle. \end{aligned}$$

Example 2. Let $C = \left\{ \begin{array}{l} \langle [0.101, 0.102, \\ 0.103], [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], [0.6, \\ 0.8, 0.10] \rangle \end{array} \right\}$ be a TNCFN. Then the score function $S(C)$, accuracy

function $H(C)$, membership uncertainty index $T(C)$ and hesitation uncertainty index $G(C)$ of a TNCFN C are defined by

$$\begin{aligned} S(C) &= \frac{\langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9] \\ &\quad + [0.7 + 0.9 + 0.11] - [0.6 + 0.8 + 0.10] \rangle}{27} = \frac{0.306(3.81 - 1.5)}{27} = \frac{0.7068}{27} = 0.0261, \\ H(C) &= \frac{\langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9] \\ &\quad + [0.7 + 0.9 + 0.11] + [0.6 + 0.8 + 0.10] \rangle}{27} = \frac{0.306(3.81 + 1.5)}{27} = \frac{1.6248}{27} = 0.0601, \\ T(C) &= \left\{ \begin{array}{l} [0.101 + 0.102 + 0.103] \langle [0.7 + 0.9 + 0.11] \\ + [0.6 + 0.8 + 0.10] - [0.5 + 0.7 + 0.9] \rangle \end{array} \right\} = 0.6(1.71 + 1.5 - 2.1) = 0.306(3.21 - 2.1) = 0.3396, \\ G(C) &= \left\{ \begin{array}{l} \langle [0.101 + 0.102 + 0.103][0.7 + 0.9 + 0.11] \\ + [0.5 + 0.7 + 0.9] - [0.6 + 0.8 + 0.10] \rangle \end{array} \right\} = 0.306(1.71 + 2.1 - 1.5) = 0.7068. \end{aligned}$$

See Figure 1.

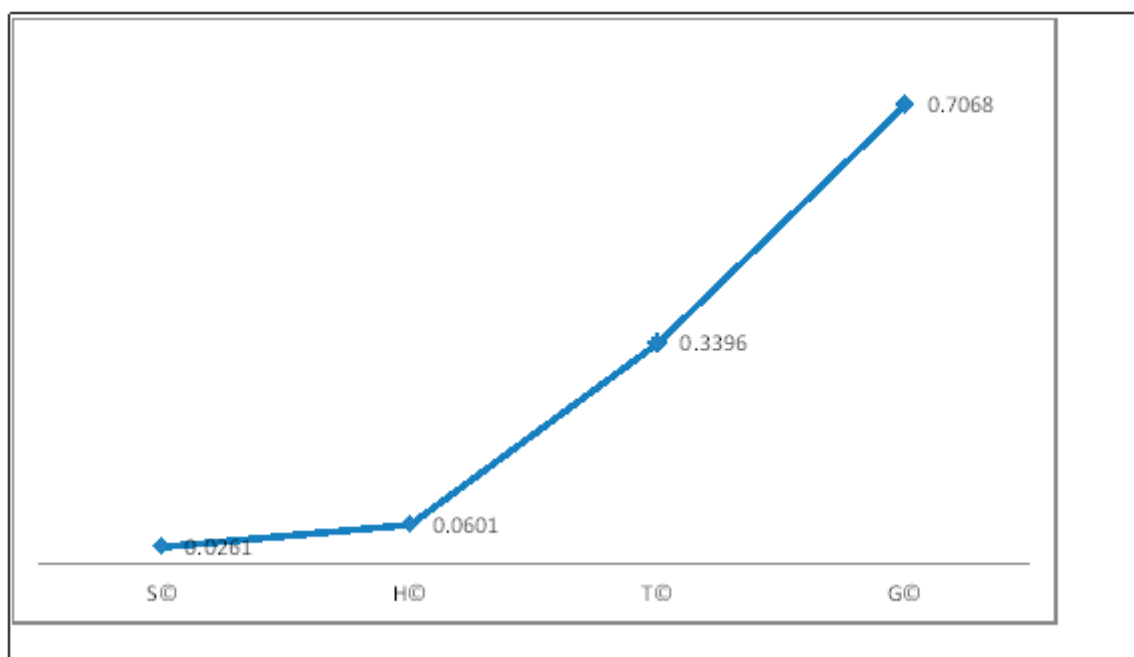


Figure 1. The score function, accuracy function, membership uncertainty index and hesitation uncertainty index are ranking of TNCFN.

3. Some Einstein Operations on TNCFNs

Definition 6. Let $C = \left\{ \begin{array}{l} [a(h), e(h), \\ G(h)], \langle [Y =^-(h), \\ k^-(h), \Gamma^-(h)], \\ [Y =^+(h), k^+(h), \\ \Gamma^+(h)], [Y = (h), \\ k(h), \Gamma(h)] \rangle \\ | h \in H \end{array} \right\}$, $C_1 = \left\{ \begin{array}{l} [a_1(h), e_1(h), \\ G_1(h)], \langle [Y =_1^-(h), \\ k_1^-(h), \Gamma_1^-(h)], \\ [Y =_1^+(h), k_1^+(h), \\ \Gamma_1^+(h)], [Y =_1(h), \\ k_1(h), \Gamma_1(h)] \rangle \\ | h \in H \end{array} \right\}$ and

$C_2 = \left\{ \begin{array}{l} [a_2(h), e_2(h), \\ G_2(h)], \langle [Y =_2^-(h), \\ k_2^-(h), \Gamma_2^-(h)], \\ [Y =_2^+(h), k_2^+(h), \\ \Gamma_2^+(h)], [Y =_2(h), \\ k_2(h), \Gamma_2(h)] \rangle \\ | h \in H \end{array} \right\}$ be any three TNCFNs. Then some Einstein operations of C_1

and C_2 can be defined as: $C_1 + C_2 = \left\langle \left[\begin{array}{l} \frac{a_1(h) + a_2(h)}{1 + a_1(h)(1 - a_2(h))}, \\ \frac{e_1(h) + e_2(h)}{1 + e_1(h)(1 - e_2(h))}, \\ \frac{G_1(h) + G_2(h)}{1 + G_1(h)(1 - G_2(h))} \end{array} \right], \left[\begin{array}{l} \frac{Y =_1^-(h) + Y =_2^-(h)}{1 + Y =_1^-(h)(1 - Y =_2^-(h))}, \\ \frac{Y =_1^+(h) + Y =_2^+(h)}{1 + Y =_1^+(h)(1 - Y =_2^+(h))}, \\ \frac{k_1^-(h) + k_2^-(h)}{1 + k_1^-(h)(1 - k_2^-(h))}, \\ \frac{k_1^+(h) + k_2^+(h)}{1 + k_1^+(h)(1 - k_2^+(h))}, \\ \frac{\Gamma_1^-(h) + \Gamma_2^-(h)}{1 + \Gamma_1^-(h)(1 - \Gamma_2^-(h))}, \\ \frac{\Gamma_1^+(h) + \Gamma_2^+(h)}{1 + \Gamma_1^+(h)(1 - \Gamma_2^+(h))} \end{array} \right] \right\rangle$,

$$\left[\begin{array}{c} \frac{Y_1(h)Y_2(h)}{(1+(1-Y_1(h))(1-Y_2(h)))}, \\ \frac{k_1(h)k_2(h)}{(1+(1-k_1(h))(1-k_2(h)))}, \\ \frac{\Gamma_1(h)\Gamma_2(h)}{(1+((1-\Gamma_1(h))(1-\Gamma_2(h))))} \end{array} \right] \rangle, \lambda C = \left\langle \left[\begin{array}{c} \frac{[1+a_C(h)]^\lambda - [1-a_C(h)]^\lambda}{[1+a_C(h)]^\lambda + [1-a_C(h)]^\lambda}, \\ \frac{[1+e_C(h)]^\lambda - [1-e_C(h)]^\lambda}{[1+e_C(h)]^\lambda + [1-e_C(h)]^\lambda}, \\ \frac{[1+G_C(h)]^\lambda - [1-G_C(h)]^\lambda}{[1+G_C(h)]^\lambda + [1-G_C(h)]^\lambda} \end{array} \right], \right. \\
\left. \left[\begin{array}{c} \frac{2[Y_C(h)]^\lambda}{[(2-Y_C(h))^\lambda + [Y_C(h)]^\lambda]}, \\ \frac{2[k_C(h)]^\lambda}{[(2-k_C(h))^\lambda + [k_C(h)]^\lambda]}, \\ \frac{2[\Gamma_C(h)]^\lambda}{[(2-\Gamma_C(h))^\lambda + [\Gamma_C(h)]^\lambda]} \end{array} \right] \right\rangle,$$

$$\left[\begin{array}{c} \frac{[1+Y_C^-(h)]^\lambda - [1-Y_C^-(h)]^\lambda}{[1+Y_C^-(h)]^\lambda + [1-Y_C^-(h)]^\lambda}, \\ \frac{[1+Y_C^+(h)]^\lambda - [1-Y_C^+(h)]^\lambda}{[1+Y_C^+(h)]^\lambda + [1-Y_C^+(h)]^\lambda}, \\ \frac{[1+k_C^-(h)]^\lambda - [1-k_C^-(h)]^\lambda}{[1+k_C^-(h)]^\lambda + [1-k_C^-(h)]^\lambda}, \\ \frac{[1+k_C^+(h)]^\lambda - [1-k_C^+(h)]^\lambda}{[1+k_C^+(h)]^\lambda + [1-k_C^+(h)]^\lambda}, \\ \frac{[1+\Gamma_C^-(h)]^\lambda - [1-\Gamma_C^-(h)]^\lambda}{[1+\Gamma_C^-(h)]^\lambda + [1-\Gamma_C^-(h)]^\lambda}, \\ \frac{[1+\Gamma_C^+(h)]^\lambda - [1-\Gamma_C^+(h)]^\lambda}{[1+\Gamma_C^+(h)]^\lambda + [1-\Gamma_C^+(h)]^\lambda} \end{array} \right],$$

Proposition 1. Let $\ddot{A}$, $\ddot{A}_1$ and $\ddot{A}_2$ be three TNCFNs, $\lambda, \lambda_1, \lambda_2 > 0$, then we have:

- (1) $\ddot{A}_1 + \ddot{A}_2 = \ddot{A}_2 + \ddot{A}_1$,
- (2) $\lambda(\ddot{A}_1 + \ddot{A}_2) = \lambda\ddot{A}_2 + \lambda\ddot{A}_1$,
- (3) $\lambda_1\ddot{A} + \lambda_2\ddot{A} = (\lambda_1 + \lambda_2)\ddot{A}$.

Proof. The proof of these propositions is provided in Appendix A. $\square$

Remark 1. If $\alpha_1 \leq_{L_{TNCFN}} \alpha_2$, then $\alpha_1 \leq \alpha_2$, the total order is the partial order on L_{TNCFN} , see Figure 2.

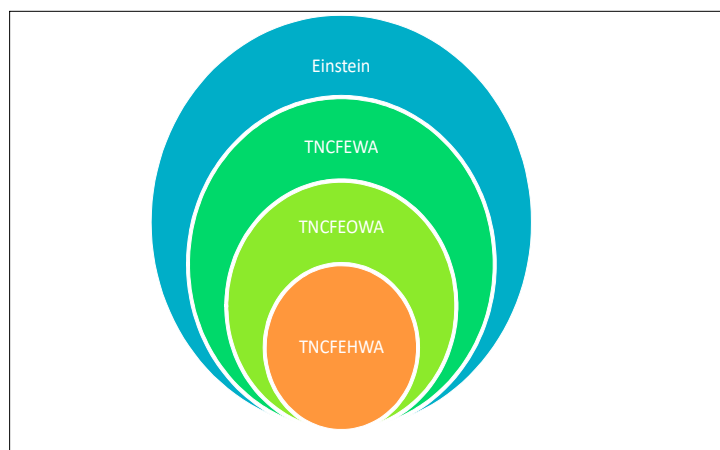


Figure 2. New extend aggregation operators, such as TNCFEWA, TNCFEOWA and TNCFEHWA operators.

4. Triangular Neutrosophic Cubic Fuzzy Averaging Operators Based on Einstein Operations

In this section, we define the aggregation operators.

4.1. Triangular Neutrosophic Cubic Fuzzy Einstein Weighted Averaging Operator

Definition 7. Let $\ddot{A} = \left\{ \begin{array}{l} [\alpha(h), \beta(h), \\ \Delta(h)] \langle [\xi_1^-(h), \\ \xi_2^-(h), \xi_3^-(h)], \\ [\xi_1^+(h), \xi_2^+(h), \\ \xi_3^+(h)], [\xi_1(h), \\ \xi_2(h), \xi_3(h)] \rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} and $\ddot{\omega} =$

$(\ddot{\omega}_1, \ddot{\omega}_2, \dots, \ddot{\omega}_n)^T$ be the weight vector, with $\ddot{\omega}_j \in [0, 1]$, $\sum_{j=1}^n \ddot{\omega}_j = 1$. Hence TNCFEWA operator of dimension n is a mapping $\text{TNCFEWA} : L_{\text{TNCFN}}^n \rightarrow L_{\text{TNCFN}}$ and defined by $\text{TNCFEWA}(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n) = \ddot{\omega}_1 \ddot{A}_1 + \ddot{\omega}_2 \ddot{A}_2, \dots, \ddot{\omega}_n \ddot{A}_n$.

If $\ddot{\omega} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$. Hence the TNCFEWA operator is reduced to TNCFEA operator of dimension n . It can be defined as follows: $\text{TNCFEA}(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n) = \frac{1}{n}(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n)$.

Theorem 1. Let $\ddot{A} = \left\{ \begin{array}{l} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle [p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} . The amassed an

incentive by utilizing the TNCFEWA operator is additionally a TNCFN and TNCFEWA.

$$(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n) = \left\langle \begin{array}{l} \frac{\prod_{j=1}^n [1+\alpha_1(h)]^\omega - \prod_{j=1}^n [1-\alpha_1(h)]^\omega}{\prod_{j=1}^n [1+\alpha_1(h)]^\omega + \prod_{j=1}^n [1-\alpha_1(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+\beta_1(h)]^\omega - \prod_{j=1}^n [1-\beta_1(h)]^\omega}{\prod_{j=1}^n [1+\beta_1(h)]^\omega + \prod_{j=1}^n [1-\beta_1(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+\Delta_1(h)]^\omega - \prod_{j=1}^n [1-\Delta_1(h)]^\omega}{\prod_{j=1}^n [1+\Delta_1(h)]^\omega + \prod_{j=1}^n [1-\Delta_1(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+p_1^+(h)]^\omega - \prod_{j=1}^n [1-p_1^+(h)]^\omega}{\prod_{j=1}^n [1+p_1^+(h)]^\omega + \prod_{j=1}^n [1-p_1^+(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+q_2^+(h)]^\omega - \prod_{j=1}^n [1-q_2^+(h)]^\omega}{\prod_{j=1}^n [1+q_2^+(h)]^\omega + \prod_{j=1}^n [1-q_2^+(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+r_3^+(h)]^\omega - \prod_{j=1}^n [1-r_3^+(h)]^\omega}{\prod_{j=1}^n [1+r_3^+(h)]^\omega + \prod_{j=1}^n [1-r_3^+(h)]^\omega} \end{array} \right\rangle, \left\langle \begin{array}{l} \frac{\prod_{j=1}^n [1+p_1^-(h)]^\omega - \prod_{j=1}^n [1-p_1^-(h)]^\omega}{\prod_{j=1}^n [1+p_1^-(h)]^\omega + \prod_{j=1}^n [1-p_1^-(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+q_2^-(h)]^\omega - \prod_{j=1}^n [1-q_2^-(h)]^\omega}{\prod_{j=1}^n [1+q_2^-(h)]^\omega + \prod_{j=1}^n [1-q_2^-(h)]^\omega}, \\ \frac{\prod_{j=1}^n [1+r_3^-(h)]^\omega - \prod_{j=1}^n [1-r_3^-(h)]^\omega}{\prod_{j=1}^n [1+r_3^-(h)]^\omega + \prod_{j=1}^n [1-r_3^-(h)]^\omega}, \\ \frac{2 \prod_{j=1}^n [p_1(h)]^\omega}{\prod_{j=1}^n [(2-p_1(h))^\omega + \prod_{j=1}^n [p_1(h)]^\omega] + \prod_{j=1}^n [p_1(h)]^\omega}, \\ \frac{2 \prod_{j=1}^n [q_2(h)]^\omega}{\prod_{j=1}^n [(2-q_2(h))^\omega + \prod_{j=1}^n [q_2(h)]^\omega] + \prod_{j=1}^n [q_2(h)]^\omega}, \\ \frac{2 \prod_{j=1}^n [r_3(h)]^\omega}{\prod_{j=1}^n [(2-r_3(h))^\omega + \prod_{j=1}^n [r_3(h)]^\omega] + \prod_{j=1}^n [r_3(h)]^\omega} \end{array} \right\rangle$$

where $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \dots, \ddot{\omega}_n)^T$ be the weight vector of $\ddot{A}_j (j = 1, 2, \dots, n)$ such that $\ddot{\omega}_j \in [0, 1]$ and $\sum_{j=1}^n \ddot{\omega}_j = 1$. If $\alpha_1(h) = \alpha_1(h)$, $\beta_1(h) = \beta_1(h)$, $\Delta_1(h) = \Delta_1(h)$, $p_1^-(h) = p_1^-(h)$, $q_2^-(h) = q_2^-(h)$, $r_3^-(h) = r_3^-(h)$, $p_1^+(h) = p_1^+(h)$, $q_2^+(h) = q_2^+(h)$, $r_3^+(h) = r_3^+(h)$ and $p_1(h) = p_1(h)$, $q_2(h) = q_2(h)$,

$r_3(h) = r_3(h)$. Then the TNCFN $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle [p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ are reduced to the triangular neutrosophic cubic fuzzy numbers $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \\ \Delta_1(h)], \langle [p_1^-(h), \\ q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ and the TNCFEWA operator is reduced to the TNCFEWA operator.

Proof. The proof of this theorem is provided in Appendix B. $\square$

Example 3. Let $C_1 = \left\{ \begin{array}{c} \langle [0.02, 0.03, 0.04], \\ [0.02, 0.04, 0.06], \\ [0.04, 0.06, 0.08], \\ [0.03, 0.05, 0.07] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.205, 0.207, 0.209], \\ [0.211, 0.213, 0.215], \\ [0.213, 0.215, 0.217], \\ [0.212, 0.214, 0.216] \rangle \end{array} \right\}$ and $C_3 = \left\{ \begin{array}{c} \langle [0.004, 0.005, 0.006], \\ [0.102, 0.104, 0.106], \\ [0.104, 0.106, 0.108], \\ [0.103, 0.105, 0.107] \rangle \end{array} \right\}$ be a TNCFN. Then the score function is defined by $S(C_1) = \frac{\left(\begin{array}{c} \langle [0.02 + 0.03 + 0.04], \\ [0.02 + 0.04 + 0.06] + \\ [0.04 + 0.06 + 0.08] - \\ [0.03 + 0.05 + 0.07] \rangle \end{array} \right)}{27} = 0.0005$, $S(C_2) = \frac{\left(\begin{array}{c} \langle [0.205 + 0.207 + 0.209], \\ [0.211 + 0.213 + 0.215] + \\ [0.213 + 0.215 + 0.217] - \\ [0.212 + 0.214 + 0.216] \rangle \end{array} \right)}{27} = 0.0147$, $S(C_3) = \frac{\left(\begin{array}{c} \langle [0.004, 0.005, 0.006], \\ [0.102, 0.104, 0.106], \\ [0.104, 0.106, 0.108], \\ [0.103, 0.105, 0.107] \rangle \end{array} \right)}{27} = \frac{0.015(0.63-0.315)}{27} = 0.0001$.

See Figure 3.

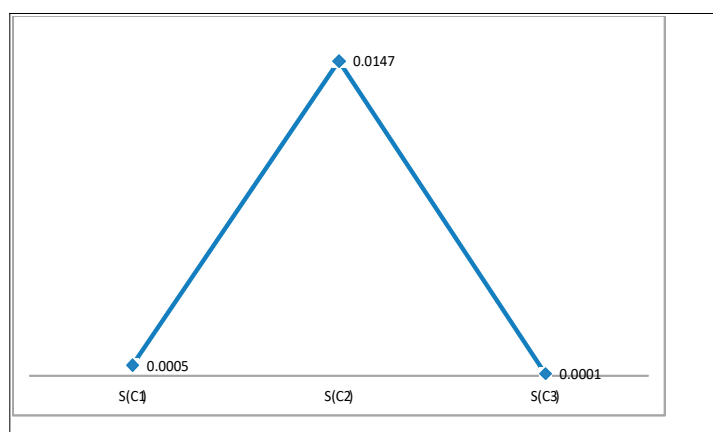


Figure 3. $S(C_2)$ is the first score value, $S(C_1)$ is the second score value and $S(C_3)$ is the third score value.

Example 4. Let $C_1 = \left\{ \begin{array}{l} \langle [0.04, 0.05, 0.06], \\ [0.01, 0.03, 0.05], \\ [0.03, 0.05, 0.07], \\ [0.02, 0.04, 0.06] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{l} \langle [0.05, 0.07, 0.09], \\ [0.11, 0.13, 0.15], \\ [0.13, 0.15, 0.17], \\ [0.12, 0.14, 0.16] \rangle \end{array} \right\}$ and $C_3 = \left\{ \begin{array}{l} \langle [0.2004, 0.2005, 0.2006], \\ [0.2102, 0.2104, 0.2106], \\ [0.2104, 0.2106, 0.2108], \\ [0.2103, 0.2105, 0.2107] \rangle \end{array} \right\}$ be a TNCFN. Then the accuracy function is defined by $H(C_1) = \frac{\langle [0.04 + 0.05 + 0.06], [0.01 + 0.03 + 0.05] + [0.03 + 0.05 + 0.07] + [0.02 + 0.04 + 0.06] \rangle}{27} = 0.002$, $H(C_2) = \frac{\langle [0.05 + 0.07 + 0.09], [0.11 + 0.13 + 0.15] + [0.13 + 0.15 + 0.17] + [0.12 + 0.14 + 0.16] \rangle}{27} = 0.0098$, $H(C_3) = \frac{\langle [0.2004 + 0.2005 + 0.2006], [0.2102 + 0.2104 + 0.2106] + [0.2104 + 0.2106 + 0.2108] + [0.2103 + 0.2105 + 0.2107] \rangle}{27} = 0.0422$.

See Figure 4.

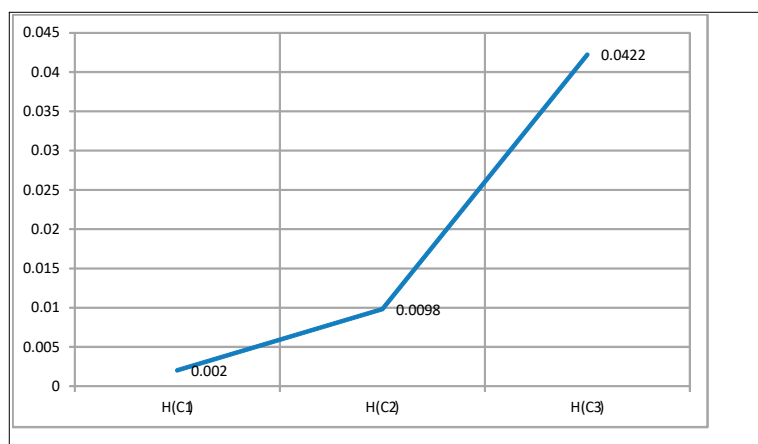


Figure 4. $H(C_2)$ is the first score value, $H(C_1)$ is the second score value and $H(C_3)$ is the third score value.

Proposition 2. Let $\ddot{A} = \left\{ \begin{array}{l} [\alpha_1(h), \beta_1(h), \Delta_1(h)] \\ \langle [p_1^-(h), q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), r_1^+(h)], \\ [p_1(h), q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} and where $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \dots, \ddot{\omega}_n)^T$ is the weight vector of $\ddot{A}_j (j = 1, 2, \dots, n)$ with $\ddot{\omega}_j \in [0, 1]$ and $\sum_{j=1}^n \ddot{\omega}_j = 1$.

Then (1) (Idempotency): If all $A_j, j = 1, 2, \dots, n$ are equal, i.e., $A_j = A$, for all $j = 1, 2, \dots, n$, then $TNCFEWA(A_1, A_2, \dots, A_n) = A$.

(2) (Boundary): If $\alpha_{\min} = \min_{1 \leq j \leq n} \alpha_j$, $\beta_{\min} = \min_{1 \leq j \leq n} \beta_j$, $\Delta_{\min} = \min_{1 \leq j \leq n} \Delta_j$, $p_{\min}^- = \min_{1 \leq j \leq n} p_j^-$, $q_{\min}^- = \min_{1 \leq j \leq n} q_j^-$, $r_{\min}^- = \min_{1 \leq j \leq n} r_j^-$, $p_{\min}^+ = \min_{1 \leq j \leq n} p_j^+$, $q_{\min}^+ = \min_{1 \leq j \leq n} q_j^+$, $r_{\min}^+ = \min_{1 \leq j \leq n} r_j^+$, $p_{\max} = \max_{1 \leq j \leq n} p_j$, $q_{\max} = \max_{1 \leq j \leq n} q_j$, $r_{\max} = \max_{1 \leq j \leq n} r_j$, $\mu_{\max} = \max_{1 \leq j \leq n} \mu_j$, $p_{\max}^- = \max_{1 \leq j \leq n} p_j^-$, $q_{\max}^- = \max_{1 \leq j \leq n} q_j^-$, $r_{\max}^- = \max_{1 \leq j \leq n} r_j^-$, $\mu_{\max}^+ = \max_{1 \leq j \leq n} \mu_j^+$, $p_{\max}^+ = \max_{1 \leq j \leq n} p_j^+$, $q_{\max}^+ = \max_{1 \leq j \leq n} q_j^+$, $r_{\max}^+ = \max_{1 \leq j \leq n} r_j^+$, $p_{\min} = \min_{1 \leq j \leq n} p_j$,

$q_{\min} = \min_{1 \leq j \leq n} q_j$, $r_{\min} = \min_{1 \leq j \leq n} r_j$, $\mu_{\min} = \min_{1 \leq j \leq n} \mu_j$ for all $j = 1, 2, \dots, n$, we can determine that

$$\left\{ \begin{array}{l} [\alpha_{\min}(h), \\ \beta_{\min}(h), \Delta_{\min}(h)] \\ \langle [p_{\min}^-(h), \\ q_{\min}^-(h), r_{\min}^-(h)] \\ [p_{\min}^+(h), \\ q_{\min}^+(h), r_{\min}^+(h)] \\ [p_{\max}(h), q_{\max}(h), \\ r_{\max}(h), s_{\max}(h)] \rangle | \\ h \in H \end{array} \right\} \leq \text{TNCFEWA}(A_1, A_2, \dots, A_n) \leq \left\{ \begin{array}{l} [\alpha_{\max}(h), \beta_{\max}(h), \\ \Delta_{\max}(h)], \langle [p_{\max}^-(h), \\ q_{\max}^-(h), r_{\max}^-(h)], \\ [p_{\max}^+(h), q_{\max}^+(h), \\ r_{\max}^+(h)], [p_{\min}(h), \\ q_{\min}(h), r_{\min}(h)] \rangle | \\ h \in H \end{array} \right\}$$

$$(3) \text{ (Monotonicity): } A = \left\{ \begin{array}{l} [\alpha_A(h), \\ \beta_A(h), \Delta_A(h)], \\ \langle [p_A^-(h), \\ q_A^-(h), r_A^-(h)] \\ [p_A^+(h), \\ q_A^+(h), r_A^+(h)] \\ [p_A(h), \\ q_A(h), r_A(h)] \rangle | \\ h \in H \end{array} \right\} \text{ and } B = \left\{ \begin{array}{l} [\alpha_B(h), \\ \beta_B(h), \Delta_B(h)], \\ \langle [p_B^-(h), \\ q_B^-(h), r_B^-(h)] \\ [p_B^+(h), \\ q_B^+(h), r_B^+(h)] \\ [p_B(h), \\ q_B(h), r_B(h)] \rangle | \\ h \in H \end{array} \right\} \text{ be two collection}$$

of TNCFNs in L_{TNCFN} and $A_j \leq L_{\text{TNCFN}} B_j$ i.e., $\alpha_A(h) \leq \alpha_B(h)$, $\beta_A(h) \leq \beta_B(h)$, $\Delta_A(h) \leq \Delta_B(h)$, $p_A^-(h) \leq p_B^-(h)$, $q_A^-(h) \leq q_B^-(h)$, $r_A^-(h) \leq r_B^-(h)$, $p_A^+(h) \leq p_B^+(h)$, $q_A^+(h) \leq q_B^+(h)$, $r_A^+(h) \leq r_B^+(h)$ and $p_A(h) \leq p_B(h)$, $q_A(h) \leq q_B(h)$, $r_A(h) \leq r_B(h)$ then $\text{TNCFEWA}(A_1, A_2, \dots, A_n) \leq \text{TNCFEWA}(B_1, B_2, \dots, B_n)$.

4.2. Triangular Neutrosophic Cubic Fuzzy Einstein Ordered Weighted Averaging Operator

Definition 8. Let $\ddot{A} = \left\{ \begin{array}{l} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \langle [p_{\ddot{A}}^-(h), \\ q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^+(h), \\ r_{\ddot{A}}^+(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h)] \rangle \\ | h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} , a TNCFEOWA

operator of dimension n is a mapping TNCFEOWA: $L_{\text{TNCFN}}^n \rightarrow L_{\text{TNCFN}}$, that has an associated vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. $\text{TNCFEOWA}(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n) = \ddot{\sigma}_1 \ddot{A}_{(\sigma)1} + \ddot{\sigma}_2 \ddot{A}_{(\sigma)2}, \dots, + \ddot{\sigma}_n \ddot{A}_{(\sigma)n}$, where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\ddot{A}_{\sigma(1)} \leq \ddot{A}_{\sigma(j-1)}$ for all $j = 2, 3, \dots, n$ (i.e., $\ddot{A}_{\sigma(j)}$ is the j the largest value in the collection $(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n)$). If $w = (w_1, w_2, \dots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$. Then the TNCFEOWA operator is reduced to the TCFA operator (2) of dimension n .

$$q_{\ddot{A}}(h) = q_{\ddot{A}}(h), r_{\ddot{A}}(h) = r_{\ddot{A}}(h). \text{ The TNCFN } \ddot{A} = \left\{ \begin{array}{l} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)] \langle [p_{\ddot{A}}^-(h), \\ q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^-(h), \\ r_{\ddot{A}}^-(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \rangle \\ |h \in H \end{array} \right\} \text{ are reduced to the triangular}$$

$$\text{neutrosophic cubic fuzzy numbers } \ddot{A} = \left\{ \begin{array}{l} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \langle [p_{\ddot{A}}^-(h), \\ q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^-(h), \\ r_{\ddot{A}}^-(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \rangle \\ |h \in H \end{array} \right\}. \text{ Then the TNCFEWA operator is reduced to}$$

the triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging operator.

Proof. The process of this proof is the same as Theorem 1. $\square$

$$\text{Example 5. Let } C_1 = \left\{ \begin{array}{l} \langle [0.01, 0.02, 0.03], \\ [0.103, 0.105, 0.107], \\ [0.105, 0.107, 0.109], \\ [0.104, 0.106, 0.108] \rangle \end{array} \right\}, C_2 = \left\{ \begin{array}{l} \langle [0.306, 0.308, 0.310], \\ [0.310, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\} \text{ and}$$

$$C_3 = \left\{ \begin{array}{l} \langle [0.44, 0.55, 0.66], \\ [0.122, 0.124, 0.126], \\ [0.124, 0.126, 0.128], \\ [0.123, 0.125, 0.127] \rangle \end{array} \right\} \text{ be a TNCFN. Then the score function is defined by } S(C_1) =$$

$$\frac{\left\{ \begin{array}{l} \langle [0.01 + 0.02 + 0.03], \\ [0.103 + 0.105 + 0.107] + \\ [0.105 + 0.107 + 0.109] - \\ [0.104 + 0.106 + 0.108] \rangle \end{array} \right\}}{27} = 0.00004, S(C_2) = \frac{\left\{ \begin{array}{l} \langle [0.306 + 0.308 + 0.310], \\ [0.310 + 0.313 + 0.315] + \\ [0.313 + 0.315 + 0.317] - \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0322, S(C_3) =$$

$$\frac{\left\{ \begin{array}{l} \langle [0.44 + 0.55 + 0.66], \\ [0.122 + 0.124 + 0.126] + \\ [0.124 + 0.126 + 0.128] - \\ [0.123 + 0.125 + 0.127] \rangle \end{array} \right\}}{27} = 0.0229.$$

See Figure 5.

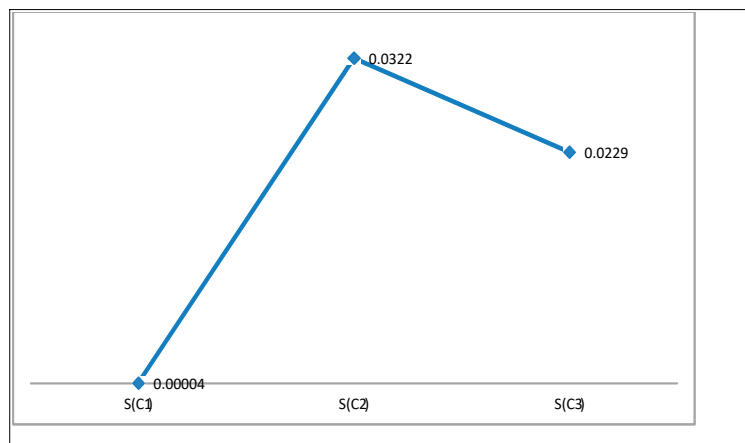


Figure 5. Different score ranking of TNCFEOWA operator.

Example 6. Let $C_1 = \left\{ \begin{array}{l} \langle [0.02, 0.03, 0.04], \\ [0.06, 0.08, 0.010], \\ [0.08, 0.010, 0.012], \\ [0.07, 0.09, 0.011] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{l} \langle [0.1105, 0.1107, 0.1109], \\ [0.1111, 0.1113, 0.1115], \\ [0.1113, 0.1115, 0.1117], \\ [0.1112, 0.1114, 0.1116] \rangle \end{array} \right\}$ and $C_3 = \left\{ \begin{array}{l} \langle [0.214, 0.215, 0.216], \\ [0.2202, 0.2204, 0.2206], \\ [0.2204, 0.2206, 0.2208], \\ [0.2203, 0.2205, 0.2207] \rangle \end{array} \right\}$ be a TNCFN. Then the accuracy function is defined by $H(C_1) = \frac{\left\{ \begin{array}{l} \langle [0.02, 0.03, 0.04], \\ [0.06, 0.08, 0.010], \\ [0.08, 0.010, 0.012], \\ [0.07, 0.09, 0.011] \rangle \end{array} \right\}}{27} = -0.0021$, $H(C_2) = \frac{\left\{ \begin{array}{l} \langle [0.1105, 0.1107, 0.1109], \\ [0.1111, 0.1113, 0.1115], \\ [0.1113, 0.1115, 0.1117], \\ [0.1112, 0.1114, 0.1116] \rangle \end{array} \right\}}{27} = 0.0041$, $H(C_3) = \frac{\left\{ \begin{array}{l} \langle [0.214, 0.215, 0.216], \\ [0.2202, 0.2204, 0.2206], \\ [0.2204, 0.2206, 0.2208], \\ [0.2203, 0.2205, 0.2207] \rangle \end{array} \right\}}{27} = 0.0158$.

See Figure 6.

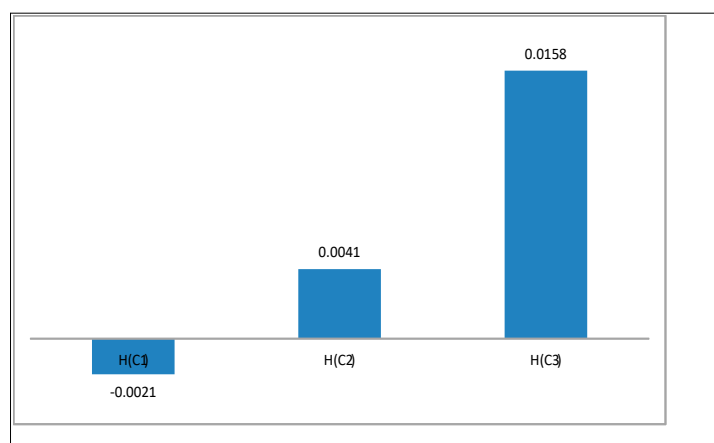


Figure 6. Different accuracy ranking of TNCFEOWA operator.

4.3. Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operator

Definition 9. Let $\ddot{A} = \left\{ \begin{array}{l} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ \ddot{A}(h)], \langle [p_{\ddot{A}}^-(h), \\ q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^-(h), \\ r_{\ddot{A}}^-(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} and $\ddot{w} =$

$(\ddot{w}_1, \ddot{w}_2, \dots, \ddot{w}_n)^T$ is the weight vector of $\ddot{A}_j (j = 1, 2, \dots, n)$ such that $\ddot{w}_j \in [0, 1]$ and $\sum_{j=1}^n \ddot{w}_j = 1$.

Then TNCFEHWA operator of dimension n is a mapping $\text{TNCFEHWA}: L_{\text{TNCFN}}^n \rightarrow L_{\text{TNCFN}}$, that is an associated vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. TNCFEHWA

$(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n) = p_1 \ddot{A}_{\sigma(1)} + p_2 \ddot{A}_{\sigma(2)}, \dots, p_n \ddot{A}_{\sigma(n)}$. If $p = \theta \ddot{w}_{\sigma(j)} + (1 - \theta)w_{\sigma(j)}$ with a balancing coefficient $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\ddot{A}_{\sigma(j)} \leq \ddot{A}_{\sigma(j-1)}$ for all $j = 2, 3, \dots, n$ (i.e., $\ddot{A}_{\sigma(j)}$ is the j th largest value in the collection $(\ddot{A}_1, \ddot{A}_2, \dots, \ddot{A}_n)$).

Example 7. Let $C_1 = \left\{ \begin{array}{l} \langle [0.11, 0.13, 0.14], \\ [0.62, 0.64, 0.66], \\ [0.64, 0.66, 0.68], \\ [0.63, 0.65, 0.67] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{l} \langle [0.51, 0.52, 0.53], \\ [0.311, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\}$ and $C_3 =$

$\left\{ \begin{array}{l} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}$ be a TNCFN. Then the score function is defined by $S(C_1) =$

$$\frac{\left\{ \begin{array}{l} \langle [0.11 + 0.13 + 0.14], \\ [0.62 + 0.64 + 0.66] + \\ [0.64 + 0.66 + 0.68] - \\ [0.63 + 0.65 + 0.67] \rangle \end{array} \right\}}{27} = 0.0281, S(C_2) = \frac{\left\{ \begin{array}{l} \langle [0.51 + 0.52 + 0.53], \\ [0.311 + 0.313 + 0.315] + \\ [0.313 + 0.315 + 0.317] - \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0543, S(C_3) =$$

$$\frac{\left\{ \begin{array}{l} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}}{27} = 0.0104.$$

See Figure 7.

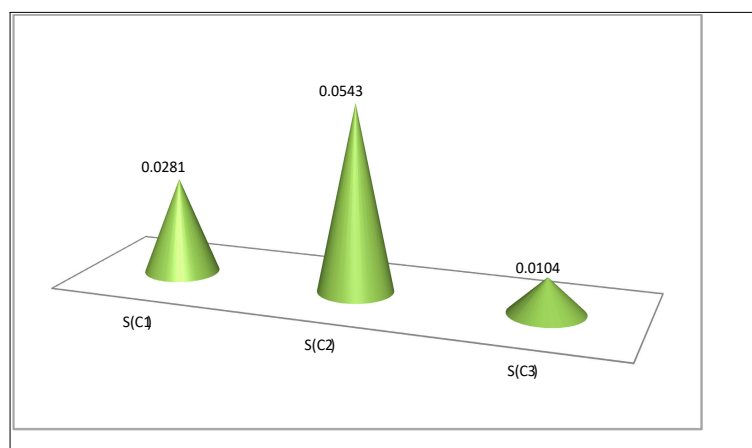


Figure 7. Different score ranking of TNCFEHWA operator.

Example 8. Let $C_1 = \left\{ \begin{array}{l} \langle [0.112, 0.113, 0.114], \\ [0.21, 0.24, 0.28], \\ [0.24, 0.28, 0.32], \\ [0.23, 0.26, 0.30] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{l} \langle [0.0019, 0.0021, 0.0034], \\ [0.1231, 0.1233, 0.1235], \\ [0.1233, 0.1235, 0.1237], \\ [0.1232, 0.1234, 0.1236] \rangle \end{array} \right\}$ and

$C_3 = \left\{ \begin{array}{l} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}$ be a TNCFN. Then the accuracy function is defined by $H(C_1) =$

$$\frac{\left\{ \begin{array}{l} \langle [0.112 + 0.113 + 0.114], \\ [0.21 + 0.24 + 0.28] + \\ [0.24 + 0.28 + 0.32] - \\ [0.23 + 0.26 + 0.30] \rangle \end{array} \right\}}{27} = 0.0111, H(C_2) = \frac{\left\{ \begin{array}{l} \langle [0.0019 + 0.0021 + 0.0034] \\ [0.1231 + 0.1233 + 0.1235] + \\ [0.1233 + 0.1235 + 0.1237] - \\ [0.1232 + 0.1234 + 0.1236] \rangle \end{array} \right\}}{27} = 0.1124,$$

$$H(C_3) = \frac{\left\{ \begin{array}{l} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}}{27} = 0.0226.$$

See Figure 8.

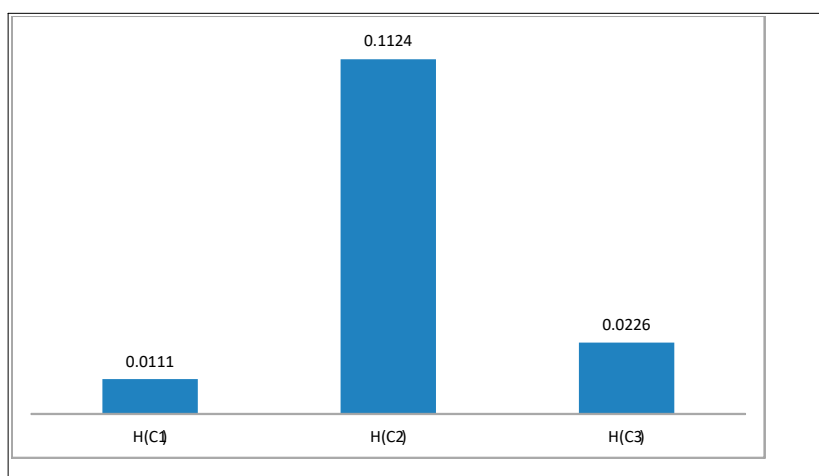


Figure 8. Different accuracy ranking of TNCFEHWA operator.

5. An Approach to MADM with TNCF Data

Let us suppose the discrete set is $h = \{h_1, h_2, \dots, h_n\}$ and $G = \{g_1, g_2, \dots, g_m\}$ are the attributes. Consider that the value of alternatives h_i ($i = 1, 2, \dots, n$) on attributes g_j ($j = 1, 2, \dots, m$) given by

decision maker are TNCFNs in L_{TNCFN} : $\ddot{A} = \left\{ \begin{array}{l} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ \ddot{A}(h)], \langle [p_{\ddot{A}}^-(h), \\ q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^+(h), \\ r_{\ddot{A}}^+(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \rangle \\ |h \in H \end{array} \right\}$, a MADM problem is expressed

in the TNCF-decision matrix $\ddot{D} = (\ddot{A}_{ij})_{m \times n} = \left\{ \begin{array}{l} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \ddot{A}(h)], \\ \langle [p_{\ddot{A}}^-(h), q_{\ddot{A}}^-(h), r_{\ddot{A}}^-(h)], \\ [p_{\ddot{A}}^+(h), q_{\ddot{A}}^+(h), r_{\ddot{A}}^+(h)], \\ p_{\ddot{A}}(h), q_{\ddot{A}}(h), r_{\ddot{A}}(h) \rangle \\ |h \in H \end{array} \right\}$.

Step 1: Calculate the TNCF decision matrix.

Step 2: Utilize the TNCFEWA operator to mix all values $\ddot{\beta}_{ij}$ ($j = 1, 2, \dots, m$) and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \dots, \ddot{\omega}_n)^T$ is the weight vector.

Step 3: Calculate the score function.

Step 4: Find the ranking.

See Figure 9.

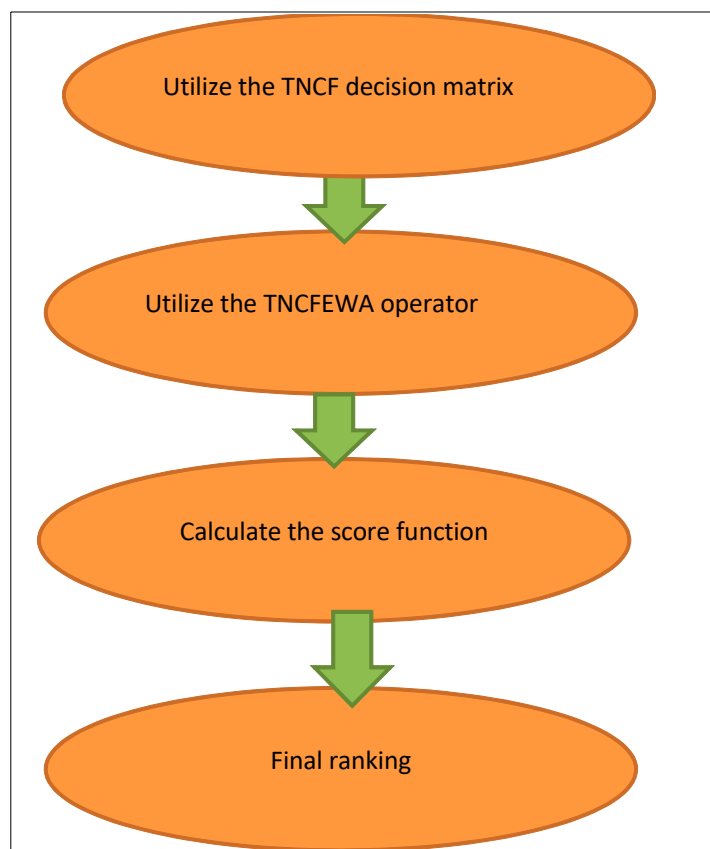


Figure 9. Proposed method.

6. Numerical Application

The inspiration structure is designed to be dependent upon an assessment that has been devised for the purpose of a stimulus/influencing technique of a twofold entire traveler dispersion to work over the Lahore in Faisalabad by lessening the adventure stage in extraordinarily brimful waterway movement. Inspiration structure choices are sure the settled of options $A = \{A_1, A_2, A_3, A_4\}$

A_1 : Old-style propeller and high trundle

A_2 : Get-up-and-go,

A_3 : Cyclonical propeller,

A_4 : Outmoded

See Figure 10.

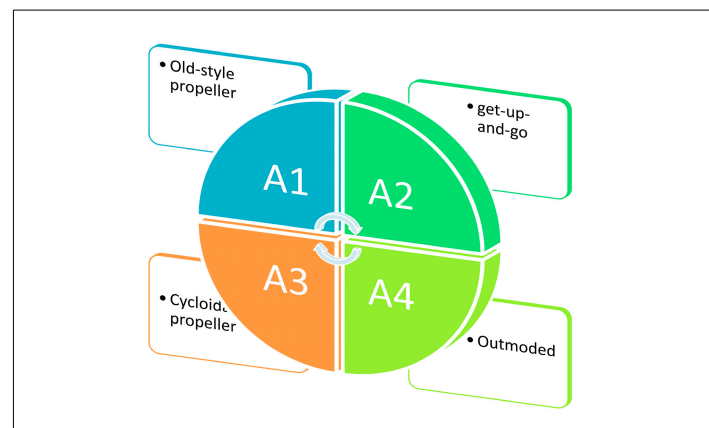


Figure 10. Four alternatives.

The ideal is prepared on the possibility of lone zone and four issue characteristics, which are as follows:

c_1 : Theory rate

c_2 : Reparation and support uses

c_3 : Agility

c_4 : Tremor and unrest.

See Figure 11.

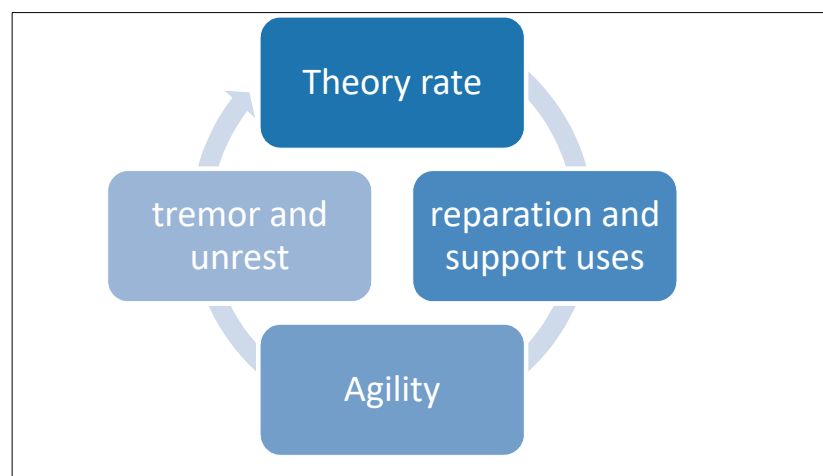


Figure 11. Different criteria.

The weight vector is $\vec{\omega} = (0.25, 0.50, 0.25)$. So, the triangular neutrosophic cubic fuzzy MADM issue is intended to choose the appropriate energy structure from between 3 choices.

Step 1: Calculate the TNCF decision matrix.

The TNCF decision matrix is as Table 2

Table 2. Triangular Neutrosophic Cubic Fuzzy Decision Matrix.

	c_1	c_2
A_1	$\left\langle \begin{matrix} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{matrix} \right\rangle$
A_2	$\left\langle \begin{matrix} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{matrix} \right\rangle$
A_3	$\left\langle \begin{matrix} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{matrix} \right\rangle$
A_4	$\left\langle \begin{matrix} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{matrix} \right\rangle$
	c_3	c_4
A_1	$\left\langle \begin{matrix} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{matrix} \right\rangle$
A_2	$\left\langle \begin{matrix} [0.3, 0.4, 0.5], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{matrix} \right\rangle$
A_3	$\left\langle \begin{matrix} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{matrix} \right\rangle$
A_4	$\left\langle \begin{matrix} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{matrix} \right\rangle$

Step 2: Calculate the TNCFEWA operator to total all the rating values and $w = (0.1, 0.2, 0.4, 0.3)^T$. The TNCFEWA operator are defined in Table 3.

Table 3. TNCFEWA Operator.

A_1	$\left\{ \left\langle \begin{matrix} [0.2539, 0.2751, 0.2965], [0.1628, 0.2513, 0.3973], \\ [0.2513, 0.3973, 0.0503], [0.7335, 0.8054, 0.6003] \end{matrix} \right\rangle \right\}$
A_2	$\left\{ \left\langle \begin{matrix} [0.1536, 0.1995, 0.2477], [0.2944, 0.3597, 0.6447], \\ [0.3597, 0.6447, 0.0988], [0.4838, 0.6067, 0.4831] \end{matrix} \right\rangle \right\}$
A_3	$\left\{ \left\langle \begin{matrix} [0.3481, 0.4499, 0.5594], [0.3626, 0.5867, 0.7852], \\ [0.5867, 0.7852, 0.7582], [0.1049, 0.2122, 0.3571] \end{matrix} \right\rangle \right\}$
A_4	$\left\{ \left\langle \begin{matrix} [0.2282, 0.2945, 0.3622], [0.3704, 0.5631, 0.7729], \\ [0.5631, 0.7729, 0.4197], [0.2593, 0.3985, 0.2735] \end{matrix} \right\rangle \right\}$

Step 3: The score value are calculated as $s_1 = -0.0192$, $s_2 = 0.0184$, $s_3 = 0.1603$, $s_4 = 0.0829$.

Step 4: Ranking $\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$.
See Figure 12.

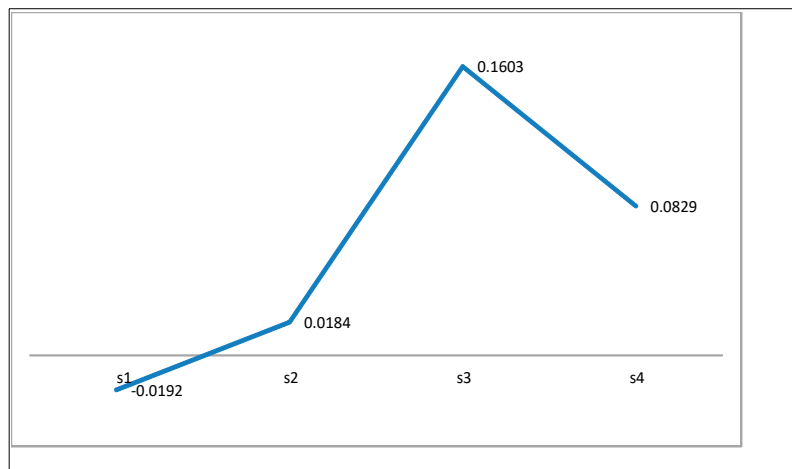


Figure 12. Rating value different range of values.

7. Comparison Analysis

So as to check the legitimacy and viability of the proposed methodology, a near report is led utilizing the techniques triangular cubic fuzzy number [21], which are unique instances of TNCFNs, to the equivalent illustrative model.

A Comparison Analysis with the Existing MCDM Method Triangular Cubic Fuzzy Number

Aliya et al [21] after transformation, the triangular cubic fuzzy information is given in Table 4.

Table 4. Triangular cubic fuzzy decision matrix.

	c_1	c_2	c_3	c_4
A_1	$\left\{ \begin{array}{l} [0.1, 0.2, 0.3], \\ \langle [0.4, 0.8], 0.7 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.22, 0.23], \\ \langle [0.5, 0.9], 0.8 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.15, 0.16, 0.17], \\ \langle [0.12, 0.16], 0.15 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.22, 0.23], \\ \langle [0.5, 0.9], 0.8 \rangle \end{array} \right\}$
A_2	$\left\{ \begin{array}{l} [0.21, 0.22, 0.23], \\ \langle [0.5, 0.9], 0.8 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.1, 0.2, 0.3], \\ \langle [0.4, 0.8], 0.7 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.3, 0.4, 0.5], \\ \langle [0.2, 0.6], 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.15, 0.16, 0.17], \\ \langle [0.12, 0.16], 0.15 \rangle \end{array} \right\}$
A_3	$\left\{ \begin{array}{l} [0.15, 0.16, 0.17], \\ \langle [0.12, 0.16], 0.15 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.3, 0.4, 0.5], \\ \langle [0.2, 0.6], 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.1, 0.2, 0.3], \\ \langle [0.4, 0.8], 0.7 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.3, 0.4, 0.5], \\ \langle [0.2, 0.6], 0.5 \rangle \end{array} \right\}$
A_4	$\left\{ \begin{array}{l} [0.3, 0.4, 0.5], \\ \langle [0.2, 0.6], 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.15, 0.16, 0.17], \\ \langle [0.12, 0.16], 0.15 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.22, 0.23], \\ \langle [0.5, 0.9], 0.8 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.1, 0.2, 0.3], \\ \langle [0.4, 0.8], 0.7 \rangle \end{array} \right\}$

Calculate the TCFA operator and $w = (0.1, 0.2, 0.4, 0.3)^T$.

The TCFA operator is presented in Table 5.

Table 5. TCFA operator.

A_1	$\langle [0.067, 0.081, 0.093], [0.1833, 0.4721], 0.1384 \rangle$
A_2	$\langle [0.152, 0.196, 0.24], [0.2672, 0.6329], 0.5073 \rangle$
A_3	$\langle [0.42, 0.464, 0.588], [0.4631, 0.7646], 0.2132 \rangle$
A_4	$\langle [0.228, 0.294, 0.36], [0.3727, 0.7771], 0.3613 \rangle$

Calculate the score function $s_1 = 0.0138$, $s_2 = 0.0256$, $s_3 = 0.1659$, $s_4 = 0.0772$.
See Figure 13.

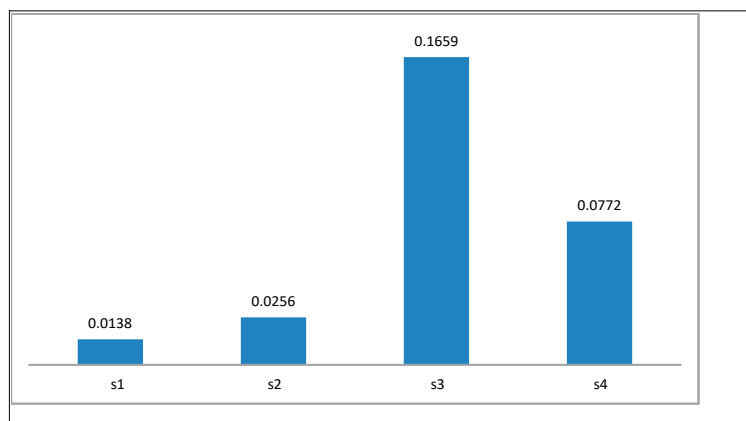


Figure 13. s_3 is the first ranking, s_4 is the 2nd ranking, s_2 is the third ranking and s_1 is the 4th ranking in the TCFN.

The existing Table 6 is as

Table 6. Comparison method with existing methods.

Method	Ranking
TNCFNs	$\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$
TCFN [21]	$\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$

See Figure 14.

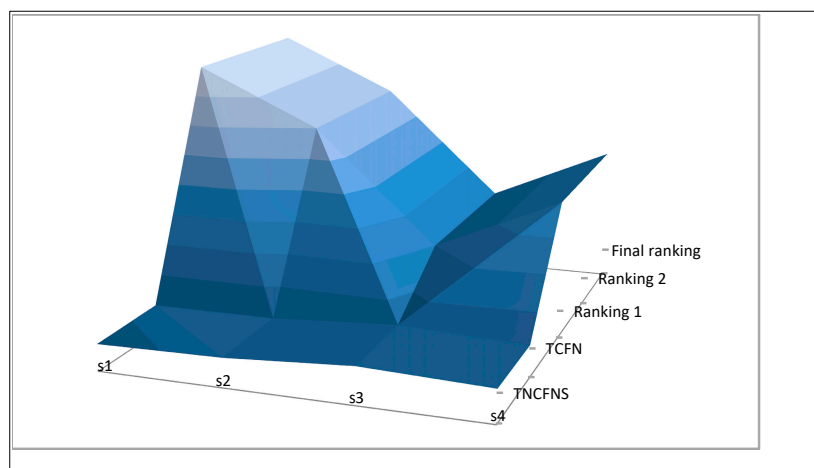


Figure 14. Comparison analysis with existing method.

The comparison method of score function is presented in Table 7.

Table 7. Comparison method with score function.

Score function	Ranking
TNCFEWA operator	$S(C_3) > S(C_2) > S(C_1)$
TNCFEOWA operator	$S(C_1) > S(C_2) > S(C_3)$
TNCFEHWa operator	$S(C_2) > S(C_1) > S(C_3)$

See Figure 15.

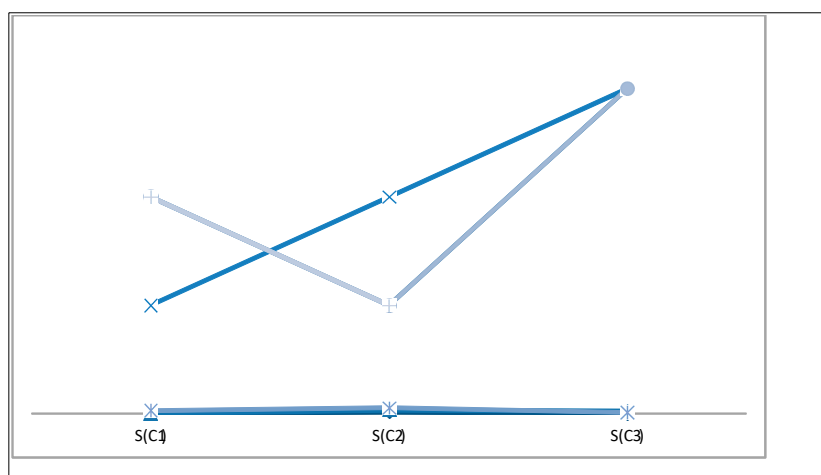


Figure 15. Different score value.

8. Conclusions

In this paper, we introduce a new concept of TNCFNs and operational laws. We introduce three aggregation operators, namely, the TNCFEWA operator, TNCFEOWA operator and TNCFEWA operator. We introduce group decision making under TNCFNs. Finally, a numerical example is provided to demonstrate the utility of the established approach. In cluster decision-making issues, consultants sometimes return from completely different specialty fields and have different backgrounds and levels of data; as such, they sometimes have branching opinions. These operators may be applied to several different fields, like data fusion, data processing, and pattern recognition, triangular neutrosophic cube like linguistic fuzzy Vikor methodology and quadrangle neutrosophic cube linguistic fuzzy Vikor methodology, which may be a suitable topic for longer term analysis, see Figure 16.

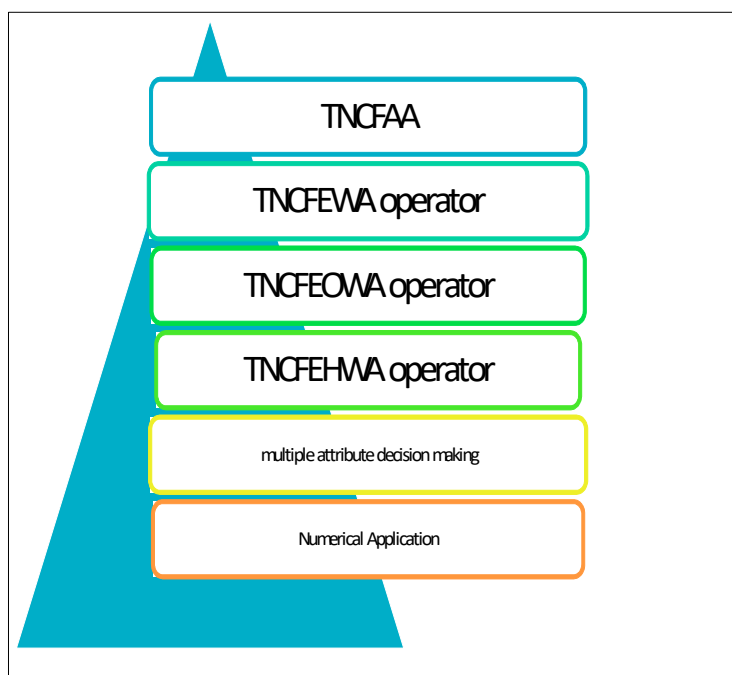


Figure 16. Flowcharts of whole papers.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Proposition 1

$$(1) A_1 + A_2 = A_2 + A_1;$$

$$A_1 + A_2 = \left\{ \left\langle \left[\frac{\alpha_1(h) + \alpha_2(h)}{(1 + \alpha_1(h))(1 - \alpha_2(h))}, \frac{\beta_1(h) + \beta_2(h)}{(1 + \beta_1(h))(1 - \beta_2(h))}, \frac{\Delta_1(h) + \Delta_2(h)}{(1 + \Delta_1(h))(1 - \Delta_2(h))} \right], \right. \right. \\ \left. \left[\frac{p_1^-(h) + p_2^-(h)}{(1 + p_1^-(h))(1 - p_2^-(h))}, \frac{q_1^-(h) + q_2^-(h)}{(1 + q_1^-(h))(1 - q_2^-(h))}, \frac{r_1^-(h) + r_2^-(h)}{(1 + r_1^-(h))(1 - r_2^-(h))} \right], \right. \\ \left. \left[\frac{p_1^+(h) + p_2^+(h)}{(1 + p_1^+(h))(1 - p_2^+(h))}, \frac{q_1^+(h) + q_2^+(h)}{(1 + q_1^+(h))(1 - q_2^+(h))}, \frac{r_1^+(h) + r_2^+(h)}{(1 + r_1^+(h))(1 - r_2^+(h))} \right] \right\rangle \\ \left. \left[\frac{p_1(h)p_2(h)}{1 + ((1 - p_1(h))(1 - p_2(h)))}, \frac{q_1(h)q_2(h)}{1 + ((1 - q_1(h))(1 - q_2(h)))}, \frac{r_1(h)r_2(h)}{1 + ((1 - r_1(h))(1 - r_2(h)))} \right] \right\rangle \Bigg\} = \\ \left\{ \left\langle \left[\frac{\alpha_2(h) + \alpha_1(h)}{(1 + \alpha_2(h))(1 - \alpha_1(h))}, \frac{\beta_2(h) + \beta_1(h)}{(1 + \beta_2(h))(1 - \beta_1(h))}, \frac{\Delta_2(h) + \Delta_1(h)}{(1 + \Delta_2(h))(1 - \Delta_1(h))} \right], \right. \right. \\ \left. \left[\frac{p_2^-(h) + p_1^-(h)}{(1 + p_2^-(h))(1 - p_1^-(h))}, \frac{q_2^-(h) + q_1^-(h)}{(1 + q_2^-(h))(1 - q_1^-(h))}, \frac{r_2^-(h) + r_1^-(h)}{(1 + r_2^-(h))(1 - r_1^-(h))} \right], \right. \\ \left. \left[\frac{p_2^+(h) + p_1^+(h)}{(1 + p_2^+(h))(1 - p_1^+(h))}, \frac{q_2^+(h) + q_1^+(h)}{(1 + q_2^+(h))(1 - q_1^+(h))}, \frac{r_2^+(h) + r_1^+(h)}{(1 + r_2^+(h))(1 - r_1^+(h))} \right] \right\rangle \\ \left. \left[\frac{p_2(h)p_1(h)}{1 + ((1 - p_2(h))(1 - p_1(h)))}, \frac{q_2(h)q_1(h)}{1 + ((1 - q_2(h))(1 - q_1(h)))}, \frac{r_2(h)r_1(h)}{1 + ((1 - r_2(h))(1 - r_1(h)))} \right] \right\rangle \Bigg\} = A_2 + A_1$$

Hence $A_1 + A_2 = A_2 + A_1$.

$$(2) \lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$$

$$\lambda(A_1 + A_2) = \left\langle \left[\frac{[(1 + \alpha_1(h))(1 - \alpha_1(h))]^\lambda [(1 + \alpha_2(h))(1 - \alpha_2(h))]^\lambda}{[(1 + \alpha_1(h))(1 - \alpha_1(h))]^\lambda [(1 + \alpha_2(h))(1 - \alpha_2(h))]^\lambda}, \right. \right. \\ \left. \frac{[(1 + \beta_1(h))(1 - \beta_1(h))]^\lambda [(1 + \beta_2(h))(1 - \beta_2(h))]^\lambda}{[(1 + \beta_1(h))(1 - \beta_1(h))]^\lambda [(1 + \beta_2(h))(1 - \beta_2(h))]^\lambda}, \right. \\ \left. \frac{[(1 + \Delta_1(h))(1 - \Delta_1(h))]^\lambda [(1 + \Delta_2(h))(1 - \Delta_2(h))]^\lambda}{[(1 + \Delta_1(h))(1 - \Delta_1(h))]^\lambda [(1 + \Delta_2(h))(1 - \Delta_2(h))]^\lambda} \right] \\ \left. \left[\frac{[(1 + p_1^-(h))(1 - p_1^-(h))]^\lambda [(1 + p_2^-(h))(1 - p_2^-(h))]^\lambda}{[(1 + p_1^-(h))(1 - p_1^-(h))]^\lambda [(1 + p_2^-(h))(1 - p_2^-(h))]^\lambda}, \right. \right. \\ \left. \frac{[(1 + q_1^-(h))(1 - q_1^-(h))]^\lambda [(1 + q_2^-(h))(1 - q_2^-(h))]^\lambda}{[(1 + q_1^-(h))(1 - q_1^-(h))]^\lambda [(1 + q_2^-(h))(1 - q_2^-(h))]^\lambda}, \right. \\ \left. \frac{[(1 + r_1^-(h))(1 - r_1^-(h))]^\lambda [(1 + r_2^-(h))(1 - r_2^-(h))]^\lambda}{[(1 + r_1^-(h))(1 - r_1^-(h))]^\lambda [(1 + r_2^-(h))(1 - r_2^-(h))]^\lambda} \right] \\ \left. \left[\frac{[(1 + p_1^+(h))(1 - p_1^+(h))]^\lambda [(1 + p_2^+(h))(1 - p_2^+(h))]^\lambda}{[(1 + p_1^+(h))(1 - p_1^+(h))]^\lambda [(1 + p_2^+(h))(1 - p_2^+(h))]^\lambda}, \right. \right. \\ \left. \frac{[(1 + q_1^+(h))(1 - q_1^+(h))]^\lambda [(1 + q_2^+(h))(1 - q_2^+(h))]^\lambda}{[(1 + q_1^+(h))(1 - q_1^+(h))]^\lambda [(1 + q_2^+(h))(1 - q_2^+(h))]^\lambda}, \right. \\ \left. \frac{[(1 + r_1^+(h))(1 - r_1^+(h))]^\lambda [(1 + r_2^+(h))(1 - r_2^+(h))]^\lambda}{[(1 + r_1^+(h))(1 - r_1^+(h))]^\lambda [(1 + r_2^+(h))(1 - r_2^+(h))]^\lambda} \right] \\ \left. \left[\frac{2[p_1(h)p_2(h)]^\lambda}{[(4 - 2p_1(h) - 2p_2(h) - p_1(h)p_2(h))]^\lambda + [p_1(h)p_2(h)]^\lambda}, \right. \right. \\ \left. \frac{2[q_1(h)q_2(h)]^\lambda}{[(4 - 2q_1(h) - 2q_2(h) - q_1(h)q_2(h))]^\lambda + [q_1(h)q_2(h)]^\lambda}, \right. \\ \left. \frac{2[r_1(h)r_2(h)]^\lambda}{[(4 - 2r_1(h) - 2r_2(h) - r_1(h)r_2(h))]^\lambda + [r_1(h)r_2(h)]^\lambda} \right] \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \left[\frac{[(1 + \alpha_1(h))]^\lambda - [(1 - \alpha_1(h))]^\lambda}{[(1 + \alpha_1(h))]^\lambda + [(1 - \alpha_1(h))]^\lambda}, \right. \right. \\ \left. \frac{[(1 + \beta_1(h))]^\lambda - [(1 - \beta_1(h))]^\lambda}{[(1 + \beta_1(h))]^\lambda + [(1 - \beta_1(h))]^\lambda}, \right. \\ \left. \frac{[(1 + \Delta_1(h))]^\lambda - [(1 - \Delta_1(h))]^\lambda}{[(1 + \Delta_1(h))]^\lambda + [(1 - \Delta_1(h))]^\lambda} \right] \\ \left. \left[\frac{[(1 + p_1^-(h))]^\lambda - (1 - p_1^-(h))^\lambda}{[(1 + p_1^-(h))]^\lambda + (1 - p_1^-(h))^\lambda}, \right. \right. \\ \left. \frac{[(1 + q_1^-(h))]^\lambda - (1 - q_1^-(h))^\lambda}{[(1 + q_1^-(h))]^\lambda + (1 - q_1^-(h))^\lambda}, \right. \\ \left. \frac{[(1 + r_1^-(h))]^\lambda - (1 - r_1^-(h))^\lambda}{[(1 + r_1^-(h))]^\lambda + (1 - r_1^-(h))^\lambda} \right] \\ \left. \left[\frac{[(1 + p_1^+(h))]^\lambda - (1 - p_1^+(h))^\lambda}{[(1 + p_1^+(h))]^\lambda + (1 - p_1^+(h))^\lambda}, \right. \right. \\ \left. \frac{[(1 + q_1^+(h))]^\lambda - (1 - q_1^+(h))^\lambda}{[(1 + q_1^+(h))]^\lambda + (1 - q_1^+(h))^\lambda}, \right. \\ \left. \frac{[(1 + r_1^+(h))]^\lambda - (1 - r_1^+(h))^\lambda}{[(1 + r_1^+(h))]^\lambda + (1 - r_1^+(h))^\lambda} \right] \\ \left. \left[\frac{2p_1^\lambda(h)}{[(2 - p_1(h))]^\lambda + [p_1(h)]^\lambda}, \right. \right. \\ \left. \frac{2q_1^\lambda(h)}{[(2 - q_1(h))]^\lambda + [q_1(h)]^\lambda}, \right. \\ \left. \frac{2r_1^\lambda(h)}{[(2 - r_1(h))]^\lambda + [r_1(h)]^\lambda} \right] \right\rangle$$

$$\lambda A_2 = \left\langle \left[\frac{[(1+\alpha_2(h))^\lambda - (1-\alpha_2(h))^\lambda]}{[(1+\alpha_2(h))^\lambda + (1-\alpha_2(h))^\lambda]}, \frac{[(1+\beta_2(h))^\lambda - (1-\beta_2(h))^\lambda]}{[(1+\beta_2(h))^\lambda + (1-\beta_2(h))^\lambda]}, \frac{[(1+\Delta_2(h))^\lambda - (1-\Delta_2(h))^\lambda]}{[(1+\Delta_2(h))^\lambda + (1-\Delta_2(h))^\lambda]} \right], \left[\frac{[(1+p_2^-(h))^\lambda - (1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda + (1-p_2^-(h))^\lambda]}, \frac{[(1+q_2^-(h))^\lambda - (1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda + (1-q_2^-(h))^\lambda]}, \frac{[(1+r_2^-(h))^\lambda - (1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda + (1-r_2^-(h))^\lambda]}, \frac{[(1+p_2^+(h))^\lambda - (1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda + (1-p_2^+(h))^\lambda]}, \frac{[(1+q_2^+(h))^\lambda - (1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda + (1-q_2^+(h))^\lambda]}, \frac{[(1+r_2^+(h))^\lambda - (1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda + (1-r_2^+(h))^\lambda]} \right], \left[\frac{2p_2^\lambda(h)}{[(2-p_2(h))^\lambda + [p_2(h)]^\lambda]}, \frac{2q_2^\lambda(h)}{[(2-q_2(h))^\lambda + [q_2(h)]^\lambda]}, \frac{2r_2^\lambda(h)}{[(2-r_2(h))^\lambda + [r_2(h)]^\lambda]} \right] \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \left[\frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda [(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda [(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}, \frac{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda [(1+\beta_1(h))(1-\beta_1(h))]^\lambda}{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda [(1+\beta_1(h))(1-\beta_1(h))]^\lambda}, \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda [(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda}{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda [(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda} \right], \left[\frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda [(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda [(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda [(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda [(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda [(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda [(1+r_1^-(h))(1-r_1^-(h))]^\lambda}, \frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda [(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda [(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda [(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda [(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda [(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda [(1+r_1^+(h))(1-r_1^+(h))]^\lambda} \right], \left[\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h))^\lambda + [p_2(h)p_1(h)]^\lambda]}, \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h))^\lambda + [q_2(h)q_1(h)]^\lambda]}, \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h))^\lambda + [r_2(h)r_1(h)]^\lambda]} \right] \right\rangle$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

$$(3) \lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2) A$$

$$\lambda_1 A = \left\langle \left[\frac{[(1+\alpha_A(h))^{\lambda_1} - (1-\alpha_A(h))^{\lambda_1}]}{[(1+\alpha_A(h))^{\lambda_1} + (1-\alpha_A(h))^{\lambda_1}]}, \frac{[(1+\beta_A(h))^{\lambda_1} - (1-\beta_A(h))^{\lambda_1}]}{[(1+\beta_A(h))^{\lambda_1} + (1-\beta_A(h))^{\lambda_1}]}, \frac{[(1+\Delta_A(h))^{\lambda_1} - (1-\Delta_A(h))^{\lambda_1}]}{[(1+\Delta_A(h))^{\lambda_1} + (1-\Delta_A(h))^{\lambda_1}]} \right], \left[\frac{[(1+p_A^-(h))^{\lambda_1} - (1-p_A^-(h))^{\lambda_1}]}{[(1+p_A^-(h))^{\lambda_1} + (1-p_A^-(h))^{\lambda_1}]}, \frac{[(1+q_A^-(h))^{\lambda_1} - (1-q_A^-(h))^{\lambda_1}]}{[(1+q_A^-(h))^{\lambda_1} + (1-q_A^-(h))^{\lambda_1}]}, \frac{[(1+r_A^-(h))^{\lambda_1} - (1-r_A^-(h))^{\lambda_1}]}{[(1+r_A^-(h))^{\lambda_1} + (1-r_A^-(h))^{\lambda_1}]}, \frac{[(1+p_A^+(h))^{\lambda_1} - (1-p_A^+(h))^{\lambda_1}]}{[(1+p_A^+(h))^{\lambda_1} + (1-p_A^+(h))^{\lambda_1}]}, \frac{[(1+q_A^+(h))^{\lambda_1} - (1-q_A^+(h))^{\lambda_1}]}{[(1+q_A^+(h))^{\lambda_1} + (1-q_A^+(h))^{\lambda_1}]}, \frac{[(1+r_A^+(h))^{\lambda_1} - (1-r_A^+(h))^{\lambda_1}]}{[(1+r_A^+(h))^{\lambda_1} + (1-r_A^+(h))^{\lambda_1}]} \right], \left[\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h))^{\lambda_1} + [p_A(h)]^{\lambda_1}]}, \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h))^{\lambda_1} + [q_A(h)]^{\lambda_1}]}, \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h))^{\lambda_1} + [r_A(h)]^{\lambda_1}]} \right] \right\rangle$$

$$\text{and } \lambda_2 A = \left\langle \left[\frac{[(1+\alpha_A(h))^{\lambda_2} - (1-\alpha_A(h))^{\lambda_2}]}{[(1+\alpha_A(h))^{\lambda_2} + (1-\alpha_A(h))^{\lambda_2}]}, \frac{[(1+\beta_A(h))^{\lambda_2} - (1-\beta_A(h))^{\lambda_2}]}{[(1+\beta_A(h))^{\lambda_2} + (1-\beta_A(h))^{\lambda_2}]}, \frac{[(1+\Delta_A(h))^{\lambda_2} - (1-\Delta_A(h))^{\lambda_2}]}{[(1+\Delta_A(h))^{\lambda_2} + (1-\Delta_A(h))^{\lambda_2}]} \right], \left[\frac{[(1+p_A^-(h))^{\lambda_2} - (1-p_A^-(h))^{\lambda_2}]}{[(1+p_A^-(h))^{\lambda_2} + (1-p_A^-(h))^{\lambda_2}]}, \frac{[(1+q_A^-(h))^{\lambda_2} - (1-q_A^-(h))^{\lambda_2}]}{[(1+q_A^-(h))^{\lambda_2} + (1-q_A^-(h))^{\lambda_2}]}, \frac{[(1+r_A^-(h))^{\lambda_2} - (1-r_A^-(h))^{\lambda_2}]}{[(1+r_A^-(h))^{\lambda_2} + (1-r_A^-(h))^{\lambda_2}]}, \frac{[(1+p_A^+(h))^{\lambda_2} - (1-p_A^+(h))^{\lambda_2}]}{[(1+p_A^+(h))^{\lambda_2} + (1-p_A^+(h))^{\lambda_2}]}, \frac{[(1+q_A^+(h))^{\lambda_2} - (1-q_A^+(h))^{\lambda_2}]}{[(1+q_A^+(h))^{\lambda_2} + (1-q_A^+(h))^{\lambda_2}]}, \frac{[(1+r_A^+(h))^{\lambda_2} - (1-r_A^+(h))^{\lambda_2}]}{[(1+r_A^+(h))^{\lambda_2} + (1-r_A^+(h))^{\lambda_2}]} \right], \left[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h))^{\lambda_2} + [p_A(h)]^{\lambda_2}]}, \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h))^{\lambda_2} + [q_A(h)]^{\lambda_2}]}, \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h))^{\lambda_2} + [r_A(h)]^{\lambda_2}]} \right] \right\rangle =$$

$$\left\langle \left[\begin{array}{c} \frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}-[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}+[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}, \frac{[1+\beta_A(h)]^{\lambda_1+\lambda_2}-[1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2}+[1-\beta_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}-[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}+[1-\Delta_A(h)]^{\lambda_1+\lambda_2}} \end{array} \right], \right. \\ \left. \left[\begin{array}{c} \frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}} \end{array} \right], \right. \\ \left. \left[\begin{array}{c} \frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}]}, \\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}]}, \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}]} \end{array} \right] \right\rangle = (\lambda_1 + \lambda_2)A.$$

Appendix B. Proof of Theorem 1

Assume that $n = 1$, TCFEWA $(A_1, A_2, \dots, A_n) = \bigoplus_{j=1}^k w_j A_1 \langle (\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1 + A_2) = \left[\begin{array}{c} \frac{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda [(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda [(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda [(1+\beta_2(h))(1-\beta_2(h))]^\lambda}{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda [(1+\beta_2(h))(1-\beta_2(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda [(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda}{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda [(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda} \end{array} \right], \left\langle \left[\begin{array}{c} \frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda}, \\ \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda}, \\ \frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda}, \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda} \end{array} \right], \left[\begin{array}{c} \frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h))]^\lambda + [p_1(h)p_2(h)]^\lambda}, \\ \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h))]^\lambda + [q_1(h)q_2(h)]^\lambda}, \\ \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h))]^\lambda + [r_1(h)r_2(h)]^\lambda} \end{array} \right] \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_1(h))^\lambda - (1-\alpha_1(h))^\lambda]}{[(1+\alpha_1(h))^\lambda + (1-\alpha_1(h))^\lambda]^\lambda}, \\ \frac{[(1+\beta_1(h))^\lambda - (1-\beta_1(h))^\lambda]}{[(1+\beta_1(h))^\lambda + (1-\beta_1(h))^\lambda]^\lambda}, \\ \frac{[(1+\Delta_1(h))^\lambda - (1-\Delta_1(h))^\lambda]}{[(1+\Delta_1(h))^\lambda + (1-\Delta_1(h))^\lambda]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{[(1+p_1^-(h))^\lambda - (1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda + (1-p_1^-(h))^\lambda]^\lambda}, \\ \frac{[(1+q_1^-(h))^\lambda - (1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda + (1-q_1^-(h))^\lambda]^\lambda}, \\ \frac{[(1+r_1^-(h))^\lambda - (1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda + (1-r_1^-(h))^\lambda]^\lambda}, \\ \frac{[(1+p_1^+(h))^\lambda - (1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda + (1-p_1^+(h))^\lambda]^\lambda}, \\ \frac{[(1+q_1^+(h))^\lambda - (1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda + (1-q_1^+(h))^\lambda]^\lambda}, \\ \frac{[(1+r_1^+(h))^\lambda - (1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda + (1-r_1^+(h))^\lambda]^\lambda}, \\ \frac{[(1+p_2^-(h))^\lambda - (1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda + (1-p_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+q_2^-(h))^\lambda - (1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda + (1-q_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+r_2^-(h))^\lambda - (1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda + (1-r_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+p_2^+(h))^\lambda - (1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda + (1-p_2^+(h))^\lambda]^\lambda}, \\ \frac{[(1+q_2^+(h))^\lambda - (1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda + (1-q_2^+(h))^\lambda]^\lambda}, \\ \frac{[(1+r_2^+(h))^\lambda - (1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda + (1-r_2^+(h))^\lambda]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{2p_1^\lambda(h)}{[(2-p_1(h))^\lambda + [p_1(h)]^\lambda]^\lambda}, \\ \frac{2q_1^\lambda(h)}{[(2-q_1(h))^\lambda + [q_1(h)]^\lambda]^\lambda}, \\ \frac{2r_1^\lambda(h)}{[(2-r_1(h))^\lambda + [r_1(h)]^\lambda]^\lambda} \end{bmatrix} \right\rangle$$

$$\lambda A_2 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))^\lambda - (1-\alpha_2(h))^\lambda]}{[(1+\alpha_2(h))^\lambda + (1-\alpha_2(h))^\lambda]^\lambda}, \\ \frac{[(1+\beta_2(h))^\lambda - (1-\beta_2(h))^\lambda]}{[(1+\beta_2(h))^\lambda + (1-\beta_2(h))^\lambda]^\lambda}, \\ \frac{[(1+\Delta_2(h))^\lambda - (1-\Delta_2(h))^\lambda]}{[(1+\Delta_2(h))^\lambda + (1-\Delta_2(h))^\lambda]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{[(1+p_2^-(h))^\lambda - (1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda + (1-p_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+q_2^-(h))^\lambda - (1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda + (1-q_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+r_2^-(h))^\lambda - (1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda + (1-r_2^-(h))^\lambda]^\lambda}, \\ \frac{[(1+p_2^+(h))^\lambda - (1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda + (1-p_2^+(h))^\lambda]^\lambda}, \\ \frac{[(1+q_2^+(h))^\lambda - (1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda + (1-q_2^+(h))^\lambda]^\lambda}, \\ \frac{[(1+r_2^+(h))^\lambda - (1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda + (1-r_2^+(h))^\lambda]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{2p_2^\lambda(h)}{[(2-p_2(h))^\lambda + [p_2(h)]^\lambda]^\lambda}, \\ \frac{2q_2^\lambda(h)}{[(2-q_2(h))^\lambda + [q_2(h)]^\lambda]^\lambda}, \\ \frac{2r_2^\lambda(h)}{[(2-r_2(h))^\lambda + [r_2(h)]^\lambda]^\lambda} \end{bmatrix} \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda [(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda [(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda [(1+\beta_1(h))(1-\beta_1(h))]^\lambda}{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda [(1+\beta_1(h))(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda [(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda}{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda [(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda [(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda [(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda [(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda [(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda [(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda [(1+r_1^-(h))(1-r_1^-(h))]^\lambda}, \\ \frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda [(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda [(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda [(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda [(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda [(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda [(1+r_1^+(h))(1-r_1^+(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} \frac{2p_2(h)p_1(h)}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h))^\lambda + [p_2(h)p_1(h)]^\lambda]^\lambda}, \\ \frac{2q_2(h)q_1(h)}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h))^\lambda + [q_2(h)q_1(h)]^\lambda]^\lambda}, \\ \frac{2r_2(h)r_1(h)}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h))^\lambda + [s_2(h)s_1(h)]^\lambda]^\lambda} \end{bmatrix} \right\rangle$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

$$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2) A$$

$$\lambda_1 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1} - [1-\alpha_A(h)]^{\lambda_1}}{[1+\alpha_A(h)]^{\lambda_1} + [1-\alpha_A(h)]^{\lambda_1}}, \\ \frac{[1+\beta_A(h)]^{\lambda_1} - [1-\beta_A(h)]^{\lambda_1}}{[1+\beta_A(h)]^{\lambda_1} + [1-\beta_A(h)]^{\lambda_1}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1} - [1-\Delta_A(h)]^{\lambda_1}}{[1+\Delta_A(h)]^{\lambda_1} + [1-\Delta_A(h)]^{\lambda_1}} \end{bmatrix}, \begin{bmatrix} \frac{[1+p_A^-(h)]^{\lambda_1} - [1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1} + [1-p_A^-(h)]^{\lambda_1}}, \\ \frac{[1+q_A^-(h)]^{\lambda_1} - [1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1} + [1-q_A^-(h)]^{\lambda_1}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1} - [1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1} + [1-r_A^-(h)]^{\lambda_1}}, \\ \frac{[1+p_A^+(h)]^{\lambda_1} - [1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1} + [1-p_A^+(h)]^{\lambda_1}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1} - [1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1} + [1-q_A^+(h)]^{\lambda_1}}, \\ \frac{[1+r_A^+(h)]^{\lambda_1} - [1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1} + [1-r_A^+(h)]^{\lambda_1}} \end{bmatrix}, \begin{bmatrix} \frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h))^\lambda + [p_A(h)]^\lambda]^\lambda}, \\ \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h))^\lambda + [q_A(h)]^\lambda]^\lambda}, \\ \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h))^\lambda + [r_A(h)]^\lambda]^\lambda} \end{bmatrix} \right\rangle$$

$$\text{and } \lambda_2 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_2} - [1-\alpha_A(h)]^{\lambda_2}}{[1+\alpha_A(h)]^{\lambda_2} + [1-\alpha_A(h)]^{\lambda_2}}, \\ \frac{[1+\beta_A(h)]^{\lambda_2} - [1-\beta_A(h)]^{\lambda_2}}{[1+\beta_A(h)]^{\lambda_2} + [1-\beta_A(h)]^{\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_2} - [1-\Delta_A(h)]^{\lambda_2}}{[1+\Delta_A(h)]^{\lambda_2} + [1-\Delta_A(h)]^{\lambda_2}} \end{bmatrix}, \begin{bmatrix} \frac{[1+p_A^-(h)]^{\lambda_2} - [1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2} + [1-p_A^-(h)]^{\lambda_2}}, \\ \frac{[1+q_A^-(h)]^{\lambda_2} - [1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2} + [1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2} - [1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2} + [1-r_A^-(h)]^{\lambda_2}}, \\ \frac{[1+p_A^+(h)]^{\lambda_2} - [1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2} + [1-p_A^+(h)]^{\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_2} - [1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2} + [1-q_A^+(h)]^{\lambda_2}}, \\ \frac{[1+r_A^+(h)]^{\lambda_2} - [1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2} + [1-r_A^+(h)]^{\lambda_2}} \end{bmatrix}, \begin{bmatrix} \frac{2p_A^\lambda(h)}{[(2-p_A(h))^\lambda + [p_A(h)]^\lambda]^\lambda}, \\ \frac{2q_A^\lambda(h)}{[(2-q_A(h))^\lambda + [q_A(h)]^\lambda]^\lambda}, \\ \frac{2r_A^\lambda(h)}{[(2-r_A(h))^\lambda + [r_A(h)]^\lambda]^\lambda} \end{bmatrix} \right\rangle$$

$$\begin{aligned}
& \left[\frac{\left[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h))^{\lambda_2} + [p_A(h)]^{\lambda_2}]}, \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h))^{\lambda_2} + [q_A(h)]^{\lambda_2}]} \right]}{\left[\frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h))^{\lambda_2} + [q_A(h)]^{\lambda_2}]} \right]} \right] \Bigg\rangle = \\
& \left\langle \left[\frac{\frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2} - [1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2} + [1-\alpha_A(h)]^{\lambda_1+\lambda_2}}, \frac{[\beta_A(h)]^{\lambda_1+\lambda_2} - [1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2} + [1-\beta_A(h)]^{\lambda_1+\lambda_2}}, \right. \right. \\
& \quad \left. \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2} - [1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2} + [1-\Delta_A(h)]^{\lambda_1+\lambda_2}} \right] \\
& \quad \left[\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2} - [1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2} + [1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2} - [1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2} + [1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \right. \\
& \quad \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2} - [1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2} + [1-r_A^-(h)]^{\lambda_1+\lambda_2}}, \left. \frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2} - [1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2} + [1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \right. \\
& \quad \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2} - [1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2} + [1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \left. \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2} - [1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2} + [1-r_A^+(h)]^{\lambda_1+\lambda_2}} \right] \\
& \quad \left[\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h))^{\lambda_1+\lambda_2} + [p_A(h)]^{\lambda_1+\lambda_2}]}, \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h))^{\lambda_1+\lambda_2} + [r_A(h)]^{\lambda_1+\lambda_2}]} \right] \Bigg\rangle = \\
& \left[\frac{[1+\alpha_1(h)]^{\lambda_1} - [1-\alpha_1(h)]^{\lambda_1}}{[1+\alpha_1(h)]^{\lambda_1} + [1-\alpha_1(h)]^{\lambda_1}}, \frac{[1+\beta_1(h)]^{\lambda_1} - [1-\beta_1(h)]^{\lambda_1}}{[1+\beta_1(h)]^{\lambda_1} + [1-\beta_1(h)]^{\lambda_1}}, \frac{[1+\Delta_1(h)]^{\lambda_1} - [1-\Delta_1(h)]^{\lambda_1}}{[1+\Delta_1(h)]^{\lambda_1} + [1-\Delta_1(h)]^{\lambda_1}} \right] \\
& \quad \left[\frac{[1+p_1^-(h)]^{\lambda_1} - [1-p_1^-(h)]^{\lambda_1}}{[1+p_1^-(h)]^{\lambda_1} + [1-p_1^-(h)]^{\lambda_1}}, \frac{[1+q_1^-(h)]^{\lambda_1} - [1-q_1^-(h)]^{\lambda_1}}{[1+q_1^-(h)]^{\lambda_1} + [1-q_1^-(h)]^{\lambda_1}}, \right. \\
& \quad \left. \frac{[1+r_1^-(h)]^{\lambda_1} - [1-r_1^-(h)]^{\lambda_1}}{[1+r_1^-(h)]^{\lambda_1} + [1-r_1^-(h)]^{\lambda_1}} \right] ; \left[\frac{[1+p_1^+(h)]^{\lambda_1} - [1-p_1^+(h)]^{\lambda_1}}{[1+p_1^+(h)]^{\lambda_1} + [1-p_1^+(h)]^{\lambda_1}}, \frac{[1+q_1^+(h)]^{\lambda_1} - [1-q_1^+(h)]^{\lambda_1}}{[1+q_1^+(h)]^{\lambda_1} + [1-q_1^+(h)]^{\lambda_1}}, \right. \\
& \quad \left. \frac{[1+r_1^+(h)]^{\lambda_1} - [1-r_1^+(h)]^{\lambda_1}}{[1+r_1^+(h)]^{\lambda_1} + [1-r_1^+(h)]^{\lambda_1}} \right] ; \\
& \quad \left[\frac{2[p_1(h)]^{\lambda_1}}{[(2-p_1(h))^{\lambda_1} + [p_1(h)]^{\lambda_1}]}, \frac{2[q_1(h)]^{\lambda_1}}{[(2-q_1(h))^{\lambda_1} + [q_1(h)]^{\lambda_1}]} \right] \\
& \quad \frac{2[r_1(h)]^{\lambda_1}}{[(2-r_1(h))^{\lambda_1} + [r_1(h)]^{\lambda_1}]} \Bigg].
\end{aligned}$$

Assume that $n = k$, TCFEWA $(A_1, A_2, \dots, A_n) = \bigoplus_{j=1}^k w_j A_j$

$$\begin{aligned}
& \left\langle \left[\frac{\left[\prod_{j=1}^k [1+\alpha_1(h)]^\omega - \prod_{j=1}^k [1-\alpha_1(h)]^\omega \right]}{\prod_{j=1}^k [1+\alpha_1(h)]^\omega + \prod_{j=1}^k [1-\alpha_1(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+\beta_1(h)]^\omega - \prod_{j=1}^k [1-\beta_1(h)]^\omega \right]}{\prod_{j=1}^k [1+\beta_1(h)]^\omega + \prod_{j=1}^k [1-\beta_1(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+\Delta_1(h)]^\omega - \prod_{j=1}^k [1-\Delta_1(h)]^\omega \right]}{\prod_{j=1}^k [1+\Delta_1(h)]^\omega + \prod_{j=1}^k [1-\Delta_1(h)]^\omega} \right] \\
& \quad \left[\frac{\left[\prod_{j=1}^k [1+p_1^-(h)]^\omega - \prod_{j=1}^k [1-p_1^-(h)]^\omega \right]}{\prod_{j=1}^k [1+p_1^-(h)]^\omega + \prod_{j=1}^k [1-p_1^-(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+q_1^-(h)]^\omega - \prod_{j=1}^k [1-q_1^-(h)]^\omega \right]}{\prod_{j=1}^k [1+q_1^-(h)]^\omega + \prod_{j=1}^k [1-q_1^-(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+r_1^-(h)]^\omega - \prod_{j=1}^k [1-r_1^-(h)]^\omega \right]}{\prod_{j=1}^k [1+r_1^-(h)]^\omega + \prod_{j=1}^k [1-r_1^-(h)]^\omega} \right] \\
& \quad \left[\frac{\left[\prod_{j=1}^k [1+p_1^+(h)]^\omega - \prod_{j=1}^k [1-p_1^+(h)]^\omega \right]}{\prod_{j=1}^k [1+p_1^+(h)]^\omega + \prod_{j=1}^k [1-p_1^+(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+q_1^+(h)]^\omega - \prod_{j=1}^k [1-q_1^+(h)]^\omega \right]}{\prod_{j=1}^k [1+q_1^+(h)]^\omega + \prod_{j=1}^k [1-q_1^+(h)]^\omega}, \frac{\left[\prod_{j=1}^k [1+r_1^+(h)]^\omega - \prod_{j=1}^k [1-r_1^+(h)]^\omega \right]}{\prod_{j=1}^k [1+r_1^+(h)]^\omega + \prod_{j=1}^k [1-r_1^+(h)]^\omega} \right] ; \\
& \quad \left[\frac{2 \prod_{j=1}^k [p_1(h)]^\omega}{\prod_{j=1}^k [(2-p_1(h))^\omega + \prod_{j=1}^k [p_1(h)]^\omega]}, \frac{2 \prod_{j=1}^k [q_1(h)]^\omega}{\prod_{j=1}^k [(2-q_1(h))^\omega + \prod_{j=1}^k [q_1(h)]^\omega]}, \frac{2 \prod_{j=1}^k [r_1(h)]^\omega}{\prod_{j=1}^k [(2-r_1(h))^\omega + \prod_{j=1}^k [r_1(h)]^\omega]} \right].
\end{aligned}$$

Then when $n = k + 1$, we have TCFEWA $(A_1, A_2, \dots, A_{k+1}) = \text{TCFEWA}(A_1, A_2, \dots, A_k) \oplus A_{k+1}$

$$\begin{aligned}
& \left\langle \left[\frac{\prod_{j=1}^k [1+\alpha_1(h)]^\omega - \prod_{j=1}^k [1-\alpha_1(h)]^\omega}{\prod_{j=1}^k [1+\alpha_1(h)]^\omega + \prod_{j=1}^k [1-\alpha_1(h)]^\omega}, \frac{\prod_{j=1}^k [1+\beta_1(h)]^\omega - \prod_{j=1}^k [1-\beta_1(h)]^\omega}{\prod_{j=1}^k [1+\beta_1(h)]^\omega + \prod_{j=1}^k [1-\beta_1(h)]^\omega}, \frac{\prod_{j=1}^k [1+\Delta_1(h)]^\omega - \prod_{j=1}^k [1-\Delta_1(h)]^\omega}{\prod_{j=1}^k [1+\Delta_1(h)]^\omega + \prod_{j=1}^k [1-\Delta_1(h)]^\omega} \right] \right. \\
& \left. \left[\frac{\prod_{j=1}^k [1+p_1^-(h)]^\omega - \prod_{j=1}^k [1-p_1^-(h)]^\omega}{\prod_{j=1}^k [1+p_1^-(h)]^\omega + \prod_{j=1}^k [1-p_1^-(h)]^\omega}, \frac{\prod_{j=1}^k [1+q_1^-(h)]^\omega - \prod_{j=1}^k [1-q_1^-(h)]^\omega}{\prod_{j=1}^k [1+q_1^-(h)]^\omega + \prod_{j=1}^k [1-q_1^-(h)]^\omega}, \frac{\prod_{j=1}^k [1+r_1^-(h)]^\omega - \prod_{j=1}^k [1-r_1^-(h)]^\omega}{\prod_{j=1}^k [1+r_1^-(h)]^\omega + \prod_{j=1}^k [1-r_1^-(h)]^\omega} \right] \right. \\
& \left. \left[\frac{\prod_{j=1}^k [1+p_1^+(h)]^\omega - \prod_{j=1}^k [1-p_1^+(h)]^\omega}{\prod_{j=1}^k [1+p_1^+(h)]^\omega + \prod_{j=1}^k [1-p_1^+(h)]^\omega}, \frac{\prod_{j=1}^k [1+q_1^+(h)]^\omega - \prod_{j=1}^k [1-q_1^+(h)]^\omega}{\prod_{j=1}^k [1+q_1^+(h)]^\omega + \prod_{j=1}^k [1-q_1^+(h)]^\omega}, \frac{\prod_{j=1}^k [1+r_1^+(h)]^\omega - \prod_{j=1}^k [1-r_1^+(h)]^\omega}{\prod_{j=1}^k [1+r_1^+(h)]^\omega + \prod_{j=1}^k [1-r_1^+(h)]^\omega} \right] \right. \\
& \left. \left[\frac{2 \prod_{j=1}^k [p_1(h)]^\omega}{\prod_{j=1}^k [(2-p_1(h))^\omega + \prod_{j=1}^k [p_1(h)]^\omega]}, \frac{2 \prod_{j=1}^k [q_1(h)]^\omega}{\prod_{j=1}^k [(2-q_1(h))^\omega + \prod_{j=1}^k [q_1(h)]^\omega]}, \frac{2 \prod_{j=1}^k [r_1(h)]^\omega}{\prod_{j=1}^k [(2-r_1(h))^\omega + \prod_{j=1}^k [r_1(h)]^\omega]} \right] \oplus_{k+1} \right. \\
& \left\langle \left[\frac{\prod_{j=1}^{k+1} [1+\alpha_1(h)]^\omega - \prod_{j=1}^{k+1} [1-\alpha_1(h)]^\omega}{\prod_{j=1}^{k+1} [1+\alpha_1(h)]^\omega + \prod_{j=1}^{k+1} [1-\alpha_1(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+\beta_1(h)]^\omega - \prod_{j=1}^{k+1} [1-\beta_1(h)]^\omega}{\prod_{j=1}^{k+1} [1+\beta_1(h)]^\omega + \prod_{j=1}^{k+1} [1-\beta_1(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+\Delta_1(h)]^\omega - \prod_{j=1}^{k+1} [1-\Delta_1(h)]^\omega}{\prod_{j=1}^{k+1} [1+\Delta_1(h)]^\omega + \prod_{j=1}^{k+1} [1-\Delta_1(h)]^\omega} \right] \right. \\
& \left. \left[\frac{\prod_{j=1}^{k+1} [1+p_1^-(h)]^\omega - \prod_{j=1}^{k+1} [1-p_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+p_1^-(h)]^\omega + \prod_{j=1}^{k+1} [1-p_1^-(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+q_1^-(h)]^\omega - \prod_{j=1}^{k+1} [1-q_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+q_1^-(h)]^\omega + \prod_{j=1}^{k+1} [1-q_1^-(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+r_1^-(h)]^\omega - \prod_{j=1}^{k+1} [1-r_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+r_1^-(h)]^\omega + \prod_{j=1}^{k+1} [1-r_1^-(h)]^\omega} \right] \right. \\
& \left. \left[\frac{\prod_{j=1}^{k+1} [1+p_1^+(h)]^\omega - \prod_{j=1}^{k+1} [1-p_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+p_1^+(h)]^\omega + \prod_{j=1}^{k+1} [1-p_1^+(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+q_1^+(h)]^\omega - \prod_{j=1}^{k+1} [1-q_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+q_1^+(h)]^\omega + \prod_{j=1}^{k+1} [1-q_1^+(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+r_1^+(h)]^\omega - \prod_{j=1}^{k+1} [1-r_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+r_1^+(h)]^\omega + \prod_{j=1}^{k+1} [1-r_1^+(h)]^\omega} \right] \right. \\
& \left. \left[\frac{2 \prod_{j=1}^{k+1} [p_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-p_1(h))^\omega + \prod_{j=1}^{k+1} [p_1(h)]^\omega]}, \frac{2 \prod_{j=1}^{k+1} [q_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-q_1(h))^\omega + \prod_{j=1}^{k+1} [q_1(h)]^\omega]}, \frac{2 \prod_{j=1}^{k+1} [r_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-r_1(h))^\omega + \prod_{j=1}^{k+1} [r_1(h)]^\omega]} \right] \right. \\
& = \left[\frac{\prod_{j=1}^{k+1} [1+p_1^-(h)]^\omega - \prod_{j=1}^k [1-p_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+p_1^-(h)]^\omega + \prod_{j=1}^k [1-p_1^-(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+q_1^-(h)]^\omega - \prod_{j=1}^k [1-q_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+q_1^-(h)]^\omega + \prod_{j=1}^k [1-q_1^-(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+r_1^-(h)]^\omega - \prod_{j=1}^k [1-r_1^-(h)]^\omega}{\prod_{j=1}^{k+1} [1+r_1^-(h)]^\omega + \prod_{j=1}^k [1-r_1^-(h)]^\omega} \right] \\
& \left[\frac{\prod_{j=1}^{k+1} [1+p_1^+(h)]^\omega - \prod_{j=1}^k [1-p_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+p_1^+(h)]^\omega + \prod_{j=1}^k [1-p_1^+(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+q_1^+(h)]^\omega - \prod_{j=1}^k [1-q_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+q_1^+(h)]^\omega + \prod_{j=1}^k [1-q_1^+(h)]^\omega}, \frac{\prod_{j=1}^{k+1} [1+r_1^+(h)]^\omega - \prod_{j=1}^k [1-r_1^+(h)]^\omega}{\prod_{j=1}^{k+1} [1+r_1^+(h)]^\omega + \prod_{j=1}^k [1-r_1^+(h)]^\omega} \right] \\
& \left[\frac{2 \prod_{j=1}^{k+1} [p_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-p_1(h))^\omega + \prod_{j=1}^{k+1} [p_1(h)]^\omega]}, \frac{2 \prod_{j=1}^{k+1} [q_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-q_1(h))^\omega + \prod_{j=1}^{k+1} [q_1(h)]^\omega]}, \frac{2 \prod_{j=1}^{k+1} [r_1(h)]^\omega}{\prod_{j=1}^{k+1} [(2-r_1(h))^\omega + \prod_{j=1}^{k+1} [r_1(h)]^\omega]} \right].
\end{aligned}$$

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the the TNCFEWA operator is reduced to the triangular neutrosophic cubic fuzzy einstein averaging operator, which is shown as follows:

$$\left\langle \left[\frac{\left[\prod_{j=1}^n [1+\alpha_1(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-\alpha_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+\alpha_1(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-\alpha_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+\beta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-\beta_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+\beta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-\beta_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+\Delta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-\Delta_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+\Delta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-\Delta_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}} \right] \\ \left[\frac{\left[\prod_{j=1}^n [1+p_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-p_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+p_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-p_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+q_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-q_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+q_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-q_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+r_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-r_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+r_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-r_1^-(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}} \right]; \\ \left[\frac{\left[\prod_{j=1}^n [1+p_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-p_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+p_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-p_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+q_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-q_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+q_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-q_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{\left[\prod_{j=1}^n [1+r_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^n [1-r_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [1+r_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^n [1-r_1^+(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}} \right]; \\ \left[\frac{2 \prod_{j=1}^n [p_1(h)]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [(2-p_1(h))]^{\frac{1}{n}} + \prod_{j=1}^n [p_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{2 \prod_{j=1}^n [q_1(h)]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [(2-q_1(h))]^{\frac{1}{n}} + \prod_{j=1}^n [q_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}}, \frac{2 \prod_{j=1}^n [r_1(h)]^{\frac{1}{n}}}{\left[\prod_{j=1}^n [(2-r_1(h))]^{\frac{1}{n}} + \prod_{j=1}^n [r_1(h)]^{\frac{1}{n}} \right]^{\frac{1}{n}}} \right] \right\}.$$

References

- Atanassov, K.T. Intuitionistic Fuzzy Sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [\[CrossRef\]](#)
- Atanassov, K.T. New Operations Defined Over the Intuitionistic Fuzzy Sets. *Fuzzy Sets Syst.* **1994**, *61*, 137–142. [\[CrossRef\]](#)
- Bustince, H.; Burillo, P. Structures on intuitionistic fuzzy relations. *Fuzzy Sets Syst.* **1996**, *78*, 293–303. [\[CrossRef\]](#)
- Deschrijver, G.; Kerre, E.E. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets Syst.* **2003**, *133*, 227–235. [\[CrossRef\]](#)
- Deschrijver, G.; Kerre, E.E. On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision. *Inf. Sci.* **2007**, *177*, 1860–1866. [\[CrossRef\]](#)
- Jun, Y.B.; Kim, C.S.; Yang, K.O. Annals of Fuzzy Mathematics and Informatics. *Cubic Sets* **2011**, *4*, 83–98.
- Muhiuddin, G.; Feng, F.; Jun, Y.B. Subalgebras of BCK/BCI-Algebras Based on Cubic Soft Sets. *Sci. World J.* **2014**, *2014*, 458638. [\[CrossRef\]](#)
- Turksen, I.B. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets Syst.* **1986**, *20*, 191–210. [\[CrossRef\]](#)
- Xu, Z.S. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
- Xu, Z.S.; Yager, R.R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *Int. J. Gener. Syst.* **2006**, *35*, 417–433. [\[CrossRef\]](#)
- Xu, Z.S.; Cai, X. Recent advances in intuitionistic fuzzy information aggregation. *Fuzzy Optim. Decis. Mak.* **2010**, *9*, 359–381. [\[CrossRef\]](#)
- Liu, H.W.; Wang, G.-J. Multi-criteria decision-making methods based on intuitionistic fuzzy sets. *Eur. J. Oper. Res.* **2007**, *179*, 220–233. [\[CrossRef\]](#)
- Zeng, S.Z.; Su, W.H. Intuitionistic fuzzy ordered weighted distance operator. *Knowl.-Based Syst.* **2011**, *24*, 1224–1232. [\[CrossRef\]](#)
- Zadeh, L.A. Fuzzy sets. *Inform. Contr.* **1965**, *8*, 338–353. [\[CrossRef\]](#)
- Zadeh, L.A. Outline of a new approach to analysis of complex systems and decision processes interval-valued fuzzy sets. *IEEE Trans. Syst. Man Cybern.* **1973**, *3*, 28–44. [\[CrossRef\]](#)
- Li, J.; Zhang, X.L.; Gong, Z.-T. Aggregating of Interval-valued Intuitionistic Uncertain Linguistic Variables based on Archimedean t-norm and It Applications in Group Decision Makings. *J. Comput. Anal. Appl.* **2018**, *24*, 874–885.
- Zhao, X.; Lin, R.; Wei, G. Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making. *Expert Syst. Appl.* **2014**, *41*, 1086–1094. [\[CrossRef\]](#)
- Xu, Z.S.; Yager, R.R. Dynamic intuitionistic fuzzy multi-attribute decision making. *Int. J. Approx. Reason.* **2008**, *48*, 246–262. [\[CrossRef\]](#)

19. Smarandache, F. Neutrosophic set—A generalization of the intuitionistic fuzzy set. In Proceedings of the 2006 IEEE International Conference on Granular Computing, Atlanta, GA, USA, 10–12 May 2006; pp. 38–42.
20. Smarandache, F. A geometric interpretation of the neutrosophic set—A generalization of the intuitionistic fuzzy set. In Proceedings of the 2011 IEEE International Conference on Granular Computing (GrC), Kaohsiung, Taiwan, 8–10 November 2011; pp. 602–606.
21. Fahmi, A.; Abdullah, S.; Amin, F.; Ali, A. Weighted average rating (WAR) method for solving group decision making problem using triangular cubic fuzzy hybrid aggregation (TCFHA). *Punjab Univ. J. Math.* **2018**, *50*, 23–34.
22. Fahmi, A.; Amin, F.; Smarandache, F.; Khan, M.; Hassan, N. Triangular Cubic Hesitant Fuzzy Einstein Hybrid Weighted Averaging Operator and Its Application to Decision Making. *Symmetry* **2018**, *10*, 658. [\[CrossRef\]](#)
23. Beg, I.; Rashid, T. Group decision making using intuitionistic hesitant fuzzy sets. *Int. J. Fuzzy Log. Intell. Syst.* **2014**, *14*, 181–187. [\[CrossRef\]](#)
24. Grzegorzewski, P. On Separability of Fuzzy Relations. *Int. J. Fuzzy Log. Intell. Syst.* **2017**, *17*, 137–144. [\[CrossRef\]](#)



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