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Asymptotic and Oscillatory Behavior of Solutions of a Class of Higher Order Differential Equation

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Abstract: The objective of this paper is to study asymptotic behavior of a class of higher-order delay differential equations with a p-Laplacian like operator. Symmetry ideas are often invisible in these studies, but they help us decide the right way to study them, and show us the correct direction for future developments. New oscillation criteria are obtained by employing a refinement of the generalized Riccati transformations and comparison principles. This new theorem complements and improves a number of results reported in the literature. Some examples are provided to illustrate the main results.

Keywords: oscillatory solutions; higher-order; delay differential equations; p-Laplacian equations

1. Introduction

In this work, we consider higher-order delay differential equations with a p-Laplacian like operator of the form

$$\left[a(\ell)\left|\left(u^{(n-1)}(\ell)\right)\right|^{p-2}u^{(n-1)}(\ell)\right]' + q(\ell)f(u(\tau(\ell))) = 0. \tag{1}$$

Throughout this paper, we assume that n is an even positive integer, $p > 1$ is a constant, $a \in C^1([\ell_0,\infty),\mathbb{R})$, $a'(\ell) \geq 0$, q, $\tau \in C([\ell_0,\infty),\mathbb{R})$, $q \geq 0$, $\tau(\ell) \leq \ell$, $\lim_{\ell\to\infty}\tau(\ell) = \infty$, $f \in C(\mathbb{R},\mathbb{R})$ and $uf(u) > 0$ for $u \neq 0$.

By a solution of (1), we mean a function $u \in C^{n-1}[L_u,\infty)$, $L_u \geq \ell_0$, which has the property $a\left|\left(u^{(n-1)}\right)\right|^{p-2}u^{(n-1)} \in C^1[L_u,\infty)$, and satisfies (1) on $[L_u,\infty)$. We consider only those solutions u of (1) which satisfy $\sup\{|u(\ell)| : \ell \geq L\} > 0$, for all $L > L_u$. We assume that (1) possesses such a solution. If u is neither positive nor negative eventually, then $u(\ell)$ is called oscillatory, or it will be nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Higher-order differential and difference equations naturally appear in models either biological or physical. Many authors were interested in a study oscillations of differential equations and suggested several ways to get oscillatory criteria for higher order differential equations. For some important work and papers on higher-order differential and difference equations, we refer the reader to the texts [1–26].

Of the early works, Grace and Lalli [11] studied the oscillation of nth order nonlinear differential equations with deviating arguments

$$u^{(n)}(\ell) + q(\ell)\, u(\tau(\ell)) = 0.$$

In the decade before last, Agarwal et al. [3] studied the oscillation of the equation

$$\left[\left|u^{(n-1)}(\ell)\right|^{\alpha-1} u^{(n-1)}(\ell)\right]' + q(\ell)\, |u(\tau(\ell))|^{\alpha-1} u(\tau(\ell)) = 0,$$

where α is positive real number. In [26], Zhang et al. studied the asymptotic properties of the solutions of equation

$$\left(a(\ell)\left(u^{(n-1)}(\ell)\right)^{\alpha}\right)' + q(\ell)\, u^{\beta}(\tau(\ell)) = 0, \tag{2}$$

where α and β are ratios of odd positive integers, $\beta \le \alpha$ and

$$\int_{\ell_0}^{\infty} a^{-1/\alpha}(s)\, \mathrm{d}s < \infty. \tag{3}$$

Zhang et al. in [25] presented some oscillation results, which improves the results in [26]. Moreover, Baculikova et al. in [5] studied the oscillation of the solutions of equation

$$\left(a(\ell)\left(u^{(n-1)}(\ell)\right)^{\alpha}\right)' + q(\ell)\, f(u(\tau(\ell))) = 0,$$

where α is a ratio of odd positive integers, f is nondecreasing,

$$-f(-uv) > f(uv) > f(u)\, f(v) \ \text{ for all } uv > 0. \tag{4}$$

and considered the two cases (3) and

$$\int_{\ell_0}^{\infty} a^{-1/\alpha}(s)\, \mathrm{d}s = \infty. \tag{5}$$

For more general equation, Bazighifan et al. [7] consider the oscillatory properties of the higher-order equation

$$\left[a(\ell)\left(u^{(n-1)}(\ell)\right)^{\alpha}\right]' + \int_c^d q(\ell,s)\, u^{\alpha}(g(\ell,s))\, d(s) = 0,$$

under the conditions (3) and (5).

As a result of numerous applications of the p-Laplace differential equations in continuum mechanics, it is interesting to study asymptotic and oscillatory behavior of solutions of Equation (1). Our aim in the present paper is to employ the Riccatti technique and new comparison principles to establish some new conditions for the oscillation of all solutions of Equation (1) under the condition

$$\int_{\ell_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty. \tag{6}$$

Some examples are provided to illustrate the main results.

The proof of our main results are essentially based on the following lemmas.

Lemma 1. *([2]) Let $\omega(\ell) \in C^m[\ell_0,\infty)$ of constant sign and $\omega^{(m)}(\ell) \neq 0$ on $[\ell_0,\infty)$ which satisfies $\omega(\ell)\,\omega^{(m)}(\ell) \le 0$. Then,*

(I) there exists a $\ell_1 \ge \ell_0$ such that the functions $\omega^{(i)}(\ell)$, $i = 1,2,...,m-1$ are of constant sign on $[\ell_0,\infty)$;

(II) *there exists a number $k \in \{1,3,5,...,m-1\}$ when m is even, $k \in \{0,2,4,...,m-1\}$ when m is odd, such that, for $\ell \geq \ell_1$,*

$$\omega(\ell)\,\omega^{(i)}(\ell) > 0,$$

for all $i = 0,1,...,k$ and

$$(-1)^{m+i+1}\,\omega(\ell)\,\omega^{(i)}(\ell) > 0,$$

for all $i = k+1,...,m$.

Lemma 2. *([4]) Let $\beta \geq 1$ be a ratio of two odd numbers. Then*

$$P^{(\beta+1)/\beta} - (P-Q)^{(\beta+1)/\beta} \leq \frac{1}{\beta} Q^{1/\beta}\left[(1+\beta)P - Q\right], \ \ PQ \geq 0,$$

and

$$Uz - Vz^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^{\beta}}, \ V > 0.$$

Lemma 3. *([14]) If the function u satisfies $u^{(j)} > 0$ for all $j = 0,1,...,n$, and $u^{(n+1)} < 0$, then*

$$\frac{n!}{\ell^n} u(\ell) - \frac{(n-1)!}{\ell^{n-1}} \frac{\mathrm{d}}{\mathrm{d}\ell} u(\ell) \geq 0.$$

Lemma 4. *([2]) Let $h \in C^n([\ell_0,\infty),(0,\infty))$. Suppose that $h^{(n)}(\ell)$ is of a fixed sign, on $[\ell_0,\infty)$, $h^{(n)}(\ell)$ not identically zero and that there exists a $\ell_1 \geq \ell_0$ such that, for all $\ell \geq \ell_1$,*

$$h^{(n-1)}(\ell)\,h^{(n)}(\ell) \leq 0.$$

If we have $\lim_{\ell\to\infty} h(\ell) \neq 0$, then there exists $\ell_\lambda \geq \ell_0$ such that

$$h(\ell) \geq \frac{\lambda}{(n-1)!} \ell^{n-1} \left| h^{(n-1)}(\ell) \right|,$$

for every $\lambda \in (0,1)$ and $\ell \geq \ell_\lambda$.

2. Main Results

In this section, we shall establish oscillation results for Equation (1).

For convenience, we denote

$$\begin{aligned}
\eta(\ell) &:= \int_\ell^\infty \frac{1}{a^{1/(p-1)}(s)} ds, \\
\rho'_+(\ell) &:= \max\left\{0, \ \rho'(\ell)\right\} \\
\psi(\ell) &:= k\rho(\ell)\, q(\ell) \left(\frac{\tau^{n-1}(\ell)}{\ell^{n-1}}\right)^{p-1} \text{ and } \ \phi(\ell) := \frac{\rho'_+(\ell)}{\rho(\ell)}.
\end{aligned}$$

In the next theorem, we establish new oscillation results for Equation (1) by using a generalized Riccati technique

Theorem 1. *Let $n \geq 2$, (6) holds and f satisfies the following condition:*

$$|f(u)| \,/\, |u|^{p-1} \geq k > 0$$

for all $u \neq 0$. Assume that there exists a positive function $\rho \in C([\ell_0, \infty))$ such that

$$\int_{\ell_0}^{\infty} \left(\psi(s) - \frac{1}{p^p} \phi^p(s) \frac{((n-1)!)^{p-1} \rho(s) a(s)}{((p-1)\mu s^{n-1})^{p-1}} - \frac{(p-1)\rho(s)}{a^{1/(p-1)}(s) \eta^p(s)} \right) ds = +\infty, \tag{7}$$

for some constant $\mu \in (0,1)$. If

$$\int_{\ell_0}^{\infty} kq(s) \frac{\tau(s)^{p-1}}{s^{p-1}} ds = \infty, \tag{8}$$

then every solution of (1) is oscillatory.

Proof. Let u be a nonoscillatory solution of Equation (1) on the interval $[\ell_0, \infty)$. Without loss of generality, we can assume that $u(\ell)$ is eventually positive. It follows from Lemma 1 that there exist two possible cases: for $\ell \geq \ell_1$, where $\ell_1 \geq \ell_0$ is sufficiently large,

(C_1) $u(\ell) > 0,\ u'(\ell) > 0, \ldots,\ u^{(n-1)}(\ell) > 0,\ u^{(n)}(\ell) < 0,$

(C_2) $u(\ell) > 0, u^{(j)}(\ell) > 0, u^{(j+1)}(\ell) < 0$ for all odd integer $j \in \{1, 2, ..., n-3\}$, $u^{(n-1)}(\ell) > 0$, $u^{(n)}(\ell) < 0$.

Assume that Case (C_1) holds. Define the function $\omega(\ell)$ by

$$\omega(\ell) := \rho(\ell) \left[\frac{a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1}}{u^{p-1}(\ell)} + \frac{1}{\eta^{p-1}(\ell)} \right], \tag{9}$$

then $\omega(\ell) > 0$ for $\ell \geq \ell_1$ and

$$\begin{aligned} \omega'(\ell) \leq\ & \rho'(\ell) \left[\frac{a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1}}{u^{p-1}(\ell)} + \frac{1}{\eta^{p-1}(\ell)} \right] + \rho(\ell) \frac{\left(a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1} \right)'}{u^{p-1}(\ell)} \\ & - \rho(\ell)(p-1) \frac{u'(\ell) a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1}}{u^p(\ell)} - \frac{(p-1)\rho(\ell)}{a^{1/(p-1)}(\ell) \eta^p(\ell)}. \end{aligned}$$

By Lemma 4, we get

$$u'(\ell) \geq \frac{\mu}{(n-2)!} \ell^{n-2} u^{(n-1)}(\ell). \tag{10}$$

Using (9) and (10) we obtain

$$\begin{aligned} \omega'(\ell) \leq\ & \rho'(\ell) \left[\frac{a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1}}{u^{p-1}(\ell)} + \frac{1}{\eta^{p-1}(\ell)} \right] + \rho(\ell) \frac{\left(a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p-1} \right)'}{u^{p-1}(\ell)} \\ & - \rho(\ell) \frac{(p-1)\mu \ell^{n-2}}{(n-2)!} \frac{a(\ell) \left| \left(u^{(n-1)}(\ell) \right) \right|^{p}}{u^p(\ell)} - \frac{(p-1)\rho(\ell)}{a^{1/(p-1)}(\ell) \eta^p(\ell)}. \end{aligned} \tag{11}$$

From Lemma 3, we have that

$$\frac{u(\ell)}{u'(\ell)} \geq \frac{\ell}{n-1}.$$

Thus, we obtain that u/ℓ^{n-1} is nonincreasing and so,

$$\frac{u(\tau(\ell))}{u(\ell)} \geq \frac{\tau^{n-1}(\ell)}{\ell^{n-1}}. \tag{12}$$

From (1) and (12), we get

$$\begin{aligned}\left[a(\ell)\left|\left(u^{(n-1)}(\ell)\right)\right|^{p-2}u^{(n-1)}(\ell)\right]' &\leq -kq(\ell)\left|u(\tau(\ell))\right|^{p-1} \\ &\leq -kq(\ell)\left(\frac{\tau^{n-1}(\ell)}{\ell^{n-1}}\right)^{p-1}u(\ell)^{p-1}.\end{aligned} \tag{13}$$

From (11) and (13), we get

$$\begin{aligned}\omega'(\ell) &\leq \frac{\rho'_+(\ell)}{\rho(\ell)}\omega(\ell) - k\rho(\ell)q(\ell)\left(\frac{\tau^{n-1}(\ell)}{\ell^{n-1}}\right)^{p-1} \\ &\quad -\rho(\ell)\frac{(p-1)\mu\ell^{n-2}}{(n-2)!}\frac{a(\ell)\left|\left(u^{(n-1)}(\ell)\right)\right|^{p}}{u^{p}(\ell)} - \frac{(p-1)\rho(\ell)}{a^{1/(p-1)}(\ell)\eta^{p}(\ell)} \\ &\leq \frac{\rho'_+(\ell)}{\rho(\ell)}\omega(\ell) - k\rho(\ell)q(\ell)\left(\frac{\tau^{n-1}(\ell)}{\ell^{n-1}}\right)^{p-1} \\ &\quad -\frac{(p-1)\mu\ell^{n-2}}{(n-2)!}\frac{\omega(\ell)^{\frac{p}{p-1}}}{(\rho(\ell)a(\ell))^{\frac{1}{p-1}}} - \frac{(p-1)\rho(\ell)}{a^{1/(p-1)}(\ell)\eta^{p}(\ell)}.\end{aligned} \tag{14}$$

Using Lemma 2 with $U = \frac{\rho'_+(\ell)}{\rho(\ell)}$, $V = \frac{(p-1)\mu\ell^{n-2}}{(n-2)!(\rho(\ell)a(\ell))^{\frac{1}{p-1}}}$ and $z = \omega(\ell)$, we get

$$\begin{aligned}&\frac{\rho'_+(\ell)}{\rho(\ell)}\omega(\ell) - \frac{(p-1)\mu\ell^{n-2}}{(n-2)!(\rho(\ell)a(\ell))^{\frac{1}{p-1}}}\omega(\ell)^{\frac{p}{p-1}} \\ &\qquad \leq \frac{1}{p^{p}}\left(\frac{\rho'_+(\ell)}{\rho(\ell)}\right)^{p}\frac{((n-2)!)^{p-1}\rho(\ell)a(\ell)}{((p-1)\mu\ell^{n-2})^{p-1}}.\end{aligned} \tag{15}$$

Thus, by (14) and (15), we obtain

$$\omega'(\ell) \leq -\psi(\ell) + \frac{1}{p^{p}}\phi^{p}(\ell)\frac{((n-2)!)^{p-1}\rho(\ell)a(\ell)}{((p-1)\mu\ell^{n-2})^{p-1}} - \frac{(p-1)\rho(\ell)}{a^{1/(p-1)}(\ell)\eta^{p}(\ell)}. \tag{16}$$

Integrating from ℓ_1 to ℓ, we get

$$\int_{\ell_1}^{\ell}\left(\psi(s) - \frac{1}{p^{p}}\phi^{p}(s)\frac{((n-2)!)^{p-1}\rho(s)a(s)}{((p-1)\mu s^{n-2})^{p-1}} + \frac{(p-1)\rho(s)}{a^{1/(p-1)}(s)\eta^{p}(s)}\right)ds \leq \omega(\ell_1),$$

for every $\mu \in (0, 1)$, which contradicts (7).

Assume that Case (C_2) holds. Integrating (1) from ℓ to ℓ_1, we obtain

$$\begin{aligned}&a(\ell_1)\left|\left(u^{(n-1)}(\ell_1)\right)\right|^{p-2}u^{(n-1)}(\ell_1) - a(\ell)\left|\left(u^{(n-1)}\ell\right)\right|^{p-2}u^{(n-1)}(\ell) \\ &\qquad + \int_{\ell}^{\ell_1}kq(s)\frac{\tau(s)^{p-1}}{s^{p-1}}u^{p-1}(s)\,ds = 0.\end{aligned}$$

By virtue of $u'(\ell) > 0$ and $\tau(\ell) \leq \ell$, we obtain

$$\begin{aligned}&a(\ell_1)\left|\left(u^{(n-1)}(\ell_1)\right)\right|^{p-2}u^{(n-1)}(\ell_1) - a(\ell)\left|\left(u^{(n-1)}\ell\right)\right|^{p-2}u^{(n-1)}(\ell) \\ &\qquad + u^{p-1}(\ell)\int_{\ell}^{\ell_1}kq(s)\frac{\tau(s)^{p-1}}{s^{p-1}}ds \leq 0.\end{aligned}$$

Letting $\ell_1 \to \infty$; we arrive at the inequality

$$-a(\ell)\left|\left(u^{(n-1)}\ell\right)\right|^{p-2} u^{(n-1)}(\ell) + u^{p-1}(\ell)\int_{\ell}^{\infty} kq(s)\frac{\tau(s)^{p-1}}{s^{p-1}}ds \leq 0.$$

i.e.,

$$\int_{\ell}^{\infty} kq(s)\frac{\tau(s)^{p-1}}{s^{p-1}}ds \leq \frac{a(\ell)\left|\left(u^{(n-1)}\ell\right)\right|^{p-2} u^{(n-1)}(\ell)}{u^{p-1}(\ell)}, \tag{17}$$

which contradicts (8).

Theorem 1 is proved. □

In the next theorem, we establish new oscillation results for (1) by using the theory comparison with the first order differential equation:

Theorem 2. *Let $n \geq 2$, (6) holds and f satisfies the condition (4). For some constant $\lambda \in (0,1)$, assume that the differential equation*

$$u'(\ell) + q(\ell) f\left(\frac{\lambda}{(n-1)!a^{1/(p-1)}(\tau(\ell))}\tau^{n-1}(\ell)\right) f\left(u^{1/(p-1)}(\tau(\ell))\right) = 0, \tag{18}$$

is oscillatory. Then every solution of (1) is oscillatory.

Proof. Let (1) has a nonoscillatory solution y. Without loss of generality, we can assume that $y(\ell) > 0$. Hence we have

$$y'(\ell) > 0,\ y^{(n-1)}(\ell) > 0 \text{ and } y^{(n)}(\ell) < 0. \tag{19}$$

From Lemma 4, we get

$$y(\tau(\ell)) \geq \frac{\lambda \tau^{n-1}(\ell)}{(n-1)!a^{1/(p-1)}(\ell)}\left(a^{1/(p-1)}(\ell)\, y^{(n-1)}\tau(\ell)\right), \tag{20}$$

for every $\lambda \in (0,1)$. Set

$$u(\ell) = a(\ell)\left[y^{(n-1)}(\ell)\right]^{p-1} > 0.$$

Using (20) in (1), we see that inequality

$$u'(\ell) + q(\ell) f\left(\frac{\lambda}{(n-1)!a^{1/(p-1)}(\tau(\ell))}\tau^{n-1}(\ell)\right) f\left(u^{1/(p-1)}(\tau(\ell))\right) \leq 0. \tag{21}$$

That is, u is a positive solution of inequality (21). From [23] (Theorem 1), we conclude that the corresponding Equation (18) also has a positive solution, which is a contradiction.

Theorem 2 is proved. □

Corollary 1. *Assume that (6) hold and let $n \geq 2$ be even. If*

$$\liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} q(s) f\left(\frac{\lambda}{(n-1)!a^{1/(p-1)}(\tau(s))}\tau^{n-1}(s)\right) ds > \frac{((n-1)!)^{p-1}}{e}, \tag{22}$$

for some constant $\lambda \in (0,1)$, then every solution of (1) is oscillatory.

Example 1. *For $t \geq 1$, consider a differential equation*

$$u^{(4)}(\ell) + \frac{v}{\ell} u\left(\frac{\ell}{3}\right) = 0, \tag{23}$$

where $v > 0$ is a constant. Let

$$n = 4,\ a(\ell) = 1,\ p = 2,\ \tau(\ell) = \ell/3,\ q(\ell) = v/\ell,$$

we get

$$\eta(s) = \int_{\ell_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty.$$

If we now set $k = \rho = 1$ then

$$\begin{aligned}
&\int_{\ell_0}^{\infty} \left(\psi(s) - \frac{1}{p^p} \phi^p(s) \frac{((n-1)!)^{p-1} \rho(s) a(s)}{((p-1)\mu s^{n-1})^{p-1}} - \frac{(p-1)\rho(s)}{a^{1/(p-1)}(s)\eta^p(s)} \right) ds \\
&= \int_{\ell_0}^{\infty} k\rho(s) q(s) \left(\frac{\tau^{n-1}(s)}{s^{n-1}} \right)^{p-1} ds \\
&= \frac{1}{3^3} \int_{\ell_0}^{\infty} \frac{v}{s} ds \\
&= \infty,
\end{aligned}$$

also

$$\begin{aligned}
\int_{\ell_0}^{\infty} kq(s) \frac{\tau(s)^{p-1}}{s^{p-1}} ds &= \frac{1}{3} \int_{\ell_0}^{\infty} \frac{v}{s} ds \\
&= \infty.
\end{aligned}$$

Thus, by Theorem 1, every solution of Equation (23) is oscillatory.

Example 2. *Consider a fourth order differential equation*

$$u^{(4)}(\ell) + \frac{\delta}{\ell^4} u\left(\frac{\ell}{4}\right) = 0, \qquad \ell \geq 1, \tag{24}$$

where $\delta > 0$ is a constant. Let

$$n = 4,\ a(\ell) = 1,\ p = 2,\ \tau(\ell) = \ell/4,\ q(\ell) = \delta/\ell^4,$$

we get

$$\eta(s) = \int_{\ell_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty.$$

If we now set $k = 1$, then

$$\begin{aligned}
&\liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} q(s) f\left(\frac{\lambda}{(n-1)! a^{1/(p-1)}(\tau(s))} \tau^{n-1}(s) \right) ds \\
&= \liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} \frac{\delta}{s^4} f\left(\frac{\lambda s^3}{384} \right) ds \\
&= \liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} \frac{\delta}{s^4} \left(\frac{\lambda s^3}{384} \right) ds \\
&= \liminf_{\ell\to\infty} \int_{\tau(\ell)}^{\ell} \frac{\delta\lambda}{384 s} ds = \frac{\delta\lambda}{384} \ln 4.
\end{aligned}$$

Thus, by Corollary 1, every solution of Equation (24) is oscillatory if $\delta > \frac{2304}{e \ln 4} \frac{1}{\lambda}$ for some constant $\lambda \in (0, 1)$.

3. Conclusions

In this paper, by using a Riccati technique and comparison principles with the first-order differential equations, we offer some new sufficient conditions which ensure that any solution of (1) oscillates under the condition $\int^{\infty} r^{-1/(p-1)}(t)\,dt = \infty$. Results in [7,8,21] cannot apply to the example. Further, we can consider the case of $z(t) = x(t) + a(t)\,x(\sigma(t))$, and we can try to get some oscillation criteria of (1) in the future work.

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