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On Some Sufficient Conditions for a Function to Be p-Valent Starlike

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Abstract: A function f analytic in a domain $D \in \mathbb{C}$ is called p-valent in D, if for every complex number w, the equation $f(z) = w$ has at most p roots in D, so that there exists a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D. The aim of this paper is to establish some sufficient conditions for a function analytic in the unit disc $\mathbb{D}$ to be p-valent starlike in $\mathbb{D}$ or to be at most p-valent in $\mathbb{D}$. Our results are proved mainly by applying Nunokawa's lemmas.

Keywords: univalent functions; starlike; convex; close-to-conve

MSC: primary 30C45; secondary 30C80

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1. Introduction

A function f analytic in a domain $D \in \mathbb{C}$ is called p-valent in D, if for every complex number w, the equation $f(z) = w$ has at most p roots in D, so that there exists a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D. We denote by $\mathcal{H}$ the class of functions f which are holomorphic in the open unit unit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}_p$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{A} = \mathcal{A}_1$. The well known Noshiro-Warschawski univalence condition, (see [1,2]) indicates that if f is analytic in a convex domain $D \subset \mathbb{C}$ and

$$\mathfrak{Re}\{e^{i\theta} f'(z)\} > 0, \quad z \in D, \tag{2}$$

for some real θ, then f is univalent in D. In [3] Ozaki extended the above result by showing that if f of the form (1) is analytic in a convex domain D and for some real θ we have

$$\mathfrak{Re}\{e^{i\theta} f^{(p)}(z)\} > 0, \quad z \in D,$$

then f is at most p-valent in D. In [4] it was proved that if $f \in \mathcal{A}_p$, $p \geq 2$, and

$$\left|\arg\{f^{(p)}(z)\}\right| < \frac{3\pi}{4}, \quad z \in \mathbb{D},$$

then f is at most p-valent in $\mathbb{D}$.

If $f \in \mathcal{H}$ satisfies $f(0) = 0$, $f'(0) = 1$ and

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D},$$

then f is said to be starlike with respect to the origin in $\mathbb{D}$ and it is denoted by $f \in \mathcal{S}^*$. It is known that $\mathcal{S}^* \subset \mathcal{S}$, where $\mathcal{S}$ denotes the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$. Moreover, let $\mathcal{S}_p^*$ and $\mathcal{C}_p$ be the subclasses of $\mathcal{A}_p$ defined as follows

$$\begin{aligned} \mathcal{S}_p^* &= \left\{f \in \mathcal{A}_p : \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ z \in \mathbb{D}\right\}, \\ \mathcal{C}_p &= \left\{f \in \mathcal{A}_p : zf'(z)/p \in \mathcal{S}_p^*\right\}. \end{aligned}$$

$\mathcal{S}_p^*$ is called the class of p-valent starlike functions and $\mathcal{C}_p$ is called the class of p-valent convex functions. Note that $\mathcal{S}_1^* = \mathcal{S}^*$ and $\mathcal{C}_1 = \mathcal{C}$, where $\mathcal{S}^*$ and $\mathcal{C}$ are usual classes of starlike and convex functions, respectively. A function $f \in \mathcal{A}_p$ is said to be an element of the class $\mathcal{K}_p$ of p-valent close-to-convex functions if there exists a function $g \in \mathcal{C}_p$ for which

$$\mathfrak{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{3}$$

In [5] (Th.1) Umezawa proved the following theorem.

Theorem 1. *If $f \in \mathcal{K}_p$, then f is at most p-valent in $\mathbb{D}$.*

Because $\mathcal{C}_p \subset \mathcal{S}_p^* \subset \mathcal{K}_p$, we have from Theorem 1 that p-valent starlike functions and p-valent convex functions are at most p-valent in $\mathbb{D}$ too.

2. Preliminaries

In this paper we need the following lemmas.

Lemma 1 ([6] (Th.5))**.** *If $f \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have*

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \ldots, p\} : \quad \mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0.$$

Lemma 2 ([7])**.** *Let p be an analytic function in $|z| < 1$, with $p(0) = 1$. If there exists a point z_0, $|z_0| < 1$, such that*

$$\mathfrak{Re}\{p(z)\} > 0 \quad for \quad |z| < |z_0|$$

and

$$p(z_0) = \pm ia$$

for some $a > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi}, \tag{4}$$

for some $k \geq (a + a^{-1})/2 \geq 1$.

Corollary 1. *Under the assumptions of Lemma 2, we have from (4)*

$$z_0 p'(z_0) = -ka \leq -\frac{1}{2}(a + a^{-1})a = -\frac{1}{2}\left(1 + |p(z_0)|^2\right). \tag{5}$$

Lemma 3 ([8] (p. 200))**.** *Assume that $q(z)$ is univalent in $\mathbb{D}$, $q(\mathbb{D})$ is a convex set and F, G are analytic in $\mathbb{D}$. If*

$$\frac{F'(z)}{G'(z)} \prec q(z), \quad z \in \mathbb{D}, \tag{6}$$

where G satisfies $G(0) = F(0)$ and

$$\mathfrak{Re}\left\{\frac{zG'(z)}{G(z)}\right\} > 0, \quad z \in \mathbb{D},$$

then we have

$$\frac{F(z)}{G(z)} \prec q(z), \quad z \in \mathbb{D}.$$

Here $\prec$ means the subordination.

Corollary 2. *Let $\alpha < 1$ be real number. If $f^{(p-1)}(z), g^{(p-1)}(z)$ are analytic in $\mathbb{D}$, $f^{(p-1)}(0) = g^{(p-1)}(0)$ and*

$$\mathfrak{Re}\left\{\frac{f^{(p)}(z)}{g^{(p)}(z)}\right\} > \alpha, \quad z \in \mathbb{D},$$

where g satisfies

$$\mathfrak{Re}\left\{\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right\} > 0, \quad z \in \mathbb{D},$$

then we have

$$\mathfrak{Re}\left\{\frac{f^{(p-1)}(z)}{g^{(p-1)}(z)}\right\} > \alpha, \quad z \in \mathbb{D}.$$

3. Main Results

Theorem 2. *Let $f, g \in \mathcal{A}_p$. Assume that*

$$\mathfrak{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \beta, \quad z \in \mathbb{D} \tag{7}$$

for some β, $0 < \beta < 1$. If

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| \leq \pi - \tan^{-1}\left\{\frac{2(1-\beta)|z| + 1 - |z|^2}{\beta(1-|z|^2)}\right\}, \quad z \in \mathbb{D}, \tag{8}$$

then we have

$$\mathfrak{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{9}$$

Proof. If we put

$$q(z) = \frac{f(z)}{g(z)}, \quad q(0) = 1,$$

then it follows that

$$f(z) = q(z)g(z), \quad f'(z) = g'(z)q(z) + q'(z)g(z),$$

and

$$\frac{f'(z)}{g'(z)} = q(z) + q'(z)\frac{g(z)}{g'(z)} = q(z) + zq'(z)\frac{g(z)}{zg'(z)}.$$

If there exists a point $z_0 \in \mathbb{D}$, such that

$$\mathfrak{Re}\{q(z)\} > 0, \quad (|z| < |z_0| < 1)$$

and

$$\mathfrak{Re}\,\{q(z_0)\} = 0,$$

then by (5), we have

$$z_0q'(z_0) = \mathfrak{Re}\{z_0q'(z_0)\} \leq -\frac{1}{2}(1+|q(z_0)|^2) < 0. \tag{10}$$

This shows that $z_0q'(z_0)$ is a negative real number. Furthermore, by (4), we have

$$\left|\frac{q(z_0)}{z_0q'(z_0)}\right| \leq 1. \tag{11}$$

Then it follows that

$$\frac{f'(z_0)}{g'(z_0)} = q(z_0) + z_0q'(z_0)\frac{g(z_0)}{z_0g'(z_0)} = \pm ia + z_0q'(z_0)\frac{g(z_0)}{z_0g'(z_0)},$$

where $q(z_0) = \pm ai$, $a > 0$ and (7), (10) give

$$\begin{aligned}\mathfrak{Re}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} &= \mathfrak{Re}\left\{z_0q'(z_0)\frac{g(z_0)}{z_0g'(z_0)}\right\} = z_0q'(z_0)\mathfrak{Re}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\} \\ &< -\frac{\beta}{2}(1+a^2) < 0.\end{aligned}$$

Next, we have

$$\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} = \mathfrak{Im}\left\{\pm ia + z_0q'(z_0)\frac{g(z_0)}{z_0g'(z_0)}\right\} = \pm a + z_0q'(z_0)\mathfrak{Im}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\}.$$

We will consider the four cases:

(i) $\arg\{q(z_0)\} = \pi/2$ (i.e., $q(z_0) = ia$, $a > 0$) and $\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} \geq 0$,

(ii) $\arg\{q(z_0)\} = \pi/2$ (i.e., $q(z_0) = ia$, $a > 0$) and $\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} < 0$,

(iii) $\arg\{q(z_0)\} = -\pi/2$ (i.e., $q(z_0) = -ia$, $a > 0$) and $\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} \geq 0$,

(iv) $\arg\{q(z_0)\} = -\pi/2$ (i.e., $q(z_0) = -ia$, $a > 0$) and $\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} < 0$.

Let us put

$$G(z) = \frac{pg(z)}{zg'(z)}, \quad G(0) = 1.$$

Then from the hypothesis, we have

$$\frac{G(z)-\beta}{1-\beta} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

and so we have

$$G(z) \prec \beta + (1-\beta)\frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

and so

$$|\mathfrak{Im}\{G(z)\}| = \left|\mathfrak{Im}\frac{pg(z)}{zg'(z)}\right| \leq (1-\beta)\frac{2|z|}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{12}$$

In the case (i) we have $\arg\{q(z_0)\} = \pi/2$, $q(z_0) = ia$, $a > 0$ and

$$\mathfrak{Im}\left\{\frac{f'(z_0)}{g'(z_0)}\right\} = |q(z_0)| + z_0q'(z_0)\left(\mathfrak{Im}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\}\right) \geq 0.$$

Therefore, we have

$$\begin{aligned}
\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\} &= \arg\left[z_0q'(z_0)\left(\mathfrak{Re}\frac{g(z_0)}{z_0g'(z_0)}\right)+i\left\{|q(z_0)|+z_0q'(z_0)\left(\mathfrak{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)\right\}\right] \\
&= \pi-\tan^{-1}\left\{\frac{|q(z_0)|+z_0q'(z_0)\left(\mathfrak{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)}{-z_0q'(z_0)\left(\mathfrak{Re}\frac{g(z_0)}{z_0g'(z_0)}\right)}\right\} \\
&> \pi-\tan^{-1}\left\{\frac{|q(z_0)|+z_0q'(z_0)\left(\mathfrak{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)}{-\beta z_0q'(z_0)}\right\} \\
&= \pi-\tan^{-1}\left\{-\frac{|q(z_0)|}{\beta z_0q'(z_0)}-\frac{\mathfrak{Im}\frac{g(z_0)}{z_0g'(z_0)}}{\beta}\right\} \\
&\geq \pi-\tan^{-1}\left\{\left|\frac{q(z_0)}{\beta z_0q'(z_0)}\right|+\left|\frac{\mathfrak{Im}\frac{g(z_0)}{z_0g'(z_0)}}{\beta}\right|\right\}.
\end{aligned}$$

Then, by (11) and (12), we have

$$\begin{aligned}
\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\} &> \pi-\tan^{-1}\left\{\frac{1}{\beta}+\frac{2(1-\beta)|z_0|}{(1-|z_0|^2)\beta}\right\} \\
&= \pi-\tan^{-1}\left[\frac{1}{\beta(1-|z_0|^2)}\left\{2(1-\beta)|z_0|+1-|z_0|^2\right\}\right].
\end{aligned}$$

This contradicts hypothesis (8). In the case (ii) when $\arg\{q(z_0)\}=\pi/2$, $q(z_0)=ia$, $a>0$, and

$$\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\}=q(z_0)+z_0q'(z_0)\left(\mathfrak{Im}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\}\right)<0$$

applying the same method as the above, we have

$$\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\} < -\pi+\tan^{-1}\left[\frac{1}{\beta(1-|z_0|^2)}\left\{2(1-\beta)|z_0|+1-|z_0|^2\right\}\right].$$

This is also a contradiction. In the case (iii) when $\arg\{q(z_0)\}=-\pi/2$, $q(z_0)=-ia$, $a>0$, and

$$q(z_0)+z_0q'(z_0)\left(\mathfrak{Im}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\}\right)>0$$

and in the case (iv) when $\arg\{q(z_0)\}=-\pi/2$, $q(z_0)=-ia$, $a>0$, and

$$q(z_0)+z_0q'(z_0)\left(\mathfrak{Im}\left\{\frac{g(z_0)}{z_0g'(z_0)}\right\}\right)<0,$$

applying the same method as in the proof of case (i) gives

$$\left|\arg\left\{\frac{f'(z_0)}{g'(z_0)}\right\}\right| > \pi-\tan^{-1}\left[\frac{1}{\beta(1-|z_0|^2)}\left\{2(1-\beta)|z_0|+1-|z_0|^2\right\}\right].$$

This is a contradiction. This completes the proof. □

Inequalities (9) and (2) show that the assumptions of Theorem 2 are sufficient for

$$\int_0^z\frac{f(\zeta)}{g(\zeta)}\mathrm{d}\zeta$$

to be univalent in $\mathbb{D}$.

Theorem 3. *Let $F, G \in \mathcal{A}_p$. Assume that there exist a positive integer k, $2 \le k \le p$ and a real β, $0 < \beta < 1$, for which*

$$\left|\arg\left\{\frac{F^{(k)}(z)}{G^{(k)}(z)}\right\}\right| < \pi - \tan^{-1}\left\{\frac{2(1-\beta)|z| + 1 - |z|^2}{\beta(1-|z|^2)}\right\}, \quad z \in \mathbb{D},$$

where G satisfies

$$\mathfrak{Re}\left\{\frac{G^{(k-1)}(z)}{zG^{(k)}(z)}\right\} > \beta, \quad z \in \mathbb{D}. \tag{13}$$

Then

$$\forall n \in \{1, \ldots, k-1\}: \quad \mathfrak{Re}\left\{\frac{F^{(n)}(z)}{G^{(n)}(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{14}$$

and $F \in \mathcal{K}_p$, F is at most p-valent in $\mathbb{D}$.

Proof. If we put $f = F^{(k-1)}$ and $g = G^{(k-1)}$ in Theorem 2 we immediately obtain

$$\mathfrak{Re}\left\{\frac{F^{(k-1)}(z)}{G^{(k-1)}(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{15}$$

Then, by Lemma 1, we obtain (14). For $n = 1$ the condition (14) is of the form

$$\mathfrak{Re}\left\{\frac{F'(z)}{G'(z)}\right\} > 0, \quad z \in \mathbb{D},$$

where G satisfies (13). Therefore, by Lemma 1 we have also

$$\mathfrak{Re}\left\{\frac{zG'(z)}{G(z)}\right\} > 0, \quad z \in \mathbb{D},$$

which by (3) implies $F \in \mathcal{K}_p$. By Theorem 1, F is at most p-valent in $\mathbb{D}$. □

Theorem 4. *Assume that $f \in \mathcal{A}_p$, $2 \le p$, and that there exists a positive integer k, $2 \le k \le p$ for which*

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > -1, \quad z \in \mathbb{D},$$

then we have

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D},$$

or f is p-valent starlike in $\mathbb{D}$.

Proof. Let us put

$$q_1(z) = \frac{1}{p-k+2}\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}, \quad q_1(0) = 1. \tag{16}$$

By (16) we have

$$\frac{zq_1'(z)}{q_1(z)} = 1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} - \frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}$$

and so

$$1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} = \frac{zq_1'(z)}{q_1(z)} + (p-k+2)q_1(z).$$

By the hypothesis, we have

$$1+\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\}=\mathfrak{Re}\left\{\frac{zq_1'(z)}{q_1(z)}+(p-k+2)q_1(z)\right\}>0, \quad z\in\mathbb{D}. \tag{17}$$

If there exists a point $z_1\in\mathbb{D}$, such that

$$\mathfrak{Re}\{q_1(z)\}>0, \quad (|z|<|z_1|<1)$$

and

$$\mathfrak{Re}\{q_1(z_1)\}=0,$$

then by Lemma 2, we have

$$\mathfrak{Re}\left\{\frac{z_1q_1'(z_1)}{q_1(z_1)}\right\}=0, \quad \frac{z_1q_1'(z_1)}{q_1(z_1)}=ik_1$$

for some real k_1, $|k_1|\geq 1$. This gives

$$1+\mathfrak{Re}\left\{\frac{zf^{(k)}(z_1)}{f^{(k-1)}(z_1)}\right\}=\mathfrak{Re}\left\{\frac{z_1q_1'(z_1)}{q_1(z_1)}+(p-k+2)q_1(z_1)\right\}=0.$$

It is contrary to inequality (17) and therefore, we have

$$\mathfrak{Re}\left\{\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}\right\}>\mathfrak{Re}\left\{\frac{1}{p-k+2}\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}\right\}=\mathfrak{Re}\{q_1(z)\}>0, \quad z\in\mathbb{D}. \tag{18}$$

Next, let us put

$$q_2(z)=\frac{1}{p-k+3}\frac{zf^{(k-2)}(z)}{f^{(k-3)}(z)}, \quad q_2(0)=1,$$

then it follows that

$$\frac{zq_2'(z)}{q_2(z)}=1+\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}-\frac{zf^{(k-2)}(z)}{f^{(k-3)}(z)}$$

and so

$$1+\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}=\frac{zq_2'(z)}{q_2(z)}+(p-k+3)q_2(z). \tag{19}$$

By (18) and (19), we have

$$1+\mathfrak{Re}\left\{\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)}\right\}=\mathfrak{Re}\left\{\frac{zq_2'(z)}{q_2(z)}+(p-k+3)q_2(z)\right\}>0, \quad z\in\mathbb{D}. \tag{20}$$

If there exists a point $z_2\in\mathbb{D}$, such that

$$\mathfrak{Re}\{q_2(z)\}>0, \quad (|z|<|z_2|<1)$$

and

$$\mathfrak{Re}\{q_2(z_2)\}=0,$$

then by Lemma 2, we have

$$\mathfrak{Re}\left\{\frac{z_2q_2'(z_2)}{q_2(z_2)}\right\}=0, \quad \frac{z_2q_2'(z_2)}{q_2(z_2)}=ik_2$$

for some real k_2, $|k_2| \geq 1$. Then, we have

$$1 + \mathfrak{Re}\left\{\frac{z_2 f^{(k-1)}(z_2)}{f^{(k-2)}(z_2)}\right\} = \mathfrak{Re}\left\{\frac{z_2 q_2'(z_2)}{q_2(z_2)} + (p-k+3)q_2(z_2)\right\} = 0.$$

It is contrary to (20) and therefore, we have

$$\mathfrak{Re}\left\{\frac{z f^{(k-2)}(z)}{f^{(k-3)}(z)}\right\} = \mathfrak{Re}\left\{(p-k+3)q_2(z)\right\} > 0, \quad z \in \mathbb{D}.$$

Applying the same method many times in succession we are able to obtain

$$\mathfrak{Re}\left\{\frac{z f^{(k-3)}(z)}{f^{(k-4)}(z)}\right\} > 0,\ \mathfrak{Re}\left\{\frac{z f^{(k-4)}(z)}{f^{(k-5)}(z)}\right\} > 0 \ldots,\ \mathfrak{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}$$

This shows that f is p-valent starlike in $\mathbb{D}$. $\square$

For some related conditions for starlikeness we refer to our papers [9,10].

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