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An E-Sequence Approach to the $3x+1$ Problem

Sanmin Wang

Faculty of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China; wangsanmin@hotmail.com;
Tel.: +86-136-5581-2543

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Abstract: For any odd positive integer x, define $(x_n)_{n\geqslant 0}$ and $(a_n)_{n\geqslant 1}$ by setting $x_0 = x,\ x_n = \dfrac{3x_{n-1}+1}{2^{a_n}}$ such that all x_n are odd. The $3x+1$ problem asserts that there is an $x_n = 1$ for all x. Usually, $(x_n)_{n\geqslant 0}$ is called the trajectory of x. In this paper, we concentrate on $(a_n)_{n\geqslant 1}$ and call it the E-sequence of x. The idea is that we generalize E-sequences to all infinite sequences $(a_n)_{n\geqslant 1}$ of positive integers and consider all these generalized E-sequences. We then define $(a_n)_{n\geqslant 1}$ to be Ω-convergent to x if it is the E-sequence of x and to be Ω-divergent if it is not the E-sequence of any odd positive integer. We prove a remarkable fact that the Ω-divergence of all non-periodic E-sequences implies the periodicity of $(x_n)_{n\geqslant 0}$ for all x_0. The principal results of this paper are to prove the Ω-divergence of several classes of non-periodic E-sequences. Especially, we prove that all non-periodic E-sequences $(a_n)_{n\geqslant 1}$ with $\overline{\lim\limits_{n\to\infty}} \dfrac{b_n}{n} > \log_2 3$ are Ω-divergent by using Wendel's inequality and the Matthews and Watts' formula $x_n = \dfrac{3^n x_0}{2^{b_n}} \prod\limits_{k=0}^{n-1} (1 + \dfrac{1}{3x_k})$, where $b_n = \sum\limits_{k=1}^{n} a_k$. These results present a possible way to prove the periodicity of trajectories of all positive integers in the $3x+1$ problem, and we call it the E-sequence approach.

Keywords: $3x+1$ problem; E-sequence approach; Ω-divergence of non-periodic E-sequences; Wendel's inequality

MSC: 11A99; 11B83

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1. Introduction

For any odd positive integer x, define two infinite sequences $(x_n)_{n\geqslant 0}$ and $(a_n)_{n\geqslant 1}$ of positive integers by setting:

$$x_0 = x, \quad x_n = \frac{3x_{n-1}+1}{2^{a_n}} \tag{1}$$

such that x_n is odd for all $n \in \mathbb{N} = \{1, 2, \ldots\}$. The $3x+1$ problem asserts that there is $n \in \mathbb{N}$ such that $x_n = 1$ for all odd positive integers x. For a survey, see [1]. For recent developments, see [2–7].

Usually, $(x_n)_{n\geqslant 0}$ is called the trajectory of x. In this paper, we concentrate on $(a_n)_{n\geqslant 1}$ and call it the E-sequence of x. The idea is that we generalize E-sequences to all infinite sequences $(a_n)_{n\geqslant 1}$ of positive integers. Given any generalized E-sequence $(a_n)_{n\geqslant 1}$, if it is the E-sequence of the odd positive integer x, it is called Ω-convergent to x and denoted by $\Omega - \lim a_n = x$; if $(a_n)_{n\geqslant 1}$ is not the E-sequence of any odd positive integer, it is called Ω-divergent and denoted by $\Omega - \lim a_n = \infty$. Subsequently, these generalized E-sequences are also called E-sequences for simplicity.

The $3x+1$ problem in the form (1.1) should be owed to Crandall and Sander et al., see [8,9]. E-sequences are some variants of Everett's parity sequences [10] and Terras' encoding representations [11]. Everett and Terras focused on finite E-sequences resulting from (1.1). What we are concerned with is the Ω-convergence and Ω-divergence of any infinite sequence of positive integers, i.e., the generalized E-sequences.

A possible way to prove the $3x+1$ problem was devised by Möller as follows (see [12]):

Conjecture 1. *(i)* $(x_n)_{n\geqslant 0}$ *is periodic for all odd positive integers* x_0*;*
(ii) $(1, 1, \cdots)$ *is the unique pure periodic trajectory.*

Usually, we can convert one claim about trajectories into the one about E-sequences. As for E-sequences, we have the following conjecture.

Conjecture 2. *Let* $b_n = \sum_{i=1}^{n} a_i$*. Then,*
(i) all non-periodic E-sequences are Ω*-divergent;*
(ii) every E-sequence $(a_n)_{n\geqslant 1}$ *satisfying* $3^n > 2^{b_n}$ *for all* $n \in \mathbb{N}$ *is* Ω*-divergent.*

Note that Conjecture 2(i) does not hold for some generalizations of the $3x+1$ problem studied by Möller, Matthews, and Watts in [12,13]; Conjecture 2(ii) implies that there is some n such that $2^{b_n} > 3^n$ in the E-sequence $(a_n)_{n\geqslant 1}$ of every odd positive integer x, which is a conjecture posed by Terras in [11] about his τ-stopping time.

A remarkable fact is that Conjecture 1(i) is a corollary of Conjecture 2(i) by Theorem 3. This means that the Ω-divergence of all non-periodic E-sequences implies the periodicity of $(x_n)_{n\geqslant 1}$ for all positive integers x. Then, Conjecture 2(i) is of significance to the study of the $3x+1$ problem. The principal results of this paper are to prove that several classes of non-periodic E-sequences are Ω-divergent. In particular, we prove that:

(i) All non-periodic E-sequences $(a_n)_{n\geqslant 1}$ with $\overline{\lim_{n\to\infty}} \dfrac{b_n}{n} > \log_2 3$ are Ω-divergent.
(ii) If $(a_n)_{n\geqslant 0}$ is $12121112\cdots$, where $a_n = 2$ if $n \in \{2^1, 2^2, 2^3, \cdots\}$ and $a_n = 1$, otherwise, then $\Omega - \lim a_n = \infty$;
(iii) Let $\theta \geqslant 1$ be an irrational number, and define $a_n = [n\theta] - [(n-1)\theta]$, then $\Omega - \lim a_n = \infty$, where $[a]$ denotes the integral part of a for any real a.

Note that we prove the above claim (i) by using Wendel's inequality and the Matthews and Watts' formula $x_n = \dfrac{3^n x_0}{2^{b_n}} \prod_{k=0}^{n-1} (1 + \frac{1}{3x_k})$. In addition, it seems that our approach cannot help to prove the conjecture 1(ii) of the unique cycle. For such a topic, see [14].

2. Preliminaries

Let $(a_n)_{n\geqslant 1}$ be an E-sequence. In most cases, there is no odd positive integer x such that $(a_n)_{n\geqslant 1}$ is the E-sequence of x, i.e., $\Omega - \lim a_n = \infty$. However, there always exists $x \in \mathbb{N}$ such that the first n terms of the E-sequence of x are $(a_1 \ldots a_n)$. Furthermore, for any $1 \leqslant u \leqslant v \leqslant n$, there always exists $x \in \mathbb{N}$ such that the first $v-u+1$ terms of the E-sequence of x are the designated block $(a_u \ldots a_v)$ of $(a_1 \ldots a_n)$, which is illustrated as $(a_1 \ldots a_{u-1})(a_u \ldots a_v)(a_{v+1} \ldots a_n)$.

Definition 1. *Define* $b_0 = 0$, $b_n = \sum_{i=1}^{n} a_i$, $B_n = \sum_{i=0}^{n-1} 3^{n-1-i} 2^{b_i}$.

Clearly, $B_1 = 1$, $B_n = 3B_{n-1} + 2^{b_{n-1}}$, $2 \nmid B_n$, $3 \nmid B_n$.

Proposition 1. *Let* $(x_n)_{n\geqslant 1}$ *and* $(a_n)_{n\geqslant 1}$ *be defined as in (1.1). Then* $x_n = \dfrac{3^n x + B_n}{2^{b_n}}$.

Proof. The proof is by a procedure similar to that of Theorem 1.1 in [11] and omitted. □

Proposition 2. *Given any positive integer* n*, there exist two integers* x_n *and* x_0 *such that* $2^{b_n} x_n - 3^n x_0 = B_n$*,* $1 \leqslant x_n < 3^n$*, and* $1 \leqslant x_0 < 2^{b_n}$*.*

Proof. By $\gcd(2^{b_n}, 3^n) = 1$, there exist two integers x_n and x_0 such that $2^{b_n}x_n - 3^n x_0 = B_n$ and $1 \leqslant x_n \leqslant 3^n$. Then, $x_n < 3^n$ by $3 \nmid B_n$. By $B_n \geqslant 1$, we have $x_0 = \dfrac{2^{b_n}x_n - B_n}{3^n} < \dfrac{2^{b_n}x_n}{3^n} < 2^{b_n}$. Thus, $x_0 < 2^{b_n}$.

By $2^{b_n}x_n - 3^n x_0 = B_n$, we have $2^{b_n}x_n \equiv B_n (\mathrm{mod} 3^n)$. Then $2^{b_{n-1}}(2^{a_n}x_n - 1) \equiv 3B_{n-1}(\mathrm{mod} 3^n)$ by $B_n = 2^{b_{n-1}} + 3B_{n-1}$. Thus, $3 | 2^{a_n}x_n - 1$. Define $x_{n-1} = \dfrac{2^{a_n}x_n - 1}{3}$. Then, $x_{n-1} \in \mathbb{Z}$, $x_n = \dfrac{3x_{n-1}+1}{2^{a_n}}$ and $2^{b_{n-1}}x_{n-1} \equiv B_{n-1}(\bmod 3^{n-1})$. Sequentially, define $x_{n-2}, \ldots, x_1$ such that $x_{n-1} = \dfrac{3x_{n-2}+1}{2^{a_{n-1}}}, \ldots, x_1 = \dfrac{3x_0+1}{2^{a_1}}$. Then, $x_i \in \mathbb{Z}$ for all $0 \leqslant i \leqslant n$.

Suppose that $x_0 < 0$. We then sequentially have $x_1 < 0, \ldots, x_n < 0$, which contradicts $x_n \geqslant 1$. Thus, $x_0 \geqslant 1$. □

Note that the validity of Proposition 2 is dependent on the structure of B_n. We formulate the middle part of the above proof as the following proposition.

Proposition 3. *Assume that $x_n, x_0 \in \mathbb{Z}$ and $2^{b_n}x_n - 3^n x_0 = B_n$. Define $x_1 = \dfrac{3x_0+1}{2^{a_1}}, \ldots, x_{n-1} = \dfrac{3x_{n-2}+1}{2^{a_{n-1}}}$. Then, $x_n = \dfrac{3x_{n-1}+1}{2^{a_n}}$ and $x_i \in \mathbb{Z}$ for all $0 \leqslant i \leqslant n$.*

Definition 2. *For any $1 \leqslant u \leqslant v$, define $b_u^{u-1} = 0$, $b_u^v = \sum_{i=u}^{v} a_i$, $B_u^{u-2} = 0$, $B_u^{u-1} = 1$, $B_u^v = 3^{v-u+1} + 3^{v-u}2^{b_u^u} + \cdots + 3^1 2^{b_u^{v-1}} + 2^{b_u^v} = \sum_{i=0}^{v-u+1} 3^{v-u+1-i}2^{b_u^{u-1+i}}$.*

Then, $b_u^u = a_u$, $b_u^{u+1} = a_u + a_{u+1}$, $B_u^u = 3 + 2^{a_u}$, $B_u^{u+1} = 3^2 + 3 \cdot 2^{a_u} + 2^{a_u + a_{u+1}}$, $B_u^v = 3B_u^{v-1} + 2^{b_u^v} = \sum_{i=u-1}^{v} 3^{v-i}2^{b_u^i}$. Clearly, b_1^n and B_1^{n-1} are the same as b_n and B_n, respectively.

Proposition 4. $B_n = 3^{n-u+1}B_1^{u-2} + 3^{n-1-v}2^{b_{u-1}}B_u^v + 2^{b_{v+1}}B_{v+2}^{n-1}$.

Proof. By $B_1^{u-2} = \sum_{i=0}^{u-2} 3^{u-2-i}2^{b_i}$ and $B_{v+2}^{n-1} = \sum_{i=v+1}^{n-1} 3^{n-1-i}2^{b_{v+2}^i}$, we have:

$$\begin{aligned} B_n = B_1^{n-1} &= \sum_{i=0}^{n-1} 3^{n-1-i}2^{b_i} = \sum_{i=0}^{u-2} 3^{n-1-i}2^{b_i} + \sum_{i=u-1}^{v} 3^{n-1-i}2^{b_i} + \sum_{i=v+1}^{n-1} 3^{n-1-i}2^{b_i} \\ &= 3^{n-u+1}\sum_{i=0}^{u-2} 3^{u-2-i}2^{b_i} + 3^{n-1-v}2^{b_{u-1}}\sum_{i=u-1}^{v} 3^{v-i}2^{b_u^i} + 2^{b_{v+1}}\sum_{i=v+1}^{n-1} 3^{n-1-i}2^{b_{v+2}^i} \\ &= 3^{n-u+1}B_1^{u-2} + 3^{n-1-v}2^{b_{u-1}}B_u^v + 2^{b_{v+1}}B_{v+2}^{n-1}. \end{aligned}$$

□

Definition 3. *For any $1 \leqslant u \leqslant v$, define two integers $x_0^{u,v}$ and $x_{v-u+1}^{u,v}$ such that $2^{b_u^v}x_{v-u+1}^{u,v} - 3^{v-u+1}x_0^{u,v} = B_u^{v-1}$, $1 \leqslant x_0^{u,v} < 2^{b_u^v}$, and $1 \leqslant x_{v-u+1}^{u,v} < 3^{v-u+1}$. Further, define $x_1^{u,v} = \dfrac{3x_0^{u,v}+1}{2^{a_u}}$, $x_2^{u,v} = \dfrac{3x_1^{u,v}+1}{2^{a_{u+1}}}, \ldots, x_{v-u}^{u,v} = \dfrac{3x_{v-u-1}^{u,v}+1}{2^{a_{v-1}}}$.*

Clearly, $x_0^{1,n}$ and $x_n^{1,n}$ are the same as x_0 and x_n in Proposition 2, respectively.

Proposition 5. (i) $\quad x_{v-u+1}^{u,v} = \dfrac{3x_{v-u}^{u,v}+1}{2^{a_v}}$;

(ii) *For any* $0 \leqslant k \leqslant v-u$, $x_k^{u,v} = \dfrac{3^k x_0^{u,v} + B_u^{u+k-2}}{2^{b_u^{u+k-1}}}$, *and*
$x_{v-u+1}^{u,v} = \dfrac{3^{v-u+1-k} x_k^{u,v} + B_{u+k}^{v-1}}{2^{b_{u+k}^{v}}}$;

(iii) $x_0^{u,v} \leqslant x_0^{u,v+1}$;

(iv) $\Omega - \lim a_n = x$ *if and only if* $\lim_{n\to\infty} x_0^{1,n} = x$;

(v) $\Omega - \lim a_n = \infty$ *if and only if* $\lim_{n\to\infty} x_0^{1,n} = \infty$.

Proof. (i) is from Proposition 3(ii), which is from (i) and Proposition 1.

(iii) By Definition 3, $2^{b_u^v} x_{v-u+1}^{u,v} - 3^{v-u+1} x_0^{u,v} = B_u^{v-1}$, $2^{b_u^{v+1}} x_{v-u+2}^{u,v+1} - 3^{v-u+2} x_0^{u,v+1} = B_u^v$. Then, $3^{v-u+1} x_0^{u,v} + B_u^{v-1} \equiv 0 \,(\mathrm{mod}\ 2^{b_u^v})$, $3^{v-u+2} x_0^{u,v+1} + B_u^v \equiv 0 \,(\mathrm{mod}\ 2^{b_u^{v+1}})$. Thus, $3^{v-u+1} x_0^{u,v+1} + B_u^{v-1} \equiv 0 \,(\mathrm{mod}\ 2^{b_u^v})$ by $B_u^v = 3B_u^{v-1} + 2^{b_u^v}$. Hence, $x_0^{u,v} \equiv x_0^{u,v+1} \,(\mathrm{mod}\ 2^{b_u^v})$. Therefore, $x_0^{u,v} \leqslant x_0^{u,v+1}$ by $1 \leqslant x_0^{u,v} < 2^{b_u^v}$ and $1 \leqslant x_0^{u,v+1} < 2^{b_u^{v+1}}$.

By (iii), $(x_0^{1,n})_{n\geqslant 1}$ is increasing, then (iv) and (v) hold trivially. □

Proposition 5(iv) shows that if $\Omega - \lim a_n = x$, then $x_0^{1,n} = x$ for all sufficiently large n. Proposition 5(v) shows the reasonableness of $\Omega - \lim a_n = \infty$.

3. Periodic E-Sequences

Definition 4. (i) $(a_n)_{n\geqslant 1}$ *is periodic if there exist two integers* $l \geqslant 0$, $r \geqslant 1$ *such that* $a_n = a_{n+r}$ *for all* $n > l$;

(ii) r *is called the period of* $(a_n)_{n\geqslant 1}$;

(iii) $(a_1 \cdots a_l)$ *and* $(a_{l+1} \cdots a_{l+r} \cdots)$ *are called the non-periodic part and periodic part of* $(a_n)_{n\geqslant 1}$, *respectively;*

(iv) $(a_n)_{n\geqslant 1}$ *is called purely periodic if* $l = 0$ *and eventually periodic if* $l > 0$;

(v) *The E-sequence is denoted by* $a_1 \cdots a_l \overline{a_{l+1} \cdots a_{l+r}}$.

Throughout the remainder of this section, define $s = b_{l+1}^{l+r}$, $B_r = B_{l+1}^{l+r-1}$, and let $k \geqslant 0$ be an integer.

Proposition 6. *Let* $a_1 \cdots a_l \overline{a_{l+1} \cdots a_{l+r}}$ *be a periodic E-sequence. Then,* $B_{rk+l} = 3^{rk} B_l + 2^{b_l} B_r \dfrac{3^{rk} - 2^{sk}}{3^r - 2^s}$.

Proof. By Proposition 4, $B_{rk+l} = B_1^{rk+l-1} = 3^{rk} B_1^{l-1} + 3^{rk-r} 2^{b_l} B_{l+1}^{l+r-1} + 3^{rk-2r} 2^{b_{l+r}} B_{l+r+1}^{l+2r-1} + \cdots + 2^{b_{l+rk-r}} B_{l+1+r(k-1)}^{l+rk-1}$. By $b_{l+r} = b_l + s, b_{l+2r} = b_l + 2s, \cdots,$ $b_{l+rk-r} = b_l + (k-1)s$, $B_1^{l-1} = B_l$, $B_{l+r+1}^{l+2r-1} = \cdots = B_{l+1+r(k-1)}^{l+rk-1} = B_r$, we have:

$$\begin{aligned} B_{rk+l} &= 3^{rk} B_l + 3^{rk-r} 2^{b_l} B_r + 3^{rk-2r} 2^{b_l} 2^s B_r + \cdots + 2^{b_l} 2^{(k-1)s} B_r \\ &= 3^{rk} B_l + 2^{b_l} B_r (3^{rk-r} 2^0 + 3^{rk-2r} 2^s + \cdots + 3^0 2^{(k-1)s}) \\ &= 3^{rk} B_l + 2^{b_l} B_r \frac{3^{rk} - 2^{sk}}{3^r - 2^s}. \end{aligned}$$

□

Proposition 7. *Let* $a_1 \cdots a_l \overline{a_{l+1} \cdots a_{l+r}}$ *be a periodic E-sequence. By Proposition 2, define two integers* x_0 *and* x_{rk+l} *such that* $2^{sk+b_l} x_{rk+l} - 3^{rk+l} x_0 = B_{rk+l}$, $1 \leqslant x_0 < 2^{sk+b_l}$ *and* $1 \leqslant x_{rk+l} < 3^{rk+l}$. *Then, there is a constant* $K \in \mathbb{N}$, *depending on* $a_1, \cdots, a_{l+r}$ *such that when* $k > K$ *and,*

(i) *if* $2^s > 3^r$, *there is* $u_{rk+l} \in \mathbb{Z}$, $0 \leqslant u_{rk+l} < (2^s - 3^r)3^l$ *such that*
$x_0 = \dfrac{2^{sk+b_l} u_{rk+l} - B_l(2^s - 3^r) + 2^{b_l} B_r}{(2^s - 3^r)3^l}$, $x_{rk+l} = \dfrac{3^{rk} u_{rk+l} + B_r}{2^s - 3^r}$;

(ii) *if* $3^r > 2^s$, *there is* $u_{rk+l} \in \mathbb{N}$, $1 \leqslant u_{rk+l} \leqslant (3^r - 2^s)3^l$ *such that*
$x_0 = \dfrac{2^{sk+b_l}u_{rk+l} - B_l(3^r - 2^s) - 2^{b_l}B_r}{(3^r - 2^s)3^l}$, $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} - B_r}{3^r - 2^s}$.

Proof. (i) $2^s > 3^r$. By $x_{rk+l} = \dfrac{3^{rk+l}x_0 + B_{rk+l}}{2^{sk+b_l}}$, we have $2^{sk+b_l}x_{rk+l} \equiv B_{rk+l}\ (mod\ 3^{rk+l})$. Then, $2^{sk+b_l}x_{rk+l} \equiv 3^{rk}B_l + 2^{b_l}B_r\dfrac{2^{sk} - 3^{rk}}{2^s - 3^r}\ (mod\ 3^{rk+l})$ by Proposition 6. Thus, $(2^s - 3^r)2^{sk+b_l}x_{rk+l} \equiv (2^s - 3^r)3^{rk}B_l + (2^{sk} - 3^{rk})2^{b_l}B_r\ (mod\ (2^s - 3^r)3^{rk+l})$. Hence, $2^{sk+b_l}((2^s - 3^r)x_{rk+l} - B_r) \equiv 3^{rk}((2^s - 3^r)B_l - 2^{b_l}B_r)\ (mod\ (2^s - 3^r)3^{rk+l})$. Define $u_{rk+l} = \dfrac{(2^s - 3^r)x_{rk+l} - B_r}{3^{rk}}$. Then, $u_{rk+l} \in \mathbb{Z}$ and $2^{sk+b_l}u_{rk+l} \equiv (2^s - 3^r)B_l - 2^{b_l}B_r\ (mod\ (2^s - 3^r)3^l)$. Hence $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} + B_r}{2^s - 3^r}$ and:

$$
\begin{aligned}
x_0 &= \frac{2^{sk+b_l}x_{rk+l} - B_{rk+l}}{3^{rk+l}} \\
&= \frac{2^{sk+b_l}\dfrac{3^{rk}u_{rk+l} + B_r}{2^s - 3^r} - 3^{rk}B_l - 2^{b_l}B_r\dfrac{2^{sk} - 3^{rk}}{2^s - 3^r}}{3^{rk+l}} \\
&= \frac{3^{rk}2^{sk+b_l}u_{rk+l} + 2^{sk+b_l}B_r - 3^{rk}B_l(2^s - 3^r) + 3^{rk}2^{b_l}B_r - 2^{sk+b_l}B_r}{(2^s - 3^r)3^{rk+l}} \\
&= \frac{2^{sk+b_l}u_{rk+l} - B_l(2^s - 3^r) + 2^{b_l}B_r}{(2^s - 3^r)3^l}.
\end{aligned}
$$

By $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} + B_r}{2^s - 3^r} < 3^{rk+l}$, we have $u_{rk+l} < \dfrac{3^{rk+l}(2^s - 3^r) - B_r}{3^{rk}} = 3^l(2^s - 3^r) - \dfrac{B_r}{3^{rk}} < 3^l(2^s - 3^r)$.
By $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} + B_r}{2^s - 3^r} > 0$, we have $u_{rk+l} > -\dfrac{B_r}{3^{rk}}$. Since $\lim_{k\to\infty} -\dfrac{B_r}{3^{rk}} = 0$ and $u_{rk+l} \in \mathbb{Z}$, there is a constant $K \in \mathbb{N}$, depending on $a_1, \cdots, a_{l+r}$ such that $u_{rk+l} \geqslant 0$ when $k > K$.

(ii) $3^r > 2^s$. By $x_{rk+l} = \dfrac{3^{rk+l}x_0 + B_{rk+l}}{2^{sk+b_l}}$, we have:

$$2^{sk+b_l}((3^r - 2^s)x_{rk+l} + B_r) \equiv 3^{rk}((3^r - 2^s)B_l + 2^{b_l}B_r)\ (mod(3^r - 2^s)3^{rk+l}).$$

Define $u_{rk+l} = \dfrac{(3^r - 2^s)x_{rk+l} + B_r}{3^{rk}}$. Then:

$$u_{rk+l} \in \mathbb{Z}, 2^{sk+b_l}u_{rk+l} \equiv (3^r - 2^s)B_l + 2^{b_l}B_r\ (mod(3^r - 2^s)3^l).$$

Thus, $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} - B_r}{3^r - 2^s}$, $x_0 = \dfrac{2^{sk+b_l}u_{rk+l} - B_l(3^r - 2^s) - 2^{b_l}B_r}{(3^r - 2^s)3^l}$. Since $x_{rk+l} = \dfrac{3^{rk}u_{rk+l} - B_r}{3^r - 2^s} > 0$, then $u_{rk+l} > \dfrac{B_r}{3^{rk}}$, and thus, $1 \leqslant u_{rk+l}$.

By $x_0 = \dfrac{2^{sk+b_l}u_{rk+l} - B_l(3^r - 2^s) - 2^{b_l}B_r}{(3^r - 2^s)3^l} < 2^{sk+b_l}$, we have $u_{rk+l} < (3^r - 2^s)3^l + \dfrac{B_l(3^r - 2^s) + 2^{b_l}B_r}{2^{sk+b_l}}$.
Since $\lim_{k\to\infty} \dfrac{B_l(3^r - 2^s) + 2^{b_l}B_r}{2^{sk+b_l}} = 0$ and $u_{rk+l} \in \mathbb{Z}$, there is a $K \in \mathbb{N}$ such that $u_{rk+l} \leqslant (3^r - 2^s)3^l$ when $k > K$.
□

Theorem 1. *If* $3^r > 2^s$, *then* $a_1 \cdots a_l\overline{a_{l+1}a_{l+2}\cdots a_{l+r-1}a_{l+r}}$ *is* Ω*-divergent.*

Proof. By Proposition 7(ii), $x_0 = \dfrac{2^{sk+b_l} u_{rk+l} - B_l(3^r - 2^s) - 2^{b_l} B_r}{(3^r - 2^s)3^l}$ and $u_{rk+l} \geqslant 1$. Then, $x_0 \to +\infty$ as $k \to \infty$. Thus, the E-sequence is Ω-divergent. □

Theorem 2. *If $a_1 \cdots a_l \overline{a_{l+1} a_{l+2} \cdots a_{l+r-1} a_{l+r}}$ is Ω-convergent to x, then $(x_n)_{n \geqslant 0}$ is periodic.*

Proof. By Theorem 1, $2^s > 3^r$. By Proposition 7(i),

$$x_0 = \frac{2^{sk+b_l} u_{rk+l} - B_l(2^s - 3^r) + 2^{b_l} B_r}{(2^s - 3^r)3^l}$$

and $u_{rk+l} \geqslant 0$ for all $k > K$. Since $x_0 = x < \infty$ for all sufficiently large k, by Proposition 5(iv), then $u_{rk+l} = 0$. Thus, $x_0 = \dfrac{2^{b_l} B_r - B_l(2^s - 3^r)}{(2^s - 3^r)3^l}$ and $x_{rk+l} = \dfrac{B_r}{2^s - 3^r}$ for all $k \geq 0$. Hence, $(x_n)_{n \geqslant 0}$ is periodic, and its non-periodic part and periodic part are $(x_0 x_1 \cdots x_l)$ and $\overline{x_{l+1} \cdots x_{l+r}}$, respectively. □

Theorem 3. *Assume that all non-periodic E-sequence are Ω-divergent. Then, the trajectory of every odd positive integer is periodic.*

Proof. Suppose that x is an odd positive integer, $(x_n)_{n \geqslant 0}$ and $(a_n)_{n \geqslant 1}$ are its trajectory and E-sequence, respectively. Then, $\Omega - \lim a_n = x$. Thus, $(a_n)_{n \geqslant 1}$ is periodic by the assumption. Hence, $(x_n)_{n \geqslant 0}$ is periodic by Theorem 2. □

4. Non-Periodic E-Sequences

For any real number α, $\{\alpha\}$ denotes its fractional part. The following lemma is due to Matthews and Watts (see Lemma 2(b) in [13]). We present its proof for the reader's convenience.

Lemma 1. *Let $(a_n)_{n \geqslant 1}$ be an E-sequence such that $\Omega - \lim\ a_n = x_0$ and $(x_n)_{n \geqslant 0}$ is unbounded. Then, $\overline{\lim\limits_{n \to \infty}} \dfrac{b_n}{n} \leqslant \log_2 3$.*

Proof. From $x_k = \dfrac{3x_{k-1} + 1}{2^{a_k}}$, we have $2^{a_k} = \dfrac{3x_{k-1} + 1}{x_k}$. Then:

$$2^{b_n} = \prod_{k=1}^{n} 2^{a_k} = \prod_{k=1}^{n} \frac{3x_{k-1} + 1}{x_k} = \frac{x_0}{x_n} \prod_{k=1}^{n} \frac{3x_{k-1} + 1}{x_{k-1}} = \frac{3^n x_0}{x_n} \prod_{k=1}^{n} (1 + \frac{1}{3x_{k-1}}).$$

Thus:

$$x_n = \frac{3^n x_0}{2^{b_n}} \prod_{k=1}^{n} (1 + \frac{1}{3x_{k-1}})$$

which we call the Matthews and Watts' formula (see Lemma 1(b) in [13]).

Since $(x_n)_{n \geqslant 1}$ is unbounded, all x_n are distinct. Then:

$$1 \leqslant x_n \leqslant \frac{3^n x_0}{2^{b_n}} \prod_{k=1}^{n} (1 + \frac{1}{3k}).$$

Thus:

$$0 \leqslant \log \frac{3^n}{2^{b_n}} + \log x_0 + \sum_{k=1}^{n} \log(1 + \frac{1}{3k}) \leqslant \log 3^n - \log 2^{b_n} + \log x_0 + \sum_{k=1}^{n} \frac{1}{3k}.$$

Hence:

$$\log 2^{b_n} \leqslant \log 3^n + \log x_0 + \frac{1}{3}\sum_{k=1}^{n}\frac{1}{k}.$$

Therefore:

$$\frac{b_n}{n} \leqslant \log_2 3 + \frac{\log_2 x_0}{n} + \frac{1}{n\log 8}\sum_{k=1}^{n}\frac{1}{k}.$$

Then:

$$\overline{\lim_{n\to\infty}}\frac{b_n}{n} \leqslant \log_2 3.$$

□

Theorem 4. *Let $(a_n)_{n\geqslant 1}$ be a non-periodic E-sequence such that $\overline{\lim\limits_{n\to\infty}}\frac{b_n}{n} > \log_2 3$. Then, $\Omega - \lim a_n = \infty$.*

Proof. Suppose that $\Omega - \lim a_n = x_0$ for some positive integer x_0. It follows from Lemma 1 and $\overline{\lim\limits_{n\to\infty}}\frac{b_n}{n} > \log_2 3$ that $(x_n)_{n\geqslant 0}$ is bounded. Then, $(x_n)_{n\geqslant 0}$ is periodic. Thus, $(a_n)_{n\geqslant 1}$ is periodic, which contradicts the non-periodicity of $(a_n)_{n\geqslant 1}$. Hence, $\Omega - \lim a_n = \infty$. □

The following lemma is the well known Wendel's inequality (see [15]). Lemma 3 is a consequence of an easy calculation.

Lemma 2. *Let x be a positive real number, and let $s \in (0,1)$. Then, $\frac{\Gamma(x+s)}{\Gamma(x)} \leqslant x^s$.*

Lemma 3. *Let a and b be two integers with $a \geqslant 1$ and $a \nmid b$. Then, $\prod\limits_{k=0}^{n}(1+\frac{z}{ak+b}) = \frac{\Gamma(\frac{b}{a})\Gamma(\frac{b+z}{a}+n+1)}{\Gamma(\frac{b+z}{a})\Gamma(\frac{b}{a}+n+1)}$.*

Lemma 4. $\prod\limits_{1\leqslant k<3n,\ k\equiv 1,5(\bmod\ 6)}(1+\frac{1}{3k}) < 1.5n^{\frac{1}{9}}$ *for all $n \geqslant 1$.*

Proof. Let $2|n$. Then:

$$\prod_{k=0}^{\frac{n}{2}-1}(1+\frac{1}{3(6k+1)}) = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{n}{2}+\frac{2}{9})}{\Gamma(\frac{2}{9})\Gamma(\frac{n}{2}+\frac{1}{6})} \leqslant \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{9})}(\frac{n}{2}+\frac{1}{6})^{\frac{1}{18}}$$

and:

$$\prod_{k=0}^{\frac{n}{2}-1}(1+\frac{1}{3(6k+5)}) = \frac{\Gamma(\frac{5}{6})\Gamma(\frac{n}{2}+\frac{8}{9})}{\Gamma(\frac{8}{9})\Gamma(\frac{n}{2}+\frac{5}{6})} \leqslant \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{8}{9})}(\frac{n}{2}+\frac{5}{6})^{\frac{1}{18}}$$

by Wendel's inequality. Thus,

$$\prod_{1\leqslant k<3n,\ k\equiv 1,5(\bmod\ 6)}(1+\frac{1}{3k}) = \prod_{k=0}^{\frac{n}{2}-1}(1+\frac{1}{3(6k+1)})\prod_{k=0}^{\frac{n}{2}-1}(1+\frac{1}{3(6k+5)}) \leqslant$$

$$\frac{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{9})\Gamma(\frac{8}{9})}(\frac{n}{2}+\frac{5}{6})^{\frac{1}{18}}(\frac{n}{2}+\frac{1}{6})^{\frac{1}{18}} \leqslant 1.4196(\frac{n^2}{3})^{\frac{1}{18}} < 1.5n^{\frac{1}{9}}.$$

Let $2 \nmid n$. Then:

$$\prod_{k=0}^{\frac{n+1}{2}-1}(1+\frac{1}{3(6k+1)}) = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{n}{2}+\frac{13}{18})}{\Gamma(\frac{2}{9})\Gamma(\frac{n}{2}+\frac{2}{3})} \leqslant \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{9})}(\frac{n}{2}+\frac{2}{3})^{\frac{1}{18}}$$

and

$$\prod_{k=0}^{\frac{n+1}{2}-2}(1+\frac{1}{3(6k+5)}) = \frac{\Gamma(\frac{5}{6})\Gamma(\frac{n}{2}+\frac{7}{18})}{\Gamma(\frac{8}{9})\Gamma(\frac{n}{2}+\frac{1}{3})} \leqslant \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{8}{9})}(\frac{n}{2}+\frac{1}{3})^{\frac{1}{18}}$$

by Wendel's inequality. Thus,

$$\prod_{1\leqslant k<3n,\, k\equiv 1,5(\bmod 6)}(1+\tfrac{1}{3k}) = \prod_{k=0}^{\frac{n+1}{2}-1}(1+\tfrac{1}{3(6k+1)})\prod_{k=0}^{\frac{n+1}{2}-2}(1+\tfrac{1}{3(6k+5)}) \leqslant$$

$$\frac{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{9})\Gamma(\frac{8}{9})}(\frac{n}{2}+\frac{2}{3})^{\frac{1}{18}}(\frac{n}{2}+\frac{1}{3})^{\frac{1}{18}} < 1.5n^{\frac{1}{9}}.$$

□

Theorem 5. *Let $2^{b_n}x_n - 3^n x_0 = B_n$ such that $1 \leqslant x_0 < 2^{b_n}, 1 \leqslant x_n < 3^n$, $3 \nmid x_0$, and $x_0, \cdots, x_{n-1}$ are distinct integers. Then, $x_0 > \dfrac{B_n}{3^n(1.5n^{\frac{1}{9}}-1)}$.*

Proof. From the Matthews and Watts' formula and Lemma 4, we have:

$$\frac{2^{b_n}x_n}{3^n x_0} = \prod_{k=1}^{n}(1+\frac{1}{3x_{k-1}}) \leqslant \prod_{1\leqslant k<3n,\ k\equiv 1,5(\bmod 6)}(1+\frac{1}{3k}) < 1.5n^{\frac{1}{9}}.$$

Then, $\dfrac{3^n x_0 + B_n}{3^n x_0} < 1.5n^{\frac{1}{9}}$. Thus, $x_0 > \dfrac{B_n}{3^n(1.5n^{\frac{1}{9}}-1)}$. □

Corollary 1. *Let $\theta \geqslant log_2 3$ be an irrational number. Define $a_n = [n\theta] - [(n-1)\theta]$. Then, $\Omega - \lim\ a_n = \infty$.*

Proof. Let $\theta = \log_2 3$. Then, $\dfrac{B_n}{3^n} = \sum_{k=1}^{n}\dfrac{2^{[(k-1)\log_2 3]}}{3^k} > \dfrac{n}{8}$ by $\dfrac{2^{[(k-1)\log_2 3]}}{3^k} > \dfrac{1}{8}$. Thus, $\dfrac{B_n}{3^n(1.5n^{\frac{1}{9}}-1)} > \dfrac{n}{8(1.5n^{\frac{1}{9}}-1)} \to \infty$, as $n \to \infty$. Hence, $\Omega - \lim\ a_n = \infty$ by Theorem 5.

Let $\theta > \log_2 3$. Then, $\lim_{n\to\infty}\dfrac{b_n}{n} = \lim_{n\to\infty}\dfrac{[\mathrm{n}\theta]}{n} = \theta > \log_2 3$. Since θ is an irrational number, $(a_n)_{n\geqslant 1}$ is non-periodic. Thus, $\Omega - \lim\ a_n = \infty$ by Theorem 4. □

Lemma 5. *Let x and n be two positive integers. Then, (i) $\prod_{k=0}^{n-1}(1+\dfrac{1}{3(x+k)}) \leq 1+\dfrac{n}{3x}$; (ii) $\prod_{k=0}^{n-1}(1+\dfrac{1}{3(x-k)}) \geq 1+\dfrac{n}{3x}$ for $x \geq n$; (iii) $\prod_{k=0}^{n-1}(1+\dfrac{1}{3(x-k)}) > \dfrac{3x}{3x-n}$ for $x \geq n \geq 2$.*

Proof. (i) The proof is by induction on n. For the base step, let $n = 1$, then $\prod_{k=0}^{n-1}(1+\dfrac{1}{3(x+k)}) = 1+\dfrac{1}{3x} = 1+\dfrac{n}{3x}$. For the induction step, assume that $\prod_{k=0}^{n-1}(1+\dfrac{1}{3(x+k)}) \leq 1+\dfrac{n}{3x}$. Then, $\prod_{k=0}^{n}(1+\dfrac{1}{3(x+k)}) \leq (1+\dfrac{n}{3x})(1+\dfrac{1}{3(x+n)}) = 1+\dfrac{n}{3x}+\dfrac{1}{3(x+n)}+\dfrac{n}{9x(x+n)} \leq 1+\dfrac{n+1}{3x}$. Thus, the inequality holds for all $n \geq 1$. The proof of (ii) is similar to that of (i) and omitted.

(iii) Let $n = 2$. Since $3x \cdot 3x - 2 \cdot 3x - 3x + 2 > 3x \cdot 3x - 3 \cdot 3x$, then $\frac{3x-1}{3x \cdot 3(x-1)} > \frac{1}{3x-2}$. Thus, $1 + \frac{1}{3x} + \frac{1}{3(x-1)} + \frac{1}{3x \cdot 3(x-1)} > \frac{3x-2+2}{3x-2} = 1 + \frac{2}{3x-2}$. Hence, $(1+\frac{1}{3x})(1+\frac{1}{3(x-1)}) > \frac{3x}{3x-2}$. Therefore, $\prod_{k=0}^{n-1}(1+\frac{1}{3(x-k)}) > \frac{3x}{3x-n}$.

Assume that $\prod_{k=0}^{n-1}(1+\frac{1}{3(x-k)}) > \frac{3x}{3x-n}$. Since $(3x-3n+1)(3x-n-1) > (3x-n)(3x-3n)$, then $\frac{3x(3x-3n)+3x}{(3x-n)(3x-3n)} > \frac{3x}{3x-n-1}$. Thus, $\prod_{k=0}^{n}(1+\frac{1}{3(x-k)}) > \frac{3x}{3x-n}(1+\frac{1}{3(x-n)}) = \frac{3x}{3x-n} + \frac{3x}{(3x-n)(3x-3n)} > \frac{3x}{3x-n-1}$. □

Lemma 6. *Let $2^{b_n}x_n - 3^n x_0 = B_n$ such that $1 \le x_0 < 2^{b_n}$, $1 \le x_n < 3^n$, $x_i \ne x_j$ for all $0 \le i < j \le n-1$. Then, (i) $\frac{B_n}{3^n} \le \frac{n}{3}$ if $x_k > x_0$ for all $1 \le k \le n-1$; (ii) $\frac{B_n}{2^{b_n}} < \frac{n}{3}$ if $x_n < x_k$ for all $0 \le k \le n-1$; (iii) $\frac{B_n}{2^{b_n}} > \frac{n}{3}$ if $x_n > x_i$ for all $0 \le i \le n-1$; (iv) $\frac{B_n}{3^n} \ge \frac{n}{3}$ if $x_0 > x_k$ for all $1 \le k \le n$.*

Proof. (i) From $\frac{2^{b_n}x_n}{3^n x_0} = \prod_{k=0}^{n-1}(1+\frac{1}{3x_k})$, we have:

$$1 + \frac{B_n}{3^n x_0} = \prod_{k=0}^{n-1}(1+\frac{1}{3x_k}) \le \prod_{k=0}^{n-1}(1+\frac{1}{3(x_0+k)}).$$

Then, $1 + \frac{B_n}{3^n x_0} \le 1 + \frac{n}{3x_0}$ by Lemma 5(i). Thus, $\frac{B_n}{3^n} \le \frac{n}{3}$.

(ii) From $\frac{2^{b_n}x_n}{3^n x_0} = \prod_{k=0}^{n-1}(1+\frac{1}{3x_k})$, we have:

$$\frac{2^{b_n}x_n - B_n}{2^{b_n}x_n} = \prod_{k=0}^{n-1}(1+\frac{1}{3x_k})^{-1} \ge \prod_{k=0}^{n-1}(1+\frac{1}{3(x_n+k)})^{-1}.$$

Then, $1 - \frac{B_n}{2^{b_n}x_n} \ge \prod_{k=0}^{n-1}(1+\frac{1}{3(x_n+k)})^{-1} \ge (1+\frac{n}{3x_n})^{-1}$ by Lemma 5(i). Thus:

$$\frac{B_n}{2^{b_n}x_n} \le 1 - (1+\frac{n}{3x_n})^{-1} = \frac{n}{3x_n+n}.$$

Hence, $\frac{B_n}{2^{b_n}} \le \frac{nx_n}{3x_n+n} < \frac{n}{3}$.

(iii) Let $n = 1$. Then, $x_1 = \frac{3x+1}{2^{a_1}} > x$. Thus, $(3-2^{a_1})x + 1 > 0$. Hence, $a_1 = 1$. Therefore, $\frac{B_n}{2^{b_n}} = \frac{B_1}{2^{b_1}} = \frac{1}{2} > \frac{1}{3} = \frac{n}{3}$.

Let $x_n \ge n \ge 2$. By Lemma 5(iii), we have $\frac{2^{b_n}x_n}{3^n x_0} = \prod_{k=0}^{n-1}(1+\frac{1}{3x_k}) \ge \prod_{k=0}^{n-1}(1+\frac{1}{3(x_n-k)}) > \frac{3x_n}{3x_n-n}$. Then, $\frac{2^{b_n}x_n}{2^{b_n}x_n - B_n} > \frac{3x_n}{3x_n-n}$. Thus, $\frac{2^{b_n}x_n - B_n}{2^{b_n}x_n} < \frac{3x_n-n}{3x_n}$. Hence, $\frac{B_n}{2^{b_n}} > \frac{n}{3}$.

(iv) By Lemma 5(ii), we have:

$$1+\frac{B_n}{3^n x_0}=\frac{2^{b_n}x_n}{3^n x_0}=\prod_{k=0}^{n-1}(1+\frac{1}{3x_k})\geq\prod_{k=0}^{n-1}(1+\frac{1}{3(x_0-k)})\geq 1+\frac{n}{3x_0}.$$

Then, $\frac{B_n}{3^n}\geq\frac{n}{3}$. □

A direct consequence of Lemma 6 is the following theorem, which may imply something unknown.

Theorem 6. *Let $2^{b_n}x_n-3^n x_0=B_n$ such that $1\leq x_0<2^{b_n}$, $1\leq x_n<3^n$, $x_i\neq x_j$ for all $0\leq i<j\leq n-1$. Then:*

(i) $\frac{B_n}{3^n}>\frac{n}{3}$ implies $x_k\leq x_0$ for some $1\leq k\leq n-1$;

(ii) $\frac{B_n}{3^n}<\frac{n}{3}$ implies $x_0\leq x_k$ for some $1\leq k\leq n$;

(iii) $\frac{B_n}{2^{b_n}}\leq\frac{n}{3}$ implies $x_n\leq x_i$ for some $0\leq i\leq n-1$;

(iv) $\frac{B_n}{2^{b_n}}\geq\frac{n}{3}$ implies $x_n\geq x_k$ for some $0\leq k\leq n-1$.

Theorem 7. *Let $(a_n)_{n\geqslant 1}$ be an E-sequence such that (i) $3^n>2^{b_n}$ for all $n\in\mathbb{N}$; (ii) there is a constant $c>\log_2 3$ such that there are infinitely many distinct pairs (k,l) of positive integers such that $l>kc$, $a_{k+1}=\cdots=a_l=1$. Then, $\Omega-\lim a_n=\infty$.*

Proof. It follows from (i) that $B_n<3^n n$ for all $n\in\mathbb{N}$ by induction on n. $B_{k+1}^{l-1}=3^{l-k}-2^{l-k}$ by Proposition 6. Let $x_l^{1,l}=\frac{3^l x_0^{1,l}+B_1^{l-1}}{2^{b_l}}$, $1\leqslant x_0^{1,l}<2^{b_l}$, $1\leqslant x_l^{1,l}<3^l$. Then, $x_k^{1,l}=\frac{3^k x_0^{1,l}+B_1^{k-1}}{2^{b_k}}$, $x_l^{1,l}=\frac{3^{l-k}x_k^{1,l}+B_{k+1}^{l-1}}{2^{b_{k+1}^l}}$ by Proposition 5(ii). By $B_{k+1}^{l-1}=3^{l-k}-2^{l-k}$, $2^{b_{k+1}^l}=2^{l-k}$, we have $2^{l-k}(x_l^{1,l}+1)=3^{l-k}(x_k^{1,l}+1)$. Thus, $x_k^{1,l}=2^{l-k}w-1$ for some $1\leqslant w$. Hence, $x_k^{1,l}=\frac{3^k x_0^{1,l}+B_1^{k-1}}{2^{b_k}}=2^{l-k}w-1$. Therefore, $x_0^{1,l}=\frac{2^{l-k}2^{b_k}w-2^{b_k}-B_1^{k-1}}{3^k}\geqslant\frac{2^l}{3^k}2^{b_k-k}-1-k\geqslant(\frac{2^c}{3})^k 2^{b_k-k}-1-k$. If there are only finitely many distinct k in all pairs (k,l), $x_0^{1,l}\geqslant\frac{2^l}{3^k}2^{b_k-k}-1-k\to\infty$, as $l\to\infty$; otherwise, $x_0^{1,l}\geqslant(\frac{2^c}{3})^k 2^{b_k-k}-1-k\to\infty$, as $k\to\infty$. Then, $\Omega-\lim a_n=\infty$. □

Corollary 2. *Let $(a_n)_{n\geqslant 1}$ be the E-sequence $12121112\cdots$, where $a_n=2$ if $n\in\{2^1,2^2,2^3,\cdots\}$ and $a_n=1$ otherwise. Then, $\Omega-\lim a_n=\infty$.*

Proof. Take $c=\frac{7}{4}>\log_2 3$, $k=2^m$, and $l=2^{m+1}-1$. Then, $a_{k+1}=\cdots=a_l=1$, $l>kc$ for all $m\geqslant 3$. Thus, $\Omega-\lim a_n=\infty$ by Theorem 7. □

Theorem 8. *Let $(a_n)_{n\geqslant 1}$ be an E-sequence such that (i) $3^n>2^{b_n}$ for all $n\in\mathbb{N}$; (ii) there is a constant $c>\log_2 3$ such that there are infinitely many distinct pairs (r,l) of positive integers such that $l>r$, $b_{l+r}>lc$, $a_{l+k}=a_k$ for all $1\leqslant k\leqslant r$, i.e., $(a_1\cdots a_r)a_{r+1}\cdots a_l(a_{l+1}\cdots a_{l+r})$ is contained in $(a_n)_{n\geqslant 1}$. Then, $\Omega-\lim a_n=\infty$.*

Proof. Let $x_{l+r}^{1,l+r}=\frac{3^{l+r}x_0^{1,l+r}+B_1^{l+r-1}}{2^{b_1^{l+r}}}$, $1\leqslant x_0^{1,l+r}<2^{b_1^{l+r}}$, $1\leqslant x_{l+r}^{1,l+r}<3^{l+r}$. Then, $x_l^{1,l+r}=\frac{3^l x_0^{1,l+r}+B_1^{l-1}}{2^{b_1^l}}$, $x_{l+r}^{1,l+r}=\frac{3^r x_l^{1,l+r}+B_{l+1}^{l+r-1}}{2^{b_{l+1}^{l+r}}}=\frac{3^r x_l^{1,l+r}+B_1^{r-1}}{2^{b_1^r}}$ by Proposition 5(ii). By $3^l>2^{b_1^l}$, we have $x_l^{1,l+r}>x_0^{1,l+r}$.

Let $x_r^{1,r} = \dfrac{3^r x_0^{1,r} + B_1^{r-1}}{2^{b_1^r}}$, $1 \leqslant x_0^{1,r} < 2^{b_1^r}$, $1 \leqslant x_r^{1,r} < 3^r$. Then, $x_0^{1,r} \equiv x_l^{1,l+r} (\bmod\ 2^{b_1^r})$. By Proposition 5(iii), we have $x_0^{1,l+r} \geqslant x_0^{1,r}$. Let $x_l^{1,l+r} = 2^{b_1^r} u + x_0^{1,r}$. Then, $u \geqslant 1$ by $x_l^{1,l+r} > x_0^{1,l+r} \geqslant x_0^{1,r}$. Thus:

$$x_0^{1,l+r} = \frac{2^{b_1^l} 2^{b_1^r} u + 2^{b_1^l} x_0^{1,r} - B_1^{l-1}}{3^l} \geqslant \frac{2^{b_1^{l+r}}}{3^l} - l \geqslant (\frac{2^c}{3})^l - l \to \infty, \ as\ l \to \infty.$$

Hence, $\Omega - \lim a_n = \infty$. □

Theorem 9. *Let $1 \leqslant \theta < \log_2 3$, and define $a_n = [n\theta] - [(n-1)\theta]$. Then, $\Omega - \lim a_n = \infty$.*

Proof. If θ is a rational number, then $(a_n)_{n\geqslant 1}$ is purely periodic, and the result follows from Theorem 1. Let θ be an irrational number in the following. By the Hurwitz theorem, there are infinite convergents $\frac{s}{r}$ of θ such that $|\theta - \frac{s}{r}| < \frac{1}{\sqrt{5}r^2}$. There are two cases to be considered.

Case 1. There are infinite convergents $\frac{s}{r}$ of θ such that $0 < \theta - \frac{s}{r} < \frac{1}{\sqrt{5}r^2}$. We prove that $[\theta n] = [\frac{s}{r}n]$ for all $1 \leqslant n \leqslant [\sqrt{5}r]$. By $1 \leqslant n \leqslant \left[\sqrt{5}r\right]$, we have $0 < \theta n - \frac{s}{r}n < \frac{n}{\sqrt{5}r^2} < \frac{\sqrt{5}r}{\sqrt{5}r^2} = \frac{1}{r}$. Then, $0 \leqslant \{\frac{s}{r}n\} < \theta n - [\frac{s}{r}n] < \frac{1}{r} + \{\frac{s}{r}n\} \leqslant 1$. Thus, $0 < \theta n - [\frac{s}{r}n] < 1$. Hence, $[\theta n] = [\frac{s}{r}n]$. Then, we have the following periodic table for $(a_n)_{1\leqslant n\leqslant\left[\sqrt{5}r\right]}$.

a_1	a_2	$\cdots$	$a_{\left[\sqrt{5}r-2r\right]}$	$\cdots$	a_r
a_{r+1}	a_{2+r}	$\cdots$	$a_{\left[\sqrt{5}r-r\right]}$	$\cdots$	a_{2r}
a_{2r+1}	a_{2+2r}	$\cdots$	$a_{\left[\sqrt{5}r\right]}$		

By Proposition 7(ii), $x_0^{1,2r} = \dfrac{2^{2[r\theta]} u_{2r} - B_r}{3^r - 2^{[r\theta]}}$ for some $u_{2r} \geqslant 1$.

By $B_r = \sum\limits_{i=0}^{r-1} 3^{r-1-i} 2^{b_i} = 3^{r-1} \sum\limits_{i=0}^{r-1} \dfrac{2^{b_i}}{3^i} \leqslant 3^{r-1} \sum\limits_{i=0}^{r-1} \dfrac{2^{[i\theta]}}{3^i} \leqslant 3^{r-1} \sum\limits_{i=0}^{r-1} \dfrac{2^{i\theta}}{3^i} = \dfrac{3^r}{3} \dfrac{1 - (\frac{2^\theta}{3})^r}{1 - \frac{2^\theta}{3}} = \dfrac{3^r - 2^{r\theta}}{3 - 2^\theta} \leqslant \dfrac{3^r}{3 - 2^\theta}$, we have:

$$x_0^{1,2r} \geqslant \frac{2^{2[r\theta]} - B_r}{3^r - 2^{[r\theta]}} \geqslant \frac{4^{r\theta - 1} - \frac{3^r}{3 - 2^\theta}}{3^r - 2^{r\theta - 1}} = \frac{\frac{1}{4}(\frac{4^\theta}{3})^r - \frac{1}{3-2^\theta}}{1 - \frac{1}{2}(\frac{2^\theta}{3})^r}.$$

Thus, $x_0^{1,2r} \to \infty$, as $r \to \infty$. Hence, $\Omega - \lim a_n = \infty$.

Case 2. There are infinite convergents $\frac{s}{r}$ of θ such that $0 < \frac{s}{r} - \theta < \frac{1}{\sqrt{5}r^2}$.

Firstly, we prove $[\theta n] = [\frac{s}{r}n]$ for all $1 \leqslant n \leqslant [\sqrt{5}r]$, $n \notin \{r, 2r\}$. By $0 < \frac{s}{r} - \theta < \frac{1}{\sqrt{5}r^2}$, we have $\frac{s}{r} - \frac{1}{\sqrt{5}r^2} < \theta < \frac{s}{r}$. Then, $\frac{s}{r}n - [\frac{s}{r}n] - \frac{n}{\sqrt{5}r^2} < \theta n - [\frac{s}{r}n] < \frac{s}{r}n - [\frac{s}{r}n] < 1$. By $1 \leqslant n \leqslant [\sqrt{5}r]$, $n \notin \{r, 2r\}$, we have $0 < \frac{1}{r} - \frac{n}{\sqrt{5}r^2} \leqslant \frac{s}{r}n - [\frac{s}{r}n] - \frac{n}{\sqrt{5}r^2}$. Then, $0 < \theta n - [\frac{s}{r}n] < 1$. Thus, $[\theta n] = [\frac{s}{r}n]$.

Secondly, we prove $[r\theta] = s-1$, $[2r\theta] = 2s-1$. By $1 \leqslant n$, $0 < \frac{s}{r} - \theta < \frac{1}{\sqrt{5}r^2}$, we have $-\frac{n}{\sqrt{5}r^2} + \frac{s}{r}n < n\theta < \frac{s}{r}n$. By $n < \sqrt{5}r$, we have $-1 < -\frac{1}{r} < -\frac{n}{\sqrt{5}r^2}$. Then, $-1 + \frac{s}{r}n < -\frac{n}{\sqrt{5}r^2} + \frac{s}{r}n < n\theta < \frac{s}{r}n$. By taking $n = r, 2r$, we have $[r\theta] = s-1$, $[2r\theta] = 2s-1$.

Let $2 \leqslant j \leqslant r-1$, then $r+2 \leqslant r+j \leqslant 2r-1$ and $r+1 \leqslant r+j-1 \leqslant 2r-2$. Thus, $a_{r+j} = [\theta(r+j)] - [\theta(r+j-1)] = [\frac{s}{r}(r+j)] - [\frac{s}{r}(r+j-1)] = [s+\frac{s}{r}j] - [s+\frac{s}{r}(j-1)] = [\frac{s}{r}j] - [\frac{s}{r}(j-1)] = a_j$.

Let $2 \leqslant j \leqslant [\sqrt{5}r] - 2r$. Then, $2r+2 \leqslant 2r+j \leqslant [\sqrt{5}r]$ and $2r+1 \leqslant 2r+j-1 \leqslant [\sqrt{5}r]-1$. Thus, $a_{2r+j} = [\theta(2r+j)] - [\theta(2r+j-1)] = [\frac{s}{r}(2r+j)] - [\frac{s}{r}(2r+j-1)] = [\frac{s}{r}j] - [\frac{s}{r}(j-1)] = a_j$.

By easy calculation, we have $a_r = a_{2r} = 1$, $a_{r+1} = a_{2r+1} = 2$.

Then, we have the following periodic table for $(a_n)_{1\leqslant n \leqslant \left[\sqrt{5}r\right]}$.

a_1	a_2	a_3	$\cdots$	$a_{\left[\sqrt{5}r\right]-2r}$	$\cdots$	a_r	a_{r+1}
	a_{2+r}	a_{3+r}	$\cdots$	$a_{\left[\sqrt{5}r\right]-r}$	$\cdots$	a_{2r}	a_{2r+1}
	a_{2+2r}	a_{3+2r}	$\cdots$	$a_{\left[\sqrt{5}r\right]}$			

Since $\theta < \log_2 3$, we then take all convergents $\frac{s}{r}$ of θ such that $\frac{s}{r} < \log_2 3$, and thus, $2^s < 3^r$. By $a_1 = 1$, $b_2^{r+1} = [r\theta] + 1 = s$ and Proposition 7(ii), we have:

$$x_0^{1,2r+1} = \frac{2^{2s+1}u_{2r+1} - (3^r - 2^s) - 2B_2^r}{3(3^r - 2^s)}$$

for some $u_{2r+1} \geqslant 1$. By $B_2^r = \sum_{i=0}^{r-1} 3^{r-1-i} 2^{b_2^{i+1}} = 3^{r-1} \sum_{i=0}^{r-1} \frac{2^{[i\theta+\theta]-1}}{3^i} \leqslant 3^{r-1}2^{\theta-1} \sum_{i=0}^{r-1} \frac{2^{i\theta}}{3^i} = 2^{\theta-1}\frac{3^r}{3}\frac{1-(\frac{2^\theta}{3})^r}{1-\frac{2^\theta}{3}} = 2^{\theta-1}\frac{3^r - 2^{r\theta}}{3-2^\theta} \leqslant C3^r$, where $C = \frac{2^{\theta-1}}{3-2^\theta}$, we have:

$$x_0^{1,2r+1} \geqslant \frac{2}{3}\frac{4^{[r\theta]+1} - C3^r}{3^r - 2^s} - \frac{1}{3} \geqslant \frac{2}{3}\frac{4^{r\theta} - C3^r}{3^r - 2^s} - \frac{1}{3} \geqslant \frac{2}{3}\frac{4^{r\theta} - C3^r}{3^r} - \frac{1}{3} = \frac{2}{3}(\frac{4^\theta}{3})^r - \frac{2}{3}C - \frac{1}{3}.$$

Thus, $\lim_{r\to\infty} x_0^{1,2r+1} = \infty$. Hence, $\Omega - \lim a_n = \infty$. □

5. Concluding Remarks and Open Problems

The results on non-periodic E-sequences in Section 4 were based on the theory of periodic E-sequences in Section 3 and the Matthews and Watts' formula. Currently, we have no other way to tackle non-periodic E-sequences. We can obtain various generalizations and analogues of Theorems 4–8. However, we need good problems to make some progress.

One seemingly simple problem that we are not able to prove is whether $(a_n)_{n\geqslant 1}$ is divergent, where $a_n = 2$ if $n \in \{2^2, 3^2, 4^2, \ldots\}$ and $a_n = 1$ otherwise, i.e., $(a_n)_{n\geqslant 1}$ is 111211112. . . .

Another interesting problem is whether $(a_n)_{n\geqslant 1}$ with infinitely many n satisfying $b_n > n\log_2 3$ is Ω-divergent. By virtue of Theorem 4, we only need to consider the case of $\overline{\lim_{n\to\infty}} \frac{b_n}{n} = \log_2 3$. Theorem 5 answers the problem if $\dfrac{B_n}{3^n(1.5n^{\frac{1}{9}}-1)} \to \infty$, as $n \to \infty$. Currently, we do not know how to tackle the other cases of the problem.

Conjecture 2(ii) is also important in some sense.

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