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Lie Symmetries, Conservation Laws and Exact Solutions for Jaulent-Miodek Equations

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Abstract: In this paper, the Lie symmetries of the Jaulent-Miodek (JM) equations are calculated and one dimensional optimal systems of Lie algebra are obtained. Furthermore, the conservation laws are constructed by using the adjoint equation method. Finally, the exact solutions of the equations are obtained by the conservation laws.

Keywords: jaulent-miodek equations; lie symmetries; optimal systems; conservation laws; exact solutions

1. Introduction

In this paper, we consider the Jaulent-Miodek (JM) equations [1]

$$\begin{cases} u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_x v_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_x v^2 = 0, \\ v_t + v_{xxx} - 6u_x v - 6uv_x - \frac{15}{2}v_x v^2 = 0. \end{cases} \tag{1}$$

which associates with the JM spectral problem [2,3] and energy-dependent Schrodinger potential [4–6]. There are a plethora of methods to solve system (1), such as exp-function method [7–10], tanh-coth and sech methods [11–14]. A numerical method is available in [15]. According to our understanding, the Lie symmetry and conservation laws of the JM equations have not been done yet. This paper will give the symmetry reduction and conservation laws of the system (1) and construct its exact solution.

The structure of this paper is as follows: In Section 2, the Lie symmetry of the Jaulent-Miodek (JM) equations are calculated and one dimensional optimal systems of Lie algebra are obtained; in Section 3, the conservation laws of the system are given by adjoint equation method; in Section 4, the exact solutions of the system are constructed by the conservation laws; and in Section 5, a brief summary is made of the full text.

2. Lie Symmetry Analysis and Optimal Systems

2.1. Lie Symmetry

In this section, we will perform Lie symmetry analysis for the system (1). We first assume that the infinitesimal generator [16] allowed of the system (1) is:

$$V = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}, \tag{2}$$

where $\xi^1(t,x,u,v), \xi^2(t,x,u,v), \eta^1(t,x,u,v), \eta^2(t,x,u,v)$ are coefficient functions to be determined. For the system (1), V satisfies the following Lie symmetry conditions as follows

$$\begin{aligned} Pr^3V(\Delta_1)|_{\Delta_1=0,\Delta_2=0} &= 0, \\ Pr^3V(\Delta_2)|_{\Delta_1=0,\Delta_2=0} &= 0, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Delta_1 &= u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_xv_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_xv^2, \\ \Delta_2 &= v_t + v_{xxx} - 6u_xv - 6uv_x - \frac{15}{2}v_xv^2. \end{aligned} \tag{4}$$

By Lie's theory, the third prolongation [17] of (2) is of the form

$$\begin{aligned} Pr^{(3)}V = V &+ \eta^{1x}\frac{\partial}{\partial u_x} + \eta^{1t}\frac{\partial}{\partial u_t} + \eta^{2x}\frac{\partial}{\partial v_x} + \eta^{2t}\frac{\partial}{\partial v_t} + \eta^{1xx}\frac{\partial}{\partial u_{xx}} + \eta^{1xt}\frac{\partial}{\partial u_{xt}} + \eta^{1tt}\frac{\partial}{\partial u_{tt}} \\ &+ \eta^{2xx}\frac{\partial}{\partial v_{xx}} + \eta^{2xt}\frac{\partial}{\partial v_{xt}} + \eta^{2tt}\frac{\partial}{\partial v_{tt}} + \eta^{1xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{1xxt}\frac{\partial}{\partial u_{xxt}} + \eta^{1xtt}\frac{\partial}{\partial u_{xtt}} \\ &+ \eta^{1ttt}\frac{\partial}{\partial u_{ttt}} + \eta^{2xxx}\frac{\partial}{\partial v_{xxx}} + \eta^{2xxt}\frac{\partial}{\partial v_{xxt}} + \eta^{2xtt}\frac{\partial}{\partial v_{xtt}} + \eta^{2ttt}\frac{\partial}{\partial v_{ttt}}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \eta^{1x} &= D_x(\eta^1 - \xi^1u_t - \xi^2u_x) + \xi^1u_{tx} + \xi^2u_{xx}, \\ \eta^{1t} &= D_t(\eta^1 - \xi^1u_t - \xi^2u_x) + \xi^1u_{tt} + \xi^2u_{xt}, \\ \eta^{2x} &= D_x(\eta^2 - \xi^1v_t - \xi^2v_x) + \xi^1v_{tx} + \xi^2v_{xx}, \\ \eta^{2t} &= D_t(\eta^2 - \xi^1v_t - \xi^2v_x) + \xi^1v_{tt} + \xi^2v_{xt}, \\ \eta^{1xx} &= D_{xx}(\eta^1 - \xi^1u_t - \xi^2u_x) + \xi^1u_{txx} + \xi^2u_{xxx}, \\ \eta^{2xx} &= D_{xx}(\eta^2 - \xi^1v_t - \xi^2v_x) + \xi^1v_{txx} + \xi^2v_{xxx}, \\ \eta^{1xxx} &= D_{xxx}(\eta^1 - \xi^1u_t - \xi^2u_x) + \xi^1u_{txxx} + \xi^2u_{xxxx}, \\ \eta^{2xxx} &= D_{xxx}(\eta^2 - \xi^1v_t - \xi^2v_x) + \xi^1v_{txxx} + \xi^2v_{xxxx}. \end{aligned} \tag{6}$$

Combining (3) and (4), we get the determining equations of system (1) as follows:

$$\begin{cases} \eta^1(-6u_x - 6vv_x) + \eta^2(\frac{3}{2}v_{xxx} - 6uv_x - 3u_xv) + \eta^{1x}(-6u - \frac{3}{2}v^2) + \eta^{1t} \\ +\eta^{2x}(\frac{9}{2}v_{xx} - 6uv) + \eta^{2xx}\frac{9}{2}v_x + \eta^{1xxx} + \eta^{2xxx}\frac{3}{2}v = 0, \\ \eta^1(-6v_x) + \eta^2(-6u_x - 15v_xv) + \eta^{1x}(-6v) + \eta^{2x}(-6u - \frac{15}{2}v^2) + \eta^{2t} + \eta^{2xxx} = 0. \end{cases} \tag{7}$$

Substituting system (6) into the equivalent condition (7), and making the coefficients of the various monomials in partial derivatives with respect to x and various powers of u equaled, one then obtains the over determining equations of system (7):

$$\begin{array}{llll} \xi^1_u = 0, & \xi^2_u = 0, & \eta^1_v = 0, & v\eta^1 - 2u\eta^2 = 0, \\ \xi^1_v = 0, & \xi^2_v = 0, & \eta^1_t = 0, & \\ \xi^1_x = 0, & \xi^2_t = 0, & \eta^1_x = 0, & \\ 3\eta^1 + 2u\xi^1_t = 0, & \eta^1 + 2u\xi^2_x = 0, & \eta^1 - u\eta^1_u = 0. & \end{array} \tag{8}$$

Solving (8), one can get

$$\begin{array}{ll} \xi^1 = -\dfrac{3}{2}c_3t + c_1, & \xi^2 = -\dfrac{c_3}{2}x + c_2, \\ \eta^1 = c_3u, & \eta^2 = \dfrac{c_3}{2}v. \end{array} \tag{9}$$

where c_1, c_2, c_3 are three arbitrary constants. Therefore,the three-dimensional Lie algebra of infinitesimal symmetries for Jaulent-Miodek (JM) equations (1) are spanned by the following three vector fields:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t}, \\ V_2 &= \frac{\partial}{\partial x}, \\ V_3 &= -\frac{3}{2}t\frac{\partial}{\partial x} - \frac{1}{2}x\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v}. \end{aligned} \tag{10}$$

2.2. Optimal System

In this section, we research the one-dimensional optimal system of (10). The basic method of constructing one-dimensional optimal system is given in ref. [18]. The adjoint transformation is expressed as the following series form:

$$Ad(e^{\epsilon V_i})V_j = V_j - \epsilon[V_i, V_j] + \tfrac{1}{2}\epsilon^2[V_i, [V_i, V_j]] - ...,$$

where ϵ is a parameter. $[V_i, V_j]$ is the usual commutator and the calculation formula is as follows:

$$[V_i, V_j] = V_iV_j - V_jV_i.$$

Hence we have the following commutator table [19] (See Table 1) and the adjoint table (See Table 2).

Table 1. Commutator table of the Lie algebra.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	$-\frac{3}{2}V_1$
V_2	0	0	$-\frac{1}{2}V_2$
V_3	$\frac{3}{2}V_1$	$\frac{1}{2}V_2$	0

Table 2. Adjoint table of the Lie algebra.

Ad	V_1	V_2	V_3
V_1	V_1	V_2	$V_3 + \frac{3}{2}\epsilon V_1$
V_2	V_1	V_2	$V_3 + \frac{1}{2}\epsilon V_2$
V_3	$e^{-\frac{3}{2}\epsilon}V_1$	$e^{-\frac{1}{2}\epsilon}V_2$	V_3

Next, according to the method of constructing one dimensional optimal system in [18], we set up the following non-zero vector field with arbitrary coefficients a_1, a_2, a_3

$$V = a_1V_1 + a_2V_2 + a_3V_3 .$$

Step 1:

Without loss of generality, supposing that $a_3 \neq 0$ and setting $a_3 = 1$, then the vector V becomes

$$V = a_1V_1 + a_2V_2 + V_3 .$$

To eliminate the coefficient of V_1, using $Ad(e^{\epsilon V_1})$ to act on above V, we gain

$$V' = Ad(e^{\epsilon V_1})V = a_1V_1 + a_2V_2 + V_3 + \tfrac{3}{2}\epsilon V_1 = (a_1 + \tfrac{3}{2}\epsilon)V_1 + a_2V_2 + V_3,$$

where the group parameter $\epsilon = -\frac{2a_1}{3}$. Therefore, $V' = a_2V_2 + V_3$. We continue to eliminate V_2 , using $Ad(e^{\epsilon V_2})$ to act on above V', we derive

$$V'' = Ad(e^{\epsilon V_2})V' = a_2V_2 + V_3 + \tfrac{1}{2}V_2 = (a_2 + \tfrac{1}{2}\epsilon)V_2 + V_3,$$

where the group parameter $\epsilon = -2a_2$. Therefore, $V'' = V_3$.

Step 2:

Supposing that $a_3 = 0, a_2 \neq 0$ and setting $a_2 = 1$, then the vector V prove to be

$$V = a_1 V_1 + V_2.$$

Based on the above method, we know that neither V_1 and V_2 can be eliminated.

Step 3:

Supposing that $a_3 = 0, a_2 = 0, a_1 \neq 0$ and setting $a_1 = 1$, then the vector V turn into

$$V = V_1.$$

Based on the adjoint representations of the vector field, we obtain the optimal systems of Lie algebra.

$$[V_1, V_3, aV_1 + V_2],$$

with a is an arbitrary constant.

For V_1, the system of (1) is reduced as follows:

$$\begin{cases} F''' + \frac{3}{2}HH''' + \frac{9}{2}H'H'' - 6FF' - 6FHH' - \frac{3}{2}F'H^2 = 0, \\ H''' - 6F'H - 6FH' - \frac{15}{2}H'H^2 = 0, \end{cases} \tag{11}$$

where, $u(x,t) = F(\xi), v(x,t) = H(\xi), \xi = x$.

For $aV_1 + V_2$, the system of (1) is reduced as follows:

$$\begin{cases} F' - a^3F''' - \frac{3a^3}{2}HH''' - \frac{9a^3}{2}H'H'' + 6aFF' + 6aFHH' - \frac{3a}{2}F'H^2 = 0, \\ H' - a^3H''' + 6aF'H + 6aFH' + \frac{15a}{2}H'H^2 = 0, \end{cases} \tag{12}$$

where, $u(x,t) = F(\xi), v(x,t) = H(\xi), \xi = t - ax$.

Obviously, Both (11) and (12) are difficult to calculate, So we take the following method to solve the system (1).

3. The Conservation Laws of Jaulennt-Miodek Equations

In this section, we construct the conservation laws by using the adjoint equations method [20–24].

3.1. Adjoint Equations and Lagrange Functions

The formal Lagrangian for the system (1) is given by

$$\begin{aligned} L = &\theta_1(u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_xv_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}u_xv^2) + \\ &\theta_2(v_t + v_{xxx} - 6u_xv - 6uv_x - \frac{15}{2}v_xv^2), \end{aligned} \tag{13}$$

where θ_1 and θ_2 are new dependent variables of t, x, u, v. The adjoint system for the Equation (1) is defined as

$$\frac{\delta L}{\delta u} = 0, \quad \frac{\delta L}{\delta v} = 0,$$

where

$$\begin{aligned}&\frac{\delta L}{\delta u}=\frac{\partial L}{\partial u}-D_t\frac{\partial L}{\partial u_t}-D_x\frac{\partial L}{\partial u_x}+D_tD_t\frac{\partial L}{\partial u_{tt}}+D_tD_x\frac{\partial L}{\partial u_{tx}}+D_xD_x\frac{\partial L}{\partial u_{xx}}-D_tD_tD_t\frac{\partial L}{\partial u_{ttt}}\\&-D_tD_xD_x\frac{\partial L}{\partial u_{txx}}-D_tD_tD_x\frac{\partial L}{\partial u_{ttx}}-D_xD_xD_x\frac{\partial L}{\partial u_{xxx}}+...,\\&\frac{\delta L}{\delta v}=\frac{\partial L}{\partial v}-D_t\frac{\partial L}{\partial v_t}-D_x\frac{\partial L}{\partial v_x}+D_tD_t\frac{\partial L}{\partial v_{tt}}+D_tD_x\frac{\partial L}{\partial v_{tx}}+D_xD_x\frac{\partial L}{\partial v_{xx}}-D_tD_tD_t\frac{\partial L}{\partial v_{ttt}}\\&-D_tD_xD_x\frac{\partial L}{\partial v_{txx}}-D_tD_tD_x\frac{\partial L}{\partial v_{ttx}}-D_xD_xD_x\frac{\partial L}{\partial v_{xxx}}+....\end{aligned}\tag{14}$$

For system (1), The adjoint equations have the following form

$$\begin{cases}F_1=\frac{\delta L}{\delta u}=-3\theta_1vv_x+6\theta_{1x}u+\frac{3}{2}\theta_{1x}v^2+6\theta_{2x}v-\theta_{1xxx}-\theta_{1t}=0,\\F_2=\frac{\delta L}{\delta v}=3\theta_1vu_x-\frac{9}{2}\theta_{1x}v_{xx}+6\theta_{1x}uv+6\theta_{2x}u+\frac{15}{2}\theta_{2x}v^2-\frac{3}{2}\theta_{1xxx}v-\theta_{2t}-\theta_{2xxx}=0,\end{cases}\tag{15}$$

where, the solution to system (15) can be $\theta_1=2$, $\theta_2=v$. So we derive the Lagrangian

$$\begin{aligned}&L=2(u_t+u_{xxx}+\frac{3}{2}vv_{xxx}+\frac{9}{2}v_xv_{xx}-6uu_x-6uvv_x-\frac{3}{2}u_xv^2)+\\&v(v_t+v_{xxx}-6u_xv-6uv_x-\frac{15}{2}v_xv^2)\end{aligned}\tag{16}$$

3.2. Conservation Laws

Every Lie symmetry provides a conservation law for system (1). The elements of the conservation vector (C^1,C^2) are defined by the following expression:

$$\begin{aligned}&C^i=\xi^iL+W^\alpha[\frac{\partial L}{\partial u_i^\alpha}-D_j\frac{\partial L}{\partial u_{ij}^\alpha}+D_jD_k\frac{\partial L}{\partial u_{ijk}^\alpha}+...]+D_j(W^\alpha)[\frac{\partial L}{\partial u_{ij}^\alpha}-D_k\frac{\partial L}{\partial u_{ijk}^\alpha}+...]\\&+D_jD_k(W^\alpha)[\frac{\partial L}{\partial u_{ijk}^\alpha}+...]+...,\end{aligned}\tag{17}$$

where, $W^\alpha=\eta^\alpha-\xi^ju_j^\alpha$. Hence, the conservation laws for system (1) are given by

$$D_i(C^i)|_{(1.1)}=0,\quad i=1,2.$$

Next we consider conservation laws in three cases.

Case 1. *For $V_1=\frac{\partial}{\partial t}$, we obtain*

$$W^1=-u_t,\quad W^2=-v_t.$$

The conservation law of system (1) is

$$D_t(-2u_t-vv_t)+D_x(12uu_t+9v^2u_t-2u_{txx}-4v_tv_{xx}+18uvv_t+\frac{15}{2}v^3v_t-5v_{tx}v_x-4vv_{txx})=0.$$

Case 2. *For $V_2=\frac{\partial}{\partial x}$, we derive*

$$W^1=-u_x,\quad W^2=-v_x.$$

The conservation law of system (1) is

$$D_t(-2u_x-vv_x)+D_x(2u_t+vv_t)=0.$$

Case 3. *For* $V_3 = -\frac{3}{2}t\frac{\partial}{\partial t} - \frac{1}{2}x\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v}$, *We have*

$$W^1 = u + \frac{3}{2}tu_t + \frac{1}{2}xu_x, \quad W^2 = \frac{1}{2}v + \frac{3}{2}tv_t + \frac{1}{2}xv_x.$$

The conservation law of system (1) is

$$D_t(2u + 3tu_t + xu_x + \frac{1}{2}v^2 + \frac{3}{2}tvv_t + \frac{1}{2}xvv_x) + D_x(-12u^2 - 18tuu_t - 6xuu_x - 18uv^2 - \frac{27}{2}tv^2u_t - \frac{9}{2}xv^2u_x + 3tu_{txx} + 4u_{xx} + xu_{xxx} + 8vv_{xx} + 6tv_tv_{xx} + \frac{9}{2}xv_xv_{xx} - 27tuvv_t - 9xuvv_x - \frac{15}{4}v^4 - \frac{45}{4}tv^3v_t - \frac{15}{4}xv^3v_x + 5v_x^2 + \frac{27}{2}tvv_{txx} + 2xv_{xxx}) = 0.$$

4. Exact Solutions

In this section, we consider the exact solutions [25–29] by using the method of conservation laws. For the conservation law of V_2, from (Section 3.2), we obtain the conservation law of V_2

$$D_t(-2u_x - vv_x) + D_x(2u_t + vv_t) = 0.$$

Let

$$D_t(-2u_x - vv_x) = 0, \quad D_x(2u_t + vv_t) = 0. \tag{18}$$

Assuming that

$$\begin{cases} -2u_x - vv_x = g(x), \\ 2u_t + vv_t = q(t). \end{cases} \tag{19}$$

Integrating first equation of system (19), we gain

$$u = -\frac{G(x)}{2} - \frac{h(t)}{2} - \frac{1}{4}v^2 - \frac{c}{2}, \tag{20}$$

where $G'(x) = g(x)$. $h(t)$ is a function of the variable t and c is a constant. Substituting (20) into the second equation of system (19), we derive

$$-h_t(t) = q(t). \tag{21}$$

By calculating the equation of (21), we get

$$h(t) = -2u - \frac{1}{2}v^2 - \frac{d}{2}, \tag{22}$$

where d is a constant. Substituting (22) into the equation of system (20), one obtain

$$g(x) = 0, \tag{23}$$

then the first equation of system (19) turn into

$$-2u_x - vv_x = 0.$$

Integrating this equation, we obtain

$$u = -\frac{1}{4}v^2 + c.$$

Solving

$$D_t(-2u_x - vv_x) + D_x(2u_t + vv_t) = 0.$$

one gets

$$u = k_1 v^2 + k_2.$$

here k_1 and k_2 are constants.

Considering the special case, $u = k_1 v^2 + k_2$, $u = -\frac{1}{4}v^2$.

Case 4. *Let $u(x,t) = u(\xi)$, $v(x,t) = v(\xi)$, where $\xi = x - ct$.*
The system (1) turn into

$$\begin{cases} -cu' + u''' + \frac{3}{2}vv''' + \frac{9}{2}v'v'' - 6uu' - 6uvv' - \frac{3}{2}u'v^2 = 0, \\ -cv' + v''' - 6u'v - 6uv' - \frac{15}{2}v'v^2 = 0. \end{cases} \tag{24}$$

Integrating the second equation of system (24), we obtain

$$v'' = cv + 6uv + \frac{5}{2}v^3 + g_1, \tag{25}$$

here g_1 is integral constant.

Multiplying $\frac{1}{2}v'$ to equation (25), we get

$$3uvv' = -\frac{1}{2}g_1 v' - \frac{1}{2}cvv' - \frac{5}{4}v^3 v' + \frac{1}{2}v'v''. \tag{26}$$

Substituting (26) into the first equation of (24) and integrating, we obtain

$$u'' = g_2 + cu + 3u^2 - \frac{3}{2}(vv'') - \frac{1}{2}g_1 v - \frac{1}{4}cv^2 - \frac{5}{16}v^4 + \frac{3}{2}uv^2 - \frac{5}{4}(v')^2, \tag{27}$$

here g_2 is integral constant.

Combining (25), the system (24) turn into

$$\begin{cases} u'' = g_2 + cu + 3u^2 - \frac{3}{2}(vv'') - \frac{1}{2}g_1 v - \frac{1}{4}cv^2 - \frac{5}{16}v^4 + \frac{3}{2}uv^2 - \frac{5}{4}(v')^2, \\ v'' = cv + 6uv + \frac{5}{2}v^3 + g_1, \end{cases} \tag{28}$$

here g_1 and g_2 are integral constants.

Substituting $u = -\frac{1}{2}v^2 - \frac{1}{4}c$ into the system of (28), we obtain

$$\begin{cases} \frac{1}{4}(v')^2 = \frac{1}{2}(\frac{1}{2}(v')^2 + \frac{1}{4}cv^2 + \frac{1}{8}v^4) - \frac{1}{8}cv^2 - \frac{1}{4}v^4, \\ v'' = -\frac{1}{2}cv - \frac{1}{2}v^3, \end{cases} \tag{29}$$

the second equation of system (29) is equivalent to the following Hamilton system.

$$\begin{cases} \frac{dv}{d\xi} = y, \\ \frac{dy}{d\xi} = -\frac{1}{2}cv - \frac{1}{2}v^3.. \end{cases} \tag{30}$$

The Hamilton function is

$$H(v,y) = \frac{1}{2}y^2 + \frac{1}{4}cv^2 + \frac{1}{8}v^4 = h. \tag{31}$$

Solving (31) and combining $u = -\frac{1}{2}v^2 - \frac{1}{4}c$, let $c = 0, 1, -1$, we get the exact solutions of system (1)

$$\begin{cases} u_1(x,t) = -\sqrt{2}JacobiSN[\frac{1}{2}(-2^{\frac{3}{4}}x + 22^{\frac{3}{4}}a_1), -1]^2, \\ v_1(x,t) = 2^{\frac{3}{4}}JacobiSN[\frac{1}{2}(-2^{\frac{3}{4}}x + 22^{\frac{3}{4}}a_1), -1], \end{cases}$$

$$\begin{cases} u_2(x,t) = -\sqrt{2}JacobiSN[\frac{1}{2}(2^{\frac{3}{4}}x + 22^{\frac{3}{4}}a_2), -1]^2, \\ v_2(x,t) = 2^{\frac{3}{4}}JacobiSN[\frac{1}{2}(2^{\frac{3}{4}}x + 22^{\frac{3}{4}}a_2), -1], \end{cases}$$

$$\begin{cases} u_3(x,t) = -\frac{1}{4} + \frac{1}{2}(1+\sqrt{401})JacobiSN[\frac{1}{2}(-i\sqrt{-1+\sqrt{401}}(x-t) + 2i\sqrt{-1+\sqrt{401}}a_3), \frac{1}{200}(-201-\sqrt{401})]^2, \\ v_3(x,t) = -i\sqrt{1+\sqrt{401}}JacobiSN[\frac{1}{2}(-i\sqrt{-1+\sqrt{401}}(x-t) + 2i\sqrt{-1+\sqrt{401}}a_3), \frac{1}{200}(-201-\sqrt{401})], \end{cases}$$

$$\begin{cases} u_4(x,t) = -\frac{1}{4} + \frac{1}{2}(1+\sqrt{401})JacobiSN[\frac{1}{2}(i\sqrt{-1+\sqrt{401}}(x-t) + 2i\sqrt{-1+\sqrt{401}}a_4), \frac{1}{200}(-201-\sqrt{401})]^2, \\ v_4(x,t) = -i\sqrt{1+\sqrt{401}}JacobiSN[\frac{1}{2}(i\sqrt{-1+\sqrt{401}}(x-t) + 2i\sqrt{-1+\sqrt{401}}a_4), \frac{1}{200}(-201-\sqrt{401})], \end{cases}$$

$$\begin{cases} u_5(x,t) = \frac{1}{4} + \coth[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_5)]^2(1 - \tanh[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_5]^2), \\ v_5(x,t) = -i\sqrt{2}\coth[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_5)]\sqrt{1 - \tanh[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_5]^2}, \end{cases}$$

$$\begin{cases} u_6(x,t) = \frac{1}{4} + \coth[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_6)]^2(1 - \tanh[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_6]^2), \\ v_6(x,t) = i\sqrt{2}\coth[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_6)]\sqrt{1 - \tanh[\frac{1}{2}(-\sqrt{2}(x+t) + 2i\sqrt{2}a_6]^2}, \end{cases}$$

$$\begin{cases} u_7(x,t) = \frac{1}{4} + \coth[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_7)]^2(1 - \tanh[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_7]^2), \\ v_7(x,t) = -i\sqrt{2}\coth[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_7)]\sqrt{1 - \tanh[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_7]^2}, \end{cases}$$

$$\begin{cases} u_8(x,t) = \frac{1}{4} + \coth[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_8)]^2(1 - \tanh[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_8]^2), \\ v_8(x,t) = -i\sqrt{2}\coth[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_8)]\sqrt{1 - \tanh[\frac{1}{2}(\sqrt{2}(x+t) + 2i\sqrt{2}a_8]^2}, \end{cases}$$

here $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ are constants. JacobiSN represent Jacobi elliptic function in $MATHEMATIC$.

Substituting $u = -\frac{1}{4}v^2$ into the system of (1), we obtain

$$\begin{cases} vv_{xxx} + 3v_xv_{xx} - \frac{1}{2}vv_t + \frac{3}{2}v_xv^3 = 0, \\ v_t + v_{xxx} - 3v_xv^2 = 0. \end{cases} \tag{32}$$

Case 5. *Let $v(x,t) = F(z)$, where $z = x - ct$.*
The system (32) turn into

$$\begin{cases} -cF' + F''' - 3F'F^2 = 0, \\ \frac{c}{2}FF' + 3FF'' + FF''' + \frac{3}{2}F^3F' = 0. \end{cases} \tag{33}$$

From the first equation of (33), one can get

$$F''' = cF' + 3F'F^2. \tag{34}$$

Substituting the (34) for the second equation of (33), we derive

$$cFF' + 2F'F'' + 3F^3F' = 0. \tag{35}$$

By calculating the equation of (35), we obtain

$$F=\sqrt{\frac{2ce^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}+4ce^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+2c}{3(e^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}-2e^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+1)}-\frac{2c}{3}},\tag{36}$$

here a and b are integral constants.

Therefore, the exact solutions of the system of (1) are

$$\begin{cases} u_9(x,t)=-\frac{1}{4}(\frac{2ce^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}+4ce^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+2c}{3(e^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}-2e^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+1)}-\frac{2c}{3}), \\ v_9(x,t)=\sqrt{\frac{2ce^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}+4ce^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+2c}{3(e^{-2\sqrt{2c}i(x-ct)-2\sqrt{2c}b}-2e^{-\sqrt{2c}i(x-ct)-\sqrt{2c}b}+1)}-\frac{2c}{3}}. \end{cases}$$

Here a and b are integral constants (see Figure 1).

Case 6. *Let $v(x,t)=F(z)$, where $z=kx-ct$.*

The system (32) turn into

$$\begin{cases} -cF'+k^3F'''-3kF'F^2=0, \\ \frac{c}{2}FF'+3k^3FF''+k^3FF'''+\frac{3}{2}kF^3F'=0. \end{cases}\tag{37}$$

From the first equation of (37), we obtain

$$F'''=\frac{c}{k^3}F'+\frac{3}{k^2}F'F^2.\tag{38}$$

Substituting the (38) for the second equation of (37), one can get

$$cFF'+2k^3F'F''+3kF^3F'=0.\tag{39}$$

By calculating the equation of (39), we obtain

$$F=\sqrt{\frac{2ce^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}+4ce^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+2c}{3(ke^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}-2ke^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+k)}-\frac{2c}{3k}},$$

here a and b are integral constants.

Therefore, the exact solutions of the system of (1) are

$$\begin{cases} u_{10}(x,t)=-\frac{1}{4}(\frac{2ce^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}+4ce^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+2c}{3(ke^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}-2ke^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+k)}-\frac{2c}{3k}), \\ v_{10}(x,t)=\sqrt{\frac{2ce^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}+4ce^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+2c}{3(ke^{-2\sqrt{\frac{2c}{k}}i(kx-ct)-2\sqrt{\frac{2c}{k}}b}-2ke^{-\sqrt{\frac{2c}{k}}i(kx-ct)-\sqrt{\frac{2c}{k}}b}+k)}-\frac{2c}{3k}}. \end{cases}$$

Here a and b are integral constants (see Figure 2).

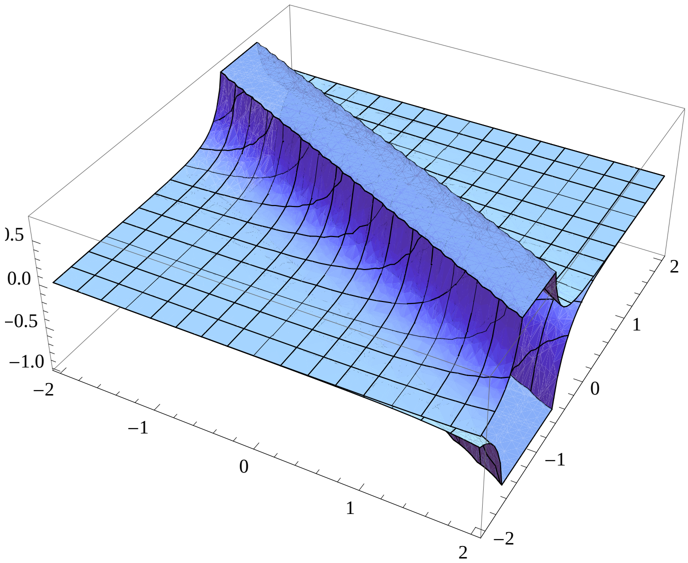

Figure 1. The 3D surface of the exact solution to system (1) by setting $b = 0,\ c = -2$.

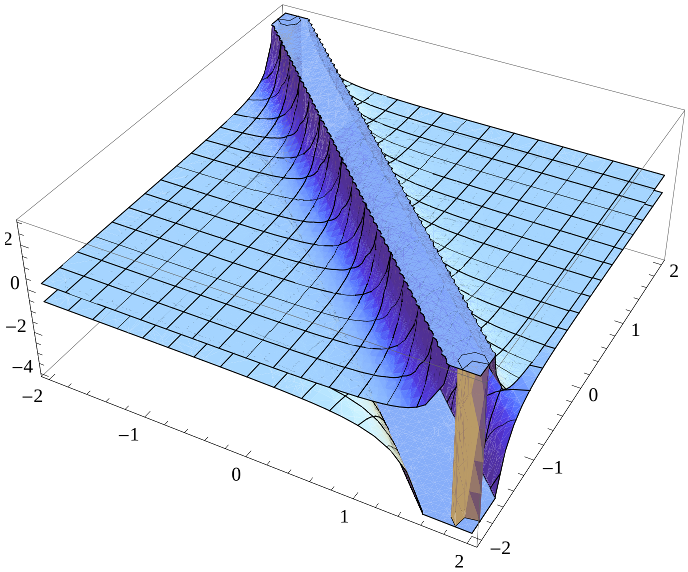

Figure 2. The 3D surface of the exact solution to system (1) by setting $b = 0,\ c = -2,\ k = 2$.

5. Conclusions

In this paper, the Lie symmetries of the Jaulent-Miodek (JM) equations are calculated and one dimensional optimal systems of Lie algebra are obtained. The conservation laws are constructed by using the adjoint equation method. Finally, the new exact solutions of the equations are constructed by the conservation laws. However, our method is special in the process of constructing exact solutions by conservation laws. More general methods require further study.

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