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On Energies of Charged Particles with Magnetic Field

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Abstract: The present paper is about magnetic curves of spherical images in Euclidean 3-space. We obtain the Lorentz forces of the spherical images and then we determine if the spherical images have a magnetic curve or not. If a spherical image has a magnetic curve, then after presenting some basic concepts about the energy of a charged particle whose trajectory is that magnetic curve and the kinetic energy of a moving particle whose trajectory is the spherical indicatrix, we find the energy of the charged particle and the kinetic energy of the moving particle.

Keywords: magnetic curve; Lorentz force; energy; spherical indicatrix; charged particle

1. Introduction

Magnetic curves are curves showing lines of a magnetic force, as between the poles of a powerful magnet. It is known as "magnetostatics" in physics terminology and it deals with stationary electric currents [1]. The static magnetic fields on $\mathbb{E}^3$ are regarded as closed 2-forms in mathematics terminology. Considering this concept on Euclidean 3-space, they can be introduced on a Riemannian manifold as closed two-forms. In the Riemannian manifold, the trajectories of the charged particles moving under the effect of the magnetic fields are magnetic curves. Magnetic curves are curves which satisfy a special equation

$$\nabla_{\gamma'}\gamma' = \phi(\gamma') \tag{1}$$

known as the Lorentz equation. Here, ϕ is Lorentz force, ∇ is the Levi–Civita connection. In other words, magnetic curves are solutions to Equation (1) [2]. When Equation (1) is zero, the Lorentz equation returns a geodesic equation. This fact shows that magnetic curves generalize the geodesic curves. So this is an important research topic in differential geometry and physics. In the last years, magnetic curves were studied in Kaehler manifolds and Sasakian manifolds, respectively, since their fundamental 2-forms provide natural examples of magnetic fields [3].

The relation between geometry and magnetic fields have a long history. It is well-known that the notion of linking number can be traced back to Gauss's work on terrestrial magnetism. The linking number connects topology and Ampere's law in magnetism. De Turck and Gluck studied magnetic curves and linking numbers in S^3 and H^3. Moreover, if magnetic trajectories have constant speed, a unit speed magnetic curve is called a normal magnetic curve and denoted by $\gamma(s)$. In comparison, studies on 3-dimensional Riemann manifolds are more specific since the 2-forms correspond to vector fields in this case. In the light of this fact, magnetic fields identified with Killing vector fields are of great importance, because they can be associated with divergence-free vector fields. Moreover, their trajectories are called Killing magnetic curves [4].

In works of classical physics, to reduce the order of the system, continuous symmetries can be used, and in some parts, its integratde completely. They may also restrict solutions to an invariant manifold which we called conservation laws along with Noether's theorem for variational problems. Thus, directly searching for symmetries in precise systems has received intensive attention in the last few decades. Another area of utilization symmetry analysis is to sort all earthly symmetry groups adopted by a differential equation with a large family. These conclusions gives us information about when a system of general form holds one or more symmetries with which circumstances [5]. Additionally, there are many works related to symmetries of charged particles [6–8].

In this paper, we study the magnetic fields of the spherical images of a regular curve in Euclidean 3-space $\mathbb{R}^3$. We use the quasi elements of a regular space curve α and give a relationship between α and the magnetic fields of its spherical images which are given with the Frenet elements. We find the Lorentz force of the spherical images of the curve α and determine if the spherical images of the curve α have a magnetic curve or not. If a spherical image has a magnetic curve, after presenting some basic concepts about the energy of a charged particle under the action of a magnetic field, we find the energy of a charged particle which has that magnetic curve as its trajectory. Moreover, after giving some basic concepts about the kinetic energy of a moving particle, we find the kinetic energy of a moving particle which has the spherical indicatrix as its trajectory.

2. Preliminaries

In this section, we present some basic concepts about magnetic fields and magnetic curves. First of all, we recall the definitions of 2-form and closed form on a Riemannian manifold.

Definition 1. *Let (M, g) be a Riemannian manifold. A 2-form η on M is a function $\eta : \chi(M)\times \chi(M) \longrightarrow C^\infty(M, \mathbb{R})$ which satisfies the following two conditions [9]:*
[i] $\eta(X, Y)$ is linear in X and in Y for all $X, Y \in \chi(M)$,
[ii] η is skew-symmetric, that is, $\eta(X, Y) = -\eta(Y, X)$ for all $X, Y \in \chi(M)$.

Definition 2. *If the exterior derivative of a form η vanishes, that is, $d\eta = 0$, then η is called a closed form [9].*

In a Riemannian manifold, the trajectories of the charged particles moving under the effect of the magnetic fields F are magnetic curves. The magnetic fields in Riemannian manifold are regarded as closed 2-forms in mathematics terminology. The Lorentz force ϕ is a transformation which satisfies a special equation

$$F(X, Y) = g(\phi(X), Y), \qquad X, Y \in \chi(M). \tag{2}$$

If the particle preserves constant energy along its trajectory then the trajectory of the particle has constant velocity [10]. For any $X, Y, Z \in \chi(M)$, the mixed product of these vector fields is defined by

$$g(X \times Y, Z) = dv_g(X, Y, Z), \tag{3}$$

where dv_g is the volume form corresponding to the metric g.

Assume that V is a Killing vector field and X is any vector field, then the Lorentz force equation is

$$\phi(X) = V \times X. \tag{4}$$

Hence, from (1) and (4), we can write

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{5}$$

Assume that γ is a unit speed magnetic curve and $\omega(s)$ is its quasislope measured with respect to V. γ is a magnetic trajectory [11], of V iff

$$V = \omega(s)T(s) + \kappa(s)B(s). \tag{6}$$

The quasi frame of a space curve $\alpha(s)$ which is parameterized with arc-length is $\{\mathbf{t}_q(s), \mathbf{n}_q(s), \mathbf{b}_q(s)\}$, where the vector fields are given as

$$\mathbf{t}_q(s) = \mathbf{T}(s), \tag{7}$$

$$\mathbf{n}_q(s) = \frac{\mathbf{T}(s) \times \overrightarrow{\mathbf{k}}}{\left\| \mathbf{T}(s) \times \overrightarrow{\mathbf{k}} \right\|}, \tag{8}$$

$$\mathbf{b}_q(s) = \mathbf{T}(s) \times \mathbf{n}_q(s). \tag{9}$$

In this paper, we choose the projection vector $\overrightarrow{\mathbf{k}} = (0, 0, 1)$. $\mathbf{n}_q(s)$ and $\mathbf{b}_q(s)$ are called the quasi normal vector field and the quasi binormal vector field of the curve $\alpha(s)$, respectively [12].

Let $\theta(s)$ be the angle between the principal normal vector field $\mathbf{N}(s)$ and the quasi normal vector field $\mathbf{n}_q(s)$. The quasi formulas are given by

$$\frac{d}{ds}\begin{bmatrix} \mathbf{t}_q(s) \\ \mathbf{n}_q(s) \\ \mathbf{b}_q(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q(s) \\ \mathbf{n}_q(s) \\ \mathbf{b}_q(s) \end{bmatrix}, \tag{10}$$

where $k_i(s)$ are called the quasi curvatures $(1 \leq i \leq 3)$ which are given by

$$k_1(s) = \boldsymbol{\kappa}(s)\cos\theta(s) = \left\langle \mathbf{t}_q'(s), \mathbf{n}_q(s) \right\rangle, \tag{11}$$

$$k_2(s) = -\boldsymbol{\kappa}(s)\sin\theta(s) = \left\langle \mathbf{t}_q'(s), \mathbf{b}_q(s) \right\rangle, \tag{12}$$

$$k_3(s) = \theta'(s) + \boldsymbol{\tau}(s) = -\left\langle \mathbf{n}_q(s), \mathbf{b}_q'(s) \right\rangle. \tag{13}$$

The relationship between the Frenet frame and the quasi frame is given by [12].

3. Magnetic Curves, Spherical Images and Energy

In this section, we give a relationship between a regular space curve which is given with the quasi frame and magnetic curves of its spherical images which are given with the Frenet frame.

3.1. **t**-*Magnetic Particles of the Tangent Indicatrix*

Let α be a regular curve according to quasi frame in Euclidean 3-space and α_1 be its tangent indicatrix. Let $\{\mathbf{t}_q, \mathbf{n}_q, \mathbf{b}_q\}$ be the quasi frame of the curve α and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of α_1.

Theorem 1. *The Lorentz force of the tangent indicatrix α_1 of the curve α can be expressed as*

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} -\sqrt{k_1^2 + k_2^2} & -\frac{k_2\Omega_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_1\Omega_1}{\sqrt{k_1^2+k_2^2}} \\ 0 & \frac{A_1k_1 - C_1\Omega_1}{\sqrt{U_1}} & \frac{A_1k_2 - B_1\Omega_1}{\sqrt{U_1}} \\ 0 & \frac{K_1k_1 - M_1\Omega_1}{\sqrt{V_1}} & \frac{K_1k_2 + L_1\Omega_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{14}$$

where $\Omega_1 = g(\phi(\mathbf{n}_q), \mathbf{b}_q)$ *and*

$$A_1 = -\left(k_1^2 + k_2^2\right)^2, \tag{15}$$

$$B_1 = k_1' k_2^2 - k_1 k_2 k_2' - k_1^2 k_2 k_3 - k_2^3 k_3, \tag{16}$$

$$C_1 = k_1^3 k_3 + k_1^2 k_2' + k_1 k_2^2 k_3 - k_1 k_1' k_2, \tag{17}$$

$$K_1 = k_1^2 k_3 + k_2^2 k_3 + k_1 k_2' - k_1' k_2, \tag{18}$$

$$L_1 = -k_2\left(k_1^2 + k_2^2\right), \tag{19}$$

$$M_1 = k_1\left(k_1^2 + k_2^2\right), \tag{20}$$

$$U_1 = \left(k_1^2 + k_2^2\right)^4 + \left(k_1^2 + k_2^2\right)\left(k_1^2 k_3 + k_2^2 k_3 + k_1 k_2' - k_1' k_2\right)^2, \tag{21}$$

$$V_1 = \left(k_1^2 + k_2^2\right)^3 + \left(k_1^2 k_3 + k_2^2 k_3 + k_1 k_2' - k_1' k_2\right)^2, \tag{22}$$

$$W_1 = 3\left(k_1\left(k_1'\right)^2 k_2 + k_1' k_2' k_2^2 - k_1^2 k_1' k_2' - k_1 k_2\left(k_2'\right)^2\right) \tag{23}$$

$$+\left(k_1^2 + k_2^2\right)\left(k_1 k_2'' + k_1^2 k_3' + k_2^2 k_3' - k_1'' k_2 - k_1 k_1' k_3 - k_2 k_2' k_3\right). \tag{24}$$

Proof of Theorem 1. According to the expression of the Frenet frame of α_1 in terms of the quasi frame of α in [13], we can write

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_2}{\sqrt{k_1^2+k_2^2}} \\ \frac{A_1}{\sqrt{U_1}} & \frac{B_1}{\sqrt{U_1}} & \frac{C_1}{\sqrt{U_1}} \\ \frac{K_1}{\sqrt{V_1}} & \frac{L_1}{\sqrt{V_1}} & \frac{M_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{25}$$

We know the following equalities from [14],

$$\begin{bmatrix} \phi(\mathbf{t}_q) \\ \phi(\mathbf{n}_q) \\ \phi(\mathbf{b}_q) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & \Omega_1 \\ -k_2 & -\Omega_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{26}$$

By the linearity of ϕ we can write

$$\phi(\mathbf{t}_q) = \frac{k_1}{\sqrt{k_1^2+k_2^2}}\phi(\mathbf{n}_q) + \frac{k_2}{\sqrt{k_1^2+k_2^2}}\phi(\mathbf{b}_q), \tag{27}$$

$$\phi(\mathbf{n}_q) = \frac{A_1}{\sqrt{U_1}}\phi(\mathbf{t}_q) + \frac{B_1}{\sqrt{U_1}}\phi(\mathbf{n}_q) + \frac{C_1}{\sqrt{U_1}}\phi(\mathbf{b}_q), \tag{28}$$

$$\phi(\mathbf{b}_q) = \frac{K_1}{\sqrt{V_1}}\phi(\mathbf{t}_q) + \frac{L_1}{\sqrt{V_1}}\phi(\mathbf{n}_q) + \frac{M_1}{\sqrt{V_1}}\phi(\mathbf{b}_q). \tag{29}$$

Since we know the equalities (26), we get

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} -\sqrt{k_1^2+k_2^2} & -\frac{k_2\Omega_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_1\Omega_1}{\sqrt{k_1^2+k_2^2}} \\ 0 & \frac{A_1k_1 - C_1\Omega_1}{\sqrt{U_1}} & \frac{A_1k_2 - B_1\Omega_1}{\sqrt{U_1}} \\ 0 & \frac{K_1k_1 - M_1\Omega_1}{\sqrt{V_1}} & \frac{K_1k_2 + L_1\Omega_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{30}$$

□

Theorem 2. *The magnetic field V of the tangent indicatrix α_1 of the regular space curve α satisfies the following equality,*

$$V = a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q, \tag{31}$$

where

$$a = \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{K_1}{\kappa_\alpha^3}, \tag{32}$$

$$b = \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{k_2}{\kappa_\alpha}, \tag{33}$$

$$c = -\sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{k_1}{\kappa_\alpha}. \tag{34}$$

Proof of Theorem 2. Since the magnetic field V corresponds to $\mathbf{t}$-magnetic curve, the equality

$$\nabla_{\mathbf{t}}\mathbf{t} = V \times \mathbf{t} \tag{35}$$

holds. So, we can write

$$(a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q) \times (\frac{k_1}{\kappa_\alpha}\mathbf{n}_q + \frac{k_2}{\kappa_\alpha}\mathbf{b}_q) = \kappa\mathbf{n}. \tag{36}$$

Using the expression of $\mathbf{n}$ in terms of the quasi elements of α, we get

$$\frac{bk_2 - ck_1}{\kappa_\alpha}\mathbf{t}_q - \frac{ak_2}{\kappa_\alpha}\mathbf{n}_q + \frac{ak_1}{\kappa_\alpha}\mathbf{b}_q = \sqrt{1 + \frac{K_1}{\kappa_\alpha^6}}(\frac{A_1}{\sqrt{U_1}}\mathbf{t}_q + \frac{B_1}{\sqrt{U_1}}\mathbf{n}_q + \frac{C_1}{\sqrt{U_1}}\mathbf{b}_q). \tag{37}$$

So, from this equality we can write the following equalities,

$$\frac{bk_2 - ck_1}{\kappa_\alpha} = \frac{A_1}{\sqrt{U_1}}\sqrt{1 + \frac{K_1}{\kappa_\alpha^6}}, \tag{38}$$

$$-\frac{ak_2}{\kappa_\alpha} = \frac{B_1}{\sqrt{U_1}}\sqrt{1 + \frac{K_1}{\kappa_\alpha^6}}, \tag{39}$$

$$\frac{ak_1}{\kappa_\alpha} = \frac{C_1}{\sqrt{U_1}}\sqrt{1 + \frac{K_1}{\kappa_\alpha^6}}. \tag{40}$$

Simple calculations give us the following equality,

$$a = \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{K_1}{\kappa_\alpha^3}. \tag{41}$$

To find b and c, we use the equality

$$\phi(V) = 0. \tag{42}$$

Using the linearity of ϕ, we can write

$$a\phi(\mathbf{t}_q) + b\phi(\mathbf{n}_q) + c\phi(\mathbf{b}_q) = 0. \tag{43}$$

So, we get

$$-(bk_1 + ck_2)\mathbf{t}_q + (ak_1 - c\Omega_1)\mathbf{n}_q + (ak_2 + b\Omega_1)\mathbf{b}_q = 0. \tag{44}$$

Thus, we can write

$$\begin{aligned} bk_1 + ck_2 &= 0, && (45) \\ ak_1 - c\Omega_1 &= 0, && (46) \\ ak_2 + b\Omega_1 &= 0. && (47) \end{aligned}$$

Now, we have two equalities to calculate b and c,

$$\begin{aligned} \frac{bk_2 - ck_1}{\kappa_\alpha} &= \frac{A_1}{\sqrt{U_1}}\sqrt{1 + \frac{K_1}{\kappa_\alpha^6}}, && (48) \\ bk_1 + ck_2 &= 0. && (49) \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} b &= \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{k_2}{\kappa_\alpha}, && (50) \\ c &= -\sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}}\frac{k_1}{\kappa_\alpha}. && (51) \end{aligned}$$

□

Corollary 1. *The function Ω_1 which is given with the equation $\Omega_1 = g(\phi(\mathbf{n}_q), \mathbf{b}_q)$ is*

$$\Omega_1 = -\frac{K_1}{\kappa_\alpha^2}. \tag{52}$$

Proof of Corollary 1. From Theorem 2, the result is obtained by direct calculations. □

3.2. The Energy of a **t***-Magnetic Particle*

Now, we give a formula to calculate the energy of a charged particle moving along a **t**-magnetic curve which is a curve where the tangent satisfies $\nabla_\mathbf{t}\mathbf{t} = V \times \mathbf{t}$. Firstly, we recall some basic concepts about this subject.

Let $\pi : TM \longrightarrow M$ be the bundle projection, $T(TM) = V \oplus H$ and $F : M \longrightarrow TM$ be a differentiable vector field. Here V is the vertical component and H is the horizontal component. Then differential dF can be separated into vertical and horizontal components as follows:

$$dF = d^vF + d^hF. \tag{53}$$

Because of the orthogonal decomposition of dF on $T(TM)$, the energy can be separated into two parts as follows:

$$E(F) = \frac{1}{2}\int_M \|dF\|^2 dx = \frac{1}{2}\int_M \|d^vF\|^2 dx + \frac{1}{2}\int_M \left\|d^hF\right\|^2 dx, \tag{54}$$

where dx shows the Riemannian volume element. Using the facts that π is a Riemannian submersion and F is a section, one can get the followings:

$$\left\|d^hF\right\|^2 = \|d\pi \circ dF\|^2 = \|id_{TM}\|^2 = m. \tag{55}$$

On the other hand, one can get

$$\|d^vF\|^2 = \|Q \circ dF\|^2 = \|\nabla F\|^2. \tag{56}$$

Thus, the energy formula becomes [15]

$$E(F) = \frac{1}{2}\int_M \|\nabla F\|^2 \, dx + \frac{m}{2} Vol(M). \tag{57}$$

Let $\sigma_1, \sigma_2 \in T(TM)$, then the Sasaki metric on $T(TM)$ is defined by the following equation:

$$g_S(\sigma_1, \sigma_2) = g(d\pi(\sigma_1), d\pi(\sigma_2)) + g(Q(\sigma_1), Q(\sigma_2)), \tag{58}$$

where $Q : T(TM) \longrightarrow TM$ is the connection map. This metric makes $\pi : TM \longrightarrow M$ a Riemannian submersion.

If V is a magnetic field which corresponds to a **t**-magnetic curve, the energy formula can be rewritten for V using the Sasaki metric as follows [16]:

$$E(V) = \frac{1}{2}\int_0^s g_S(dV, dV)ds. \tag{59}$$

Now, we give a formula to calculate the total kinetic energy of a particle traveling along a curve γ with the speed directed by γ. Firstly, we recall some basic concepts about this subject.

Definition 3. *Let M be a Riemannian manifold and $c : [0, a] \longrightarrow M$ be a piecewise differentiable curve. A variation of c is a continuous mapping $f : (-\varepsilon, \varepsilon) \times [0, a] \longrightarrow M$ such that:*
[i] $f(0, t) = c(t), t \in [0, a]$,
[ii] there exists a subdivision of $[0, a]$ by points $0 = t_0 < t_1 < \cdots < t_{k+1} = a$, such that the restriction of f to each $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$, $i = 0, 1, \ldots, k$, is differentiable.

For each $s \in (-\varepsilon, \varepsilon)$, the parametrized curve $f_s : [0, a] \longrightarrow M$ given by $f_s(t) = f(s, t)$ is called a curve in the variation. Thus, a variation determines a family $f_s(t)$ of neighboring curves of $f_0(t) = c(t)$.

A function $L : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is defined by

$$L(s) = \int_0^a \left\| \frac{\partial f}{\partial t}(s, t) \right\| dt, \; s \in (-\varepsilon, \varepsilon). \tag{60}$$

This function is used to compare the arc length of c with the arc length of neighboring curves in a variation $f : (-\varepsilon, \varepsilon) \times [0, a] \longrightarrow M$ of c. That is, $L(s)$ is the length of the curve $f_s(t)$.

The kinetic energy function $E_k(s)$ is defined by

$$E_k(s) = \int_0^a \left\| \frac{\partial f}{\partial t}(s, t) \right\|^2 dt, \; s \in (-\varepsilon, \varepsilon). \tag{61}$$

This function measures the total kinetic energy of a particle traveling along the curve $f_s(t)$ with the speed directed by $f_s(t)$.

Let $c : [0, a] \longrightarrow M$ be a curve and let

$$L(c) = \int_0^a \left\| \frac{dc}{dt} \right\| dt \text{ and } E_k(c) = \int_0^a \left\| \frac{dc}{dt} \right\|^2 dt. \tag{62}$$

Putting $f = 1$ and $g = \left\| \frac{dc}{dt} \right\|$ in the Schwarz inequality:

$$(\int_0^a fg dt)^2 \leq \int_0^a f^2 dt . \int_0^a g^2 dt, \tag{63}$$

the following inequality is obtained:

$$L(c)^2 \leq aE_k(c), \tag{64}$$

where equality occurs if and only if g is constant, that is, if and only if t is proportional to arc length [17].

Theorem 3. *The energy of the particle which has* $\mathbf{t}$*-magnetic curve of the tangent indicatrix* α_1 *of a space curve* α *under the action of the magnetic field* V *is*

$$E(V) = \frac{1}{2} \int_0^s (1 + (a')^2 + (b' + ak_1 - ck_3)^2 + (c' + ak_2 + bk_3)^2) ds, \tag{65}$$

where

$$a = \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}} \frac{K_1}{\kappa_\alpha^3}, \tag{66}$$

$$b = \sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}} \frac{k_2}{\kappa_\alpha}, \tag{67}$$

$$c = -\sqrt{\frac{\kappa_\alpha^6 + K_1}{V_1}} \frac{k_1}{\kappa_\alpha}. \tag{68}$$

Proof of Theorem 3. The $\mathbf{t}$-magnetic curve of the tangent indicatrix α_1 of a space curve α is the trajectory of α_1 under the action of the magnetic field V. To calculate the energy of the particle, we use the energy Formula (59). By the definition of the Sasaki metric, we can write

$$g_S(dV, dV) = g(d\pi(dV(\mathbf{t}_q)), d\pi(dV(\mathbf{t}_q))) + g(Q(dV(\mathbf{t}_q)), Q(dV(\mathbf{t}_q))). \tag{69}$$

Since V is a section, we get

$$d\pi \circ dV = d(\pi \circ V) = d(id_M) = id_{TM}. \tag{70}$$

So, using this fact, we find

$$g(d\pi(dV(\mathbf{t}_q)), d\pi(dV(\mathbf{t}_q))) = g(\mathbf{t}_q, \mathbf{t}_q) = 1. \tag{71}$$

On the other hand, one can get

$$Q(dV(\mathbf{t}_q)) = \nabla_{\mathbf{t}_q} V = (a')^2 + (b' + ak_1 - ck_3)^2 + (c' + ak_2 + bk_3)^2. \tag{72}$$

Thus, putting these values in the energy Formula (59), we obtain

$$E(V) = \frac{1}{2} \int_0^s (1 + (a')^2 + (b' + ak_1 - ck_3)^2 + (c' + ak_2 + bk_3)^2) ds. \tag{73}$$

□

Theorem 4. *The total kinetic energy of the moving particle which has the tangent indicatrix α_1 of a space curve α as a trajectory is*

$$E_k(\alpha_1) = \int_0^s \kappa_\alpha^2 ds. \tag{74}$$

Proof of Theorem 4. The tangent indicatrix α_1 of a space curve α is the trajectory of the particle. To calculate the total kinetic energy of the particle, we use the energy Formula (62). Since $\alpha_1(s) = \mathbf{t}_q(s)$,

$$\frac{d\alpha_1(s)}{ds} = k_1\mathbf{n}_q + k_2\mathbf{b}_q \text{ and } \left\| \frac{d\alpha_1(s)}{ds} \right\| = \sqrt{k_1^2 + k_2^2}. \tag{75}$$

We know,

$$k_1 = \kappa_\alpha \cos\theta \text{ and } k_2 = -\kappa_\alpha \sin\theta, \tag{76}$$

where θ is the angle between the principal normal $\mathbf{n}$ and the quasi normal $\mathbf{n}_q$. So, we get

$$k_1^2 + k_2^2 = \kappa_\alpha^2. \tag{77}$$

Thus, putting this value in the kinetic energy Formula (62), we obtain

$$E_k(\alpha_1) = \int_0^s \left\| \frac{d\alpha_1(s)}{ds} \right\|^2 ds = \int_0^s \kappa_\alpha^2 ds. \tag{78}$$

□

3.3. n-Magnetic Particles of the Quasi Normal Indicatrix

Let α be a regular curve according to quasi frame in Euclidean 3-space and α_2 be its quasi normal indicatrix. Let $\{\mathbf{t}_q, \mathbf{n}_q, \mathbf{b}_q\}$ be the quasi frame of the curve α and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of α_2.

Theorem 5. *The Lorentz force of the quasi normal indicatrix α_2 of the curve α can be expressed as*

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} -\frac{k_3\Omega_2}{\sqrt{k_1^2+k_3^2}} & -\sqrt{k_1^2+k_3^2} & -\frac{k_1\Omega_2}{\sqrt{k_1^2+k_3^2}} \\ -\frac{B_2k_1+C_2\Omega_2}{\sqrt{U_2}} & 0 & \frac{B_2k_3+A_2\Omega_2}{\sqrt{U_2}} \\ -\frac{L_2k_1+M_2\Omega_2}{\sqrt{V_2}} & 0 & \frac{L_2k_3+K_2\Omega_2}{\sqrt{V_2}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{79}$$

where $\Omega_2 = g(\phi(\mathbf{b}_q), \mathbf{t}_q)$.

Proof of Theorem 5. According to the expression of the Frenet frame of α_2 in terms of the quasi frame of α in [13], we can write

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{\sqrt{k_1^2+k_3^2}} & 0 & \frac{k_3}{\sqrt{k_1^2+k_3^2}} \\ \frac{A_2}{\sqrt{U_2}} & \frac{B_2}{\sqrt{U_2}} & \frac{C_2}{\sqrt{U_2}} \\ \frac{K_2}{\sqrt{V_2}} & \frac{L_2}{\sqrt{V_2}} & \frac{M_2}{\sqrt{V_2}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{80}$$

We know the following equalities from [14],

$$\begin{bmatrix} \phi(\mathbf{t}_q) \\ \phi(\mathbf{n}_q) \\ \phi(\mathbf{b}_q) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & \Omega_2 \\ -k_1 & 0 & k_3 \\ -\Omega_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{81}$$

By the linearity of ϕ we can write

$$\begin{aligned}
\phi(\mathbf{t}) &= \frac{-k_1}{\sqrt{k_1^2+k_3^2}}\phi(\mathbf{n}_q) + \frac{k_3}{\sqrt{k_1^2+k_3^2}}\phi(\mathbf{b}_q), &(82)\\
\phi(\mathbf{n}) &= \frac{A_2}{\sqrt{U_2}}\phi(\mathbf{t}_q) + \frac{B_2}{\sqrt{U_2}}\phi(\mathbf{n}_q) + \frac{C_2}{\sqrt{U_2}}\phi(\mathbf{b}_q), &(83)\\
\phi(\mathbf{b}) &= \frac{K_2}{\sqrt{V_2}}\phi(\mathbf{t}_q) + \frac{L_2}{\sqrt{V_2}}\phi(\mathbf{n}_q) + \frac{M_2}{\sqrt{V_2}}\phi(\mathbf{b}_q). &(84)
\end{aligned}$$

Since we know the equalities (81), we get

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} -\frac{k_3\Omega_2}{\sqrt{k_1^2+k_3^2}} & -\sqrt{k_1^2+k_3^2} & -\frac{k_1\Omega_2}{\sqrt{k_1^2+k_3^2}} \\ -\frac{B_2k_1+C_2\Omega_2}{\sqrt{U_2}} & 0 & \frac{B_2k_3+A_2\Omega_2}{\sqrt{U_2}} \\ -\frac{L_2k_1+M_2\Omega_2}{\sqrt{V_2}} & 0 & \frac{L_2k_3+K_2\Omega_2}{\sqrt{V_2}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \quad (85)$$

□

Theorem 6. *There is not n-magnetic curve which is a curve where the tangent satisfies $\nabla_{\mathbf{t}}\mathbf{n} = V \times \mathbf{n}$, of the quasi normal indicatrix of a regular space curve.*

Proof of Theorem 6. If there was a magnetic curve it must have a magnetic field V such as

$$V = a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q, \quad (86)$$

which satisfies the following equality,

$$\nabla_{\mathbf{t}}\mathbf{n} = V \times \mathbf{n}. \quad (87)$$

So, we can write

$$(a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q) \times (\frac{A_2}{\sqrt{U_2}}\mathbf{t}_q + \frac{B_2}{\sqrt{U_2}}\mathbf{n}_q + \frac{C_2}{\sqrt{U_2}}\mathbf{b}_q) = -\kappa\mathbf{t} + \tau\mathbf{b}. \quad (88)$$

Using the expressions of $\mathbf{t}$, $\mathbf{b}$, κ and τ in terms of the quasi elements of α, we get

$$\begin{aligned}
\frac{bC_2-cB_2}{\sqrt{U_2}}\mathbf{t}_q + \frac{cA_2-aC_2}{\sqrt{U_2}}\mathbf{n}_q + \frac{aB_2-bA_2}{\sqrt{U_2}}\mathbf{b}_q &= ((1+\frac{K_2}{(k_1^2+k_3^2)^3})^{\frac{1}{2}}\frac{k_1}{\sqrt{k_1^2+k_3^2}} &(89)\\
&+\frac{W_2K_2}{V_2\sqrt{V_2}})\mathbf{t}_q + \frac{W_2L_2}{V_2\sqrt{V_2}}\mathbf{n}_q &(90)\\
&+(\frac{W_2M_2}{V_2\sqrt{V_2}} - (1+\frac{K_2}{(k_1^2+k_3^2)^3})^{\frac{1}{2}}\frac{k_1}{\sqrt{k_1^2+k_3^2}})\mathbf{b}_q. &(91)
\end{aligned}$$

So, from this equality we can write the following equalities,

$$\begin{aligned}
\frac{bC_2-cB_2}{\sqrt{U_2}} &= \frac{W_2K_2}{V_2\sqrt{V_2}} + (1+\frac{K_2}{(k_1^2+k_3^2)^3})^{\frac{1}{2}}\frac{k_1}{\sqrt{k_1^2+k_3^2}}, &(92)\\
\frac{cA_2-aC_2}{\sqrt{U_2}} &= \frac{W_2L_2}{V_2\sqrt{V_2}}, &(93)\\
\frac{aB_2-bA_2}{\sqrt{U_2}} &= \frac{W_2M_2}{V_2\sqrt{V_2}} - (1+\frac{K_2}{(k_1^2+k_3^2)^3})^{\frac{1}{2}}\frac{k_1}{\sqrt{k_1^2+k_3^2}}. &(94)
\end{aligned}$$

Simple calculations give us the following system,

$$\begin{bmatrix} B_2 & -A_2 & 0 \\ -C_2 & 0 & A_2 \\ 0 & C_2 & -B_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sqrt{U_2}\left(\frac{W_2 M_2}{V_2\sqrt{V_2}} - \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) \\ \sqrt{U_2}\frac{W_2 L_2}{V_2\sqrt{V_2}} \\ \sqrt{U_2}\left(\frac{W_2 K_2}{V_2\sqrt{V_2}} + \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) \end{bmatrix}. \tag{95}$$

We want to solve this system according to the Crammer rule, so we must compute the determinants Δ and $\Delta_1, \Delta_2, \Delta_3$, where

$$\Delta = \begin{vmatrix} B_2 & -A_2 & 0 \\ -C_2 & 0 & A_2 \\ 0 & C_2 & -B_2 \end{vmatrix} \tag{96}$$

and

$$\Delta_1 = \begin{vmatrix} \sqrt{U_2}\left(\frac{W_2 M_2}{V_2\sqrt{V_2}} - \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) & -A_2 & 0 \\ \sqrt{U_2}\frac{W_2 L_2}{V_2\sqrt{V_2}} & 0 & A_2 \\ \sqrt{U_2}\left(\frac{W_2 K_2}{V_2\sqrt{V_2}} + \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) & C_2 & -B_2 \end{vmatrix}, \tag{97}$$

$$\Delta_2 = \begin{vmatrix} B_2 & \sqrt{U_2}\left(\frac{W_2 M_2}{V_2\sqrt{V_2}} - \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) & 0 \\ -C_2 & \sqrt{U_2}\frac{W_2 L_2}{V_2\sqrt{V_2}} & A_2 \\ 0 & \sqrt{U_2}\left(\frac{W_2 K_2}{V_2\sqrt{V_2}} + \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) & -B_2 \end{vmatrix}, \tag{98}$$

$$\Delta_3 = \begin{vmatrix} B_2 & -A_2 & \sqrt{U_2}\left(\frac{W_2 M_2}{V_2\sqrt{V_2}} - \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) \\ -C_2 & 0 & \sqrt{U_2}\frac{W_2 L_2}{V_2\sqrt{V_2}} \\ 0 & C_2 & \sqrt{U_2}\left(\frac{W_2 K_2}{V_2\sqrt{V_2}} + \left(1+\frac{K_2}{(k_1^2+k_3^2)^3}\right)^{\frac{1}{2}} \frac{k_1}{\sqrt{k_1^2+k_3^2}}\right) \end{vmatrix}. \tag{99}$$

Since $\Delta = 0$ and $\Delta_3 \neq 0$, the system (95) does not have a solution. This means that there is not magnetic curve of the quasi normal indicatrix of a regular space curve. □

*3.4. **b**-Magnetic Particles of the Quasi Binormal Indicatrix*

Let α be a regular curve according to quasi frame in Euclidean 3-space and α_3 be its quasi binormal indicatrix. Let $\{\mathbf{t}_q, \mathbf{n}_q, \mathbf{b}_q\}$ be the quasi frame of the curve α and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of α_3.

Theorem 7. *The Lorentz force of the quasi binormal indicatrix α_3 of the curve α can be expressed as*

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} \frac{k_2 k_3}{\sqrt{k_2^2+k_3^2}} & \frac{k_3^2 - k_2\Omega_3}{\sqrt{k_2^2+k_3^2}} & -\frac{k_2^2}{\sqrt{k_2^2+k_3^2}} \\ -\frac{C_3 k_2 + B_3\Omega_3}{\sqrt{U_3}} & \frac{A_3\Omega_3 - C_3 k_3}{\sqrt{U_3}} & 0 \\ -\frac{M_3 k_2 + L_3\Omega_3}{\sqrt{V_3}} & \frac{K_3\Omega_3 - M_3 k_3}{\sqrt{V_3}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{100}$$

where $\Omega_3 = g(\phi(\mathbf{t}_q), \mathbf{n}_q)$.

Proof of Theorem 7. According to the expression of the Frenet frame of α_3 in terms of the quasi frame of α in [13], we can write

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \frac{-k_2}{\sqrt{k_2^2+k_3^2}} & 0 & \frac{-k_3}{\sqrt{k_2^2+k_3^2}} \\ \frac{A_3}{\sqrt{U_3}} & \frac{B_3}{\sqrt{U_3}} & \frac{C_3}{\sqrt{U_3}} \\ \frac{K_3}{\sqrt{V_3}} & \frac{L_3}{\sqrt{V_3}} & \frac{M_3}{\sqrt{V_3}} \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{101}$$

We know the following equalities from [14],

$$\begin{bmatrix} \phi(\mathbf{t}_q) \\ \phi(\mathbf{n}_q) \\ \phi(\mathbf{b}_q) \end{bmatrix} = \begin{bmatrix} 0 & \Omega_3 & k_2 \\ -\Omega_3 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{102}$$

By the linearity of ϕ we can write

$$\phi(\mathbf{t}) = \frac{-k_2}{\sqrt{k_2^2+k_3^2}}\phi(\mathbf{t}_q) + \frac{-k_3}{\sqrt{k_2^2+k_3^2}}\phi(\mathbf{b}_q), \tag{103}$$

$$\phi(\mathbf{n}) = \frac{A_3}{\sqrt{U_3}}\phi(\mathbf{t}_q) + \frac{B_3}{\sqrt{U_3}}\phi(\mathbf{n}_q) + \frac{C_3}{\sqrt{U_3}}\phi(\mathbf{b}_q), \tag{104}$$

$$\phi(\mathbf{b}) = \frac{K_3}{\sqrt{V_3}}\phi(\mathbf{t}_q) + \frac{L_3}{\sqrt{V_3}}\phi(\mathbf{n}_q) + \frac{M_3}{\sqrt{V_3}}\phi(\mathbf{b}_q). \tag{105}$$

Since we know the equalities (102), we get

$$\begin{bmatrix} \phi(\mathbf{t}) \\ \phi(\mathbf{n}) \\ \phi(\mathbf{b}) \end{bmatrix} = \begin{bmatrix} \frac{k_2k_3}{\sqrt{k_2^2+k_3^2}} & \frac{k_3^2-k_2\Omega_3}{\sqrt{k_2^2+k_3^2}} & -\frac{k_2^2}{\sqrt{k_2^2+k_3^2}} \\ -\frac{C_3k_2+B_3\Omega_3}{\sqrt{U_3}} & \frac{A_3\Omega_3-C_3k_3}{\sqrt{U_3}} & 0 \\ -\frac{M_3k_2+L_3\Omega_3}{\sqrt{V_3}} & \frac{K_3\Omega_3-M_3k_3}{\sqrt{V_3}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_q \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{106}$$

□

Theorem 8. *There is not b-magnetic which is a curve where the tangent satisfies $\nabla_{\mathbf{t}}\mathbf{b} = V \times \mathbf{b}$, curve of the quasi binormal indicatrix of a regular space curve.*

Proof of Theorem 8. If there was a magnetic curve it must have a magnetic field V such as

$$V = a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q \tag{107}$$

which satisfies the following equality,

$$\nabla_{\mathbf{t}}\mathbf{b} = V \times \mathbf{b}. \tag{108}$$

So, we can write

$$(a\mathbf{t}_q + b\mathbf{n}_q + c\mathbf{b}_q) \times (\frac{K_3}{\sqrt{V_3}}\mathbf{t}_q + \frac{L_3}{\sqrt{V_3}}\mathbf{n}_q + \frac{M_3}{\sqrt{V_3}}\mathbf{b}_q) = -\tau\mathbf{n}. \tag{109}$$

Using the expressions of $\mathbf{n}$ and τ in terms of the quasi elements of α, we get

$$\frac{bM_3 - cL_3}{\sqrt{V_3}}\mathbf{t}_q + \frac{cK_3 - aM_3}{\sqrt{V_3}}\mathbf{n}_q + \frac{aL_3 - bK_3}{\sqrt{V_3}}\mathbf{b}_q = \frac{-W_3A_3}{V_3\sqrt{U_3}}\mathbf{t}_q + \frac{-W_3B_3}{V_3\sqrt{U_3}}\mathbf{n}_q + \frac{-W_3C_3}{V_3\sqrt{U_3}}\mathbf{b}_q. \tag{110}$$

So, from this equality we can write the following equalities,

$$\frac{bM_3 - cL_3}{\sqrt{V_3}} = \frac{-W_3A_3}{V_3\sqrt{U_3}}, \tag{111}$$

$$\frac{cK_3 - aM_3}{\sqrt{V_3}} = \frac{-W_3B_3}{V_3\sqrt{U_3}}, \tag{112}$$

$$\frac{aL_3 - bK_3}{\sqrt{V_3}} = \frac{-W_3C_3}{V_3\sqrt{U_3}}. \tag{113}$$

Simple calculations give us the following system,

$$\begin{bmatrix} 0 & M_3 & -L_3 \\ -M_3 & 0 & K_3 \\ L_3 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{-W_3A_3}{\sqrt{V_3U_3}} \\ \frac{-W_3B_3}{\sqrt{V_3U_3}} \\ \frac{-W_3C_3}{\sqrt{V_3U_3}} \end{bmatrix}. \tag{114}$$

We want to solve this system according to the Crammer rule, so we must compute the determinants Δ and $\Delta_1, \Delta_2, \Delta_3$, where

$$\Delta = \begin{vmatrix} 0 & M_3 & -L_3 \\ -M_3 & 0 & K_3 \\ L_3 & -K_3 & 0 \end{vmatrix} \tag{115}$$

and

$$\Delta_1 = \begin{vmatrix} \frac{-W_3A_3}{\sqrt{V_3U_3}} & M_3 & -L_3 \\ \frac{-W_3B_3}{\sqrt{V_3U_3}} & 0 & K_3 \\ \frac{-W_3C_3}{\sqrt{V_3U_3}} & -K_3 & 0 \end{vmatrix}, \tag{116}$$

$$\Delta_2 = \begin{vmatrix} 0 & \frac{-W_3A_3}{\sqrt{V_3U_3}} & -L_3 \\ -M_3 & \frac{-W_3B_3}{\sqrt{V_3U_3}} & K_3 \\ L_3 & \frac{-W_3C_3}{\sqrt{V_3U_3}} & 0 \end{vmatrix}, \tag{117}$$

$$\Delta_3 = \begin{vmatrix} 0 & M_3 & \frac{-W_3A_3}{\sqrt{V_3U_3}} \\ -M_3 & 0 & \frac{-W_3B_3}{\sqrt{V_3U_3}} \\ L_3 & -K_3 & \frac{-W_3C_3}{\sqrt{V_3U_3}} \end{vmatrix}. \tag{118}$$

Since $\Delta = 0$ and $\Delta_1 \neq 0$, the system (114) does not have a solution. This means that there is not magnetic curve of the quasi binormal indicatrix of a regular space curve. □

4. Conclusions

Magnetic fields and magnetic curves are studied interdisciplinary, especially in physics and differential geometry. The Lorentz force Equation (5) can be applied in some areas such as in protons, cancer therapy, and velocity selectors [18]. Firstly, we mention about what they mean in physics. By the view of differential geometry, we consider the advantages of the quasi frame of a space curve and study magnetic particles of the spherical images of a regular space curve given with the quasi frame. Also, we calculate the energy of a charged particle whose trajectory is a **t**-magnetic field, and the total kinetic energy of a moving particle whose trajectory is the tangent indicatrix. It is well known that the Lorentz formula generalizes the geodesic concept. Magnetic curves have many application areas in physics such as in Kirchhoff elastic rods, etc. For example, in his study, Munteanu mentioned the energy levels in models of atoms with closed geodesic [19]. Thus, magnetic curves are important for physics, and differential geometry is vital to study them.

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