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Harmonic Superspace Approach to the Effective Action in Six-Dimensional Supersymmetric Gauge Theories

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Abstract: We review the recent progress in studying the quantum structure of $6D, \mathcal{N} = (1,0)$, and $\mathcal{N} = (1,1)$ supersymmetric gauge theories formulated through unconstrained harmonic superfields. The harmonic superfield approach allows one to carry out the quantization and calculations of the quantum corrections in a manifestly $\mathcal{N} = (1,0)$ supersymmetric way. The quantum effective action is constructed with the help of the background field method that secures the manifest gauge invariance of the results. Although the theories under consideration are not renormalizable, the extended supersymmetry essentially improves the ultraviolet behavior of the lowest-order loops. The $\mathcal{N} = (1,1)$ supersymmetric Yang–Mills theory turns out to be finite in the one-loop approximation in the minimal gauge. Furthermore, some two-loop divergences are shown to be absent in this theory. Analysis of the divergences is performed both in terms of harmonic supergraphs and by the manifestly gauge covariant superfield proper-time method. The finite one-loop leading low-energy effective action is calculated and analyzed. Furthermore, in the Abelian case, we discuss the gauge dependence of the quantum corrections and present its precise form for the one-loop divergent part of the effective action.

Keywords: supersymmetry; harmonic superspace; quantum corrections; effective action

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Contents

1. Introduction

The higher-dimensional supersymmetric gauge theories attract significant interest due to their remarkable properties in classical and quantum domains and profound links with string/brane theory. The various aspects of the quantum structure of such theories were intensively investigated for a long time (see, e.g., [1–8] and the references therein). Although these theories are not renormalizable because of the dimensionful coupling constant [9,10], it is very interesting to understand to what extent a large number of (super)symmetries can improve the ultraviolet behavior. It is expected that supersymmetries sometimes can help with canceling divergences in the lowest loops, but in higher orders, the divergences still appear even in the maximally-extended supersymmetric models [11]. This looks very similar to what happens in the case of the supergravity theories, but from the technical point of view, the calculations in higher-dimensional gauge theories are much simpler.

If we wish to understand how the given symmetry improves the ultraviolet properties of some theory, it is obviously of importance to use a regularization and the quantization procedure, which preserve this symmetry. For the higher-dimensional supersymmetric Yang–Mills (SYM) theories with matter, it is highly desirable to keep unbroken the gauge invariance and off-shell supersymmetry. For example, quantizing $4D$, $\mathcal{N}=1$ supersymmetric theories in superspace, we ensure a manifest gauge invariance and supersymmetry at all steps of quantum calculations [9,10]. Unfortunately, sometimes, it is impossible to quantize a theory in such a way that all supersymmetries are off-shell and manifest. For example, $4D$, $\mathcal{N}=4$ SYMtheory cannot be quantized in an $\mathcal{N}=4$ supersymmetric manner since the manifest $\mathcal{N}=4$ formulation of this theory is still lacking. However, $4D$, $\mathcal{N}=2$ supersymmetry can be kept manifest within the harmonic superspace formalism [12–17]. This approach can be generalized to the $6D$ case with $\mathcal{N}=(1,0)$ supersymmetry as a manifest symmetry [18–23]. Note that, although $6D$, $\mathcal{N}=(1,0)$ supersymmetric theories look very similar to their $4D$, $\mathcal{N}=2$ counterparts, there is an essential difference between the two: in the generic case $6D$, $\mathcal{N}=(1,0)$, theories are anomalous [24–27]). However, for the $6D$, $\mathcal{N}=(1,1)$ theory, the anomalies are canceled. The manifest gauge symmetry is ensured within the background field method formulated in harmonic superspace [16,28].

In this paper, we briefly review some recent results [29–34] concerning the structure of the ultraviolet divergences and low-energy effective action in $6D$, $\mathcal{N}=(1,1)$ and $\mathcal{N}=(1,0)$ SYM theories in the harmonic superspace approach. (The maximally supersymmetric Yang-Mills theories can be constructed in a manifestly supersymmetric way in the pure spinor superfield formalism [35,36]. However, the quantum aspects of this formulation have not been worked out for the time being, and for this reason, we do not discuss this here.) The main purpose of this study is to reveal the structure of the off-shell divergences in the harmonic superspace approach and to find them explicitly in the lowest loops following the proposals of [8]. Such calculations can be done using either the formalism of harmonic supergraphs or the harmonic superspace generalization of the proper-time method of [37,38]. The proper-time method is a powerful tool for performing the one-loop calculations. In particular, it is well suited to calculating the finite contributions to effective action in the manifestly gauge invariant and supersymmetric way. We explicitly demonstrate the advantages of the harmonic superspace approach for studying the quantum structure of $6D$ SYM theories. Though these theories are not renormalizable because of the dimensionful coupling constant, we will see that in the one-loop approximation, $\mathcal{N}=(1,1)$ SYM theory is finite, if the calculations are performed in the Feynman gauge. The absence of divergences in a minimal gauge and their presence in the non-minimal gauges were already encountered in some other calculation (see, e.g., [39]).

The paper has the following structure. In Section 2, we introduce $6D$, $\mathcal{N} = (1,0)$ harmonic superspace and explain how it can be used for formulating supersymmetric gauge theories. Actually, we consider $\mathcal{N} = (1,0)$ SYM theory interacting with a massless matter hypermultiplet, which belongs to an arbitrary representation of the gauge group. The simplest Abelian theory of this type is investigated in Section 3 at the quantum level. First, in Section 3.1, we describe the harmonic superspace quantization, give an account of the Feynman rules, and deduce the Ward identities encoding the gauge invariance at the quantum level. The next Section 3.2 is devoted to the calculation of the one-loop divergences and the study of their gauge dependence in the Abelian case. In particular, we construct the total divergent part of the one-loop effective action and verify that its gauge-dependent part vanishes on shell. One-loop quantum corrections in the non-Abelian case are investigated in Section 4. We start, in Section 4.2, with the quantization procedure described in Section 4.1 and then calculate the one-loop divergences, employing the Feynman gauge. In particular, we demonstrate that in this gauge, $\mathcal{N} = (1,1)$ SYM theory is finite in the one-loop approximation. The two-loop divergence of the two-point hypermultiplet Green function (also in the Feynman gauge) is calculated in Section 4.3. We show that for $\mathcal{N} = (1,1)$ SYM theory, this Green function involves no divergences. The calculation of the one-loop divergences by the harmonic superspace generalization of the proper-time method is given in Section 4.4. This method is also applied for calculating the finite contributions to the one-loop effective action in Section 4.5, where the leading low-energy structure of this action was found. It is worth pointing out that such an effective action is closely related to the on-shell amplitudes in $6D$ maximally-extended supersymmetric Yang–Mills theories (see, e.g., [2] and the references therein) and to the so-called little strings [40–42].

2. Harmonic Superspace Formulation of $6D$ Supersymmetric Gauge Theories

The conventional $6D$, $\mathcal{N} = (1,0)$ superspace is parametrized by the coordinates $z \equiv (x^M, \theta^a_i)$, where x^M with $M = 0, \ldots 5$ are the ordinary space-time coordinates and θ^a_i with $a = 1, \ldots 4$ and $i = 1,2$ are the Grassmann (i.e., anticommuting) variables forming a $6D$ left-handed spinor. The harmonic superspace is obtained from the $\mathcal{N} = (1,0)$ superspace just defined by adding to its coordinates the harmonic variables $u^{\pm i}$, such that $u^{+i}u^-_i = 1$ and $u^-_i = (u^{+i})^*$.

The basic novel feature of the harmonic superspace is the existence of an analytic subspace in it, with the coordinates:

$$x^M_A \equiv x^M + \frac{i}{2}\theta^- \gamma^M \theta^+; \qquad \theta^{\pm a} \equiv u^\pm_i \theta^{ai}; \qquad u^\pm_i. \tag{1}$$

This subspace is closed under $6D$, $\mathcal{N} = (1,0)$ supersymmetry transformations.

For the integration measures on the harmonic superspace and its analytic subspace, we will use the notation:

$$\int d^{14}z = \int d^6x\, d^8\theta; \qquad \int d\zeta^{(-4)} \equiv \int d^6x\, d^4\theta^+. \tag{2}$$

Furthermore, we introduce the spinor covariant derivatives:

$$D^+_a = u^+_i D^i_a; \qquad D^-_a = u^-_i D^i_a, \tag{3}$$

which satisfy the relation $\{D^+_a, D^-_b\} = i(\gamma^M)_{ab}\partial_M$, and define:

$$(D^+)^4 = -\frac{1}{24}\varepsilon^{abcd} D^+_a D^+_b D^+_c D^+_d. \tag{4}$$

The integration measures are related by the useful identity:

$$\int d^{14}z = \int d\zeta^{(-4)}(D^+)^4. \tag{5}$$

An important ingredient of the approach is the harmonic derivatives:

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}; \qquad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}; \qquad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}, \tag{6}$$

which constitute the algebra $SU(2)$,

$$[D^{++}, D^{--}] = D^0, \quad [D^0, D^{\pm\pm}] = \pm D^{\pm\pm}. \tag{7}$$

In the analytic basis $(x^M_A, \theta^{\pm a}, u^{\pm}_i)$, the harmonic derivatives acquire some additional terms, the precise form of which can be found in [43].

The harmonic superspace analog of the gauge field is the analytic superfield $V^{++}(z,u)$, which satisfies the condition:

$$D^+_a V^{++} = 0 \tag{8}$$

and is real with respect to the "tilde" conjugation, $\widetilde{V^{++}} = V^{++}$. Geometrically, this object is the gauge connection covariantizing the harmonic derivative D^{++},

$$D^{++} \;\Rightarrow\; \nabla^{++} = D^{++} + iV^{++}. \tag{9}$$

The pure $6D$, $\mathcal{N} = (1,0)$ SYM theory is described by the harmonic superspace action [20]:

$$S_{\text{SYM}} = \frac{1}{f_0^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\text{tr}\int d^{14}z\, du_1 \ldots du_n\, \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}. \tag{10}$$

In this expression, f_0 is the bare coupling constant. The crucial difference of the $6D$ case from the similar $4D$ case is that the coupling constant f_0 in six dimensions is dimensionful, $[f_0] = m^{-1}$. Obviously, this gives rise to the lack of good renormalization properties at the quantum level.

In the notation accepted in this paper, we will always assume that the gauge superfield in the pure Yang–Mills action (10) is decomposed over the generators of the fundamental representation, $V^{++}(z,u) = V^{++A}t^A$. The generators t^A satisfy the conditions:

$$\text{tr}(t^At^B) = \frac{1}{2}\delta^{AB}; \qquad [t^A, t^B] = if^{ABC}t^C, \tag{11}$$

where f^{ABC} are the gauge group structure constants. Just as in the non-supersymmetric case, only terms quadratic in the gauge superfield V^{++} survive in the action (10) for the Abelian gauge group $G = U(1)$.

General $6D$, $\mathcal{N} = (1,0)$ gauge theories also involve the hypermultiplets minimally coupled to the gauge superfield V^{++}. In the harmonic superspace approach, the hypermultiplets are described by analytic superfields q^+ and their tilde-conjugated $\widetilde{q}^+$,

$$D^+_a q^+ = 0; \qquad D^+_a \widetilde{q}^+ = 0. \tag{12}$$

The full action of the gauge theory with hypermultiplets reads:

$$S = \frac{1}{f_0^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\text{tr}\int d^{14}z\, du_1 \ldots du_n\, \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)} - \int d\zeta^{(-4)}du\, \widetilde{q}^+\nabla^{++}q^+. \tag{13}$$

Note that the covariant harmonic derivative in the second piece of this action,

$$\nabla^{++} = D^{++} + iV^{++} = D^{++} + iV^{++A}T^A, \tag{14}$$

includes the generators T^A corresponding to the representation R to which the hypermultiplet superfields q^+ belong. These generators satisfy the relations analogous to (11):

$$\mathrm{tr}(T^A T^B) = T(R)\delta^{AB}; \qquad [T^A, T^B] = i f^{ABC} T^C. \tag{15}$$

Assuming that the gauge group G is simple, we also define C_2 and $C(R)_i{}^j$ as:

$$f^{ACD} f^{BCD} = C_2 \delta^{AB}; \qquad C(R)_i{}^j = (T^A T^A)_i{}^j. \tag{16}$$

Note that $C(R)_i{}^j$ is proportional to δ_i^j only for an irreducible representation R. In particular, for the adjoint representation of a simple group, we have:

$$T(Adj) = C_2; \qquad C(Adj)_i{}^j = C_2 \delta_i^j. \tag{17}$$

If the hypermultiplet belongs to the adjoint representation, $R = Adj$, the action (13) describes $\mathcal{N} = (1,1)$ SYM theory, which possesses a hidden $\mathcal{N} = (0,1)$ supersymmetry in addition to the manifest $\mathcal{N} = (1,0)$ one. This theory is the $6D$ analog of the $4D$, $\mathcal{N} = 4$ SYM theory. The $4D$, $\mathcal{N} = 4$ SYM theory is known to possess unique properties in the quantum domain since it is a completely finite quantum field theory [44–47]. One can expect that the quantum $6D$, $\mathcal{N} = (1,1)$ SYM theory possesses some remarkable properties, as well.

The general $\mathcal{N} = (1,0)$ gauge theory described by the action (13) is invariant under the gauge transformations:

$$V^{++} \to e^{i\lambda} V^{++} e^{-i\lambda} - i e^{i\lambda} D^{++} e^{-i\lambda}; \qquad q^+ \to e^{i\lambda} q^+; \qquad \widetilde{q}^+ \to \widetilde{q}^+ e^{-i\lambda} \tag{18}$$

parametrized by an analytic superfield λ, such that $\lambda = \lambda^A t^A$ for $V^{++} = V^{++A} t^A$ (in the gauge part of the total action) and $\lambda = \lambda^A T^A$ for $V^{++} = V^{++} T^A$, q^+, and $\widetilde{q}^+$ (in the hypermultiplet part).

Furthermore, we will need the non-analytic gauge superfield:

$$V^{--}(z,u) \equiv \sum_{n=1}^{\infty} (-i)^{n+1} \int du_1 \, \ldots \, du_n \, \frac{V^{++}(z,u_1) \ldots V^{++}(z,u_n)}{(u^+ u_1^+)(u_1^+ u_2^+) \ldots (u_n^+ u^+)}, \tag{19}$$

which covariantizes the harmonic derivative D^{--} and satisfies the "harmonic flatness condition":

$$D^{++} V^{--} - D^{--} V^{++} + i[V^{++}, V^{--}] = 0. \tag{20}$$

An important object is the analytic superfield strength:

$$F^{++} \equiv (D^+)^4 V^{--}, \tag{21}$$

which obeys the off-shell constraint:

$$\nabla^{++} F^{++} = 0, \tag{22}$$

as a consequence of (20) and the analyticity of V^{++}. One more useful quantity is a non-analytic superfield q^-, which is defined by the equation:

$$q^+ = \nabla^{++} q^- = (D^{++} + i V^{++}) q^-. \tag{23}$$

The solution of this equation is given by the series:

$$
\begin{aligned}
q^{-} &= \int \frac{du_1}{(u^+u_1^+)} q_1^+ - i\int \frac{du_1\, du_2}{(u^+u_1^+)(u_1^+u_2^+)} V_1^{++} q_2^+ - \int \frac{du_1\, du_2\, du_3}{(u^+u_1^+)(u_1^+u_2^+)(u_2^+u_3^+)} V_1^{++} V_2^{++} q_3^+ + \ldots \\
&= (-i)^{n-1} \sum_{n=1}^{\infty} \int du_1 \ldots du_n \, \frac{V_1^{++} \ldots V_{n-1}^{++}}{(u^+u_1^+) \ldots (u_{n-1}^+u_n^+)} q_n^+ .
\end{aligned} \tag{24}
$$

The gauge transformations of the superfields V^{--}, F^{++}, and q^- defined above are as follows:

$$
V^{--} \to e^{i\lambda} V^{--} e^{-i\lambda} - i e^{i\lambda} D^{--} e^{-i\lambda}; \qquad F^{++} \to e^{i\lambda} F^{++} e^{-i\lambda}; \qquad q^{-} \to e^{i\lambda} q^{-}. \tag{25}
$$

The simplest particular case of the theory (13) corresponds to the gauge group $U(1)$. The corresponding Abelian gauge theory is the $6D$, $\mathcal{N} = (1,0)$ supersymmetric analog of QED, and it is described by the action:

$$
S = \frac{1}{4f_0^2} \int d^{14}z \, \frac{du_1 du_2}{(u_1^+u_2^+)^2} V^{++}(z,u_1) V^{++}(z,u_2) - \int d\zeta^{(-4)} du \, \widetilde{q}^+ \nabla^{++} q^+, \tag{26}
$$

with $\nabla^{++} = D^{++} + iV^{++}$. In the Abelian case, the gauge transformations acquire the form:

$$
V^{++} \to V^{++} - D^{++}\lambda; \qquad V^{--} \to V^{--} - D^{--}\lambda; \qquad q^+ \to e^{i\lambda} q^+; \qquad F^{++} \to F^{++}, \tag{27}
$$

and the expression for V^{--} is considerably simplified,

$$
V^{--}(z,u) = \int du_1 \, \frac{V^{++}(z,u_1)}{(u^+u_1^+)^2}. \tag{28}
$$

3. Quantum Corrections in $6D$, $\mathcal{N} = (1,0)$ Supersymmetric Electrodynamics

3.1. Quantization, Feynman Rules, and Ward Identities in the Abelian Case

We will start investigating quantum properties of $6D$, $\mathcal{N} = (1,0)$ gauge theories in harmonic superspace by considering the simplest Abelian theory with the action (26). The quantization procedure in the Abelian case requires fixing the gauge. The harmonic superspace analog of the well-known ξ-gauges in QED is obtained by adding, to the original action, the gauge-fixing term,

$$
S_{\text{gf}} = -\frac{1}{4f_0^2\xi_0} \int d^{14}z \, du_1 du_2 \frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3} D_1^{++} V^{++}(z,u_1) \, D_2^{++} V^{++}(z,u_2), \tag{29}
$$

where ξ_0 is an arbitrary parameter. As usual, the normalization was chosen so that the Feynman gauge corresponds to $\xi_0 = 1$. Taking into account the absence of the Faddeev–Popov ghosts in the Abelian case, the generating functional of the theory under consideration has the form:

$$
Z = \exp(iW) = \int DV^{++} \, D\widetilde{q}^+ \, Dq^+ \, \exp\Big\{ i(S + S_{\text{gf}} + S_{\text{sources}}) \Big\} \tag{30}
$$

(as is well known, $W = -i\ln Z$ is the generating functional for the connected Green functions). In harmonic superspace, the source term can be written as:

$$
\int d\zeta^{(-4)} \, du \, \Big[V^{++} J^{(+2)} + j^{(+3)} q^+ + \widetilde{j}^{(+3)} \widetilde{q}^+ \Big], \tag{31}
$$

where the analytic superfields $J^{(+2)}$, $j^{(+3)}$, and $\widetilde{j}^{(+3)}$ are the sources for V^{++}, q^+, and $\widetilde{q}^+$, respectively.

The 1PIGreen functions are generated by the effective action:

$$\Gamma = W - S_{\text{sources}}, \tag{32}$$

with the sources being expressed in terms of the basic superfields by the equations:

$$V^{++} = \frac{\delta W}{\delta J^{(+2)}}; \qquad q^{+} = \frac{\delta W}{\delta j^{(+3)}}; \qquad \widetilde{q}^{+} = \frac{\delta W}{\delta \widetilde{j}^{(+3)}}. \tag{33}$$

Using the standard technique and starting from the functional (30), one can construct the Feynman rules for the considered theory (the detailed analysis of the similar $4D$, $\mathcal{N} = 2$ case has been accomplished in [13,14]). Namely, we represent the total classical action as a sum of the free part $S^{(2)}$, which is quadratic in the involved superfields and the interaction part S_I, which encompasses all terms of the higher orders,

$$S + S_{\text{gf}} \equiv S^{(2)} + S_I. \tag{34}$$

This allows us to write the generating functional in the form:

$$Z = \exp\left\{ iS_I\Big(V^{++} \to \frac{1}{i}\frac{\delta}{\delta J^{++}}, \widetilde{q}^{+} \to \frac{1}{i}\frac{\delta}{\delta \widetilde{j}^{(+3)}}, q^{+} \to \frac{1}{i}\frac{\delta}{\delta j^{(+3)}}\Big)\right\} Z_0, \tag{35}$$

where the generating functional of the free theory is given by the Gaussian integral:

$$Z_0 \equiv \int DV^{++}\, D\widetilde{q}^{+}\, Dq^{+}\, \exp\Big\{ i\big(S^{(2)} + S_{\text{sources}}\big)\Big\}. \tag{36}$$

Then, the expression for S_I produces the vertices, while all propagators are encoded in Z_0.
For the theory (26), the free part of the action and the interaction term read:

$$S^{(2)} = \frac{1}{4f_0^2}\Big(1 - \frac{1}{\xi_0}\Big) \int d^{14}z\, du_1 du_2 \frac{1}{(u_1^+ u_2^+)^2} V^{++}(z,u_1)V^{++}(z,u_2) + \frac{1}{4f_0^2\xi_0}\int d\zeta^{(-4)}\, du\, V^{++}(z,u)\partial^2 V^{++}(z,u); \tag{37}$$

$$S_I = -i\int d\zeta^{(-4)}\, du\, \widetilde{q}^{+} V^{++} q^{+}. \tag{38}$$

From the interaction (38), we conclude that there is only one interaction vertex in the theory. It is depicted in Figure 1.

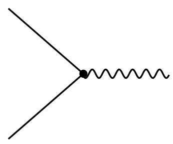

Figure 1. The only interaction vertex of the Abelian $6D, \mathcal{N} = (1,0)$ SQED.

TO calculate the Gaussian integral in (36), we solve the free equations of motion (see [31] for details) and substitute the result into the argument of the exponential. This gives:

$$Z_0 = \exp\left\{ \frac{i}{2}\int d\zeta_1^{(-4)}\, du_1\, d\zeta_2^{(-4)}\, du_2\, J^{++}(z_1,u_1) G_V^{(2,2)}(z_1,u_1;z_2,u_2) J^{++}(z_2,u_2) + i\int d\zeta_1^{(-4)}\, du_1\, d\zeta_2^{(-4)}\, du_2\, j_1^{(+3)} G_q^{(1,1)}(z_1,u_1;z_2,u_2) \widetilde{j}_2^{(+3)} \right\}. \tag{39}$$

Here, the propagators of the gauge superfield and of the hypermultiplet are given, respectively, by the expressions:

$$G_V^{(2,2)}(z_1,u_1;z_2,u_2) = -2f_0^2\Big(\frac{\xi_0}{\partial^2}(D_1^+)^4\delta^{(2,-2)}(u_2,u_1) \\ -\frac{\xi_0-1}{\partial^4}(D_1^+)^4(D_2^+)^4\frac{1}{(u_1^+u_2^+)^2}\Big)\delta^{14}(z_1-z_2); \tag{40}$$

$$G_q^{(1,1)}(z_1,u_1;z_2,u_2) = (D_1^+)^4(D_2^+)^4\frac{1}{\partial^2}\delta^{14}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3}, \tag{41}$$

with:

$$\delta^{14}(z_1-z_2) \equiv \delta^6(x_1-x_2)\delta^8(\theta_1-\theta_2). \tag{42}$$

Graphically, the V^{++} propagator is denoted by a wavy line, while the hypermultiplet propagator by a solid line. They are depicted on the left and right sides of Figure 2, respectively.

Figure 2. The propagators of the gauge superfield V^{++} and the hypermultiplets.

It is obvious that the Feynman diagrams containing closed loops are divergent. Their superficial degree of divergence has been found in [29]. It is defined by the equation:

$$\omega = 2L - N_q - \frac{1}{2}N_D, \tag{43}$$

where the number of loops is denoted by L, the number of external hypermultiplet lines by N_q, and N_D denotes the number of spinor covariant derivatives acting on the external lines. From Equation (43), one can directly conclude that in the one-loop approximation, divergent diagrams should either contain two external hypermultiplet lines or not contain such external lines at all.

At the quantum level, the gauge invariance of the given theory leads to some relations between the Green functions. In the Abelian case, these are the Ward identities [48]. Their non-Abelian generalization is the Slavnov–Taylor identities [49,50]. The harmonic superspace Ward identities were constructed in [33] by making the transformation (18) in the generating functional (30). Using the notation:

$$\Delta\Gamma = \Gamma - S_{\rm gf}, \tag{44}$$

the generating Ward identity amounts to the equation:

$$D^{++}\frac{\delta\Delta\Gamma}{\delta V^{++}} = -iq^+\frac{\delta\Delta\Gamma}{\delta q^+} + i\widetilde{q}^+\frac{\delta\Delta\Gamma}{\delta\widetilde{q}^+}. \tag{45}$$

The adjective "generating" refers to the fact that in this equation, the (super)field arguments are not put equal to zero in advance. Therefore, Equation (45) encompasses an infinite set of identities, which relate the longitudinal parts of $(n+1)$-point Green functions to n-point Green functions.

The lowest-order Ward identity leads to the transversality of quantum corrections to the two-point function of the gauge (super)field. In the harmonic superspace language, it can be obtained by differentiating Equation (45) twice with respect to V^{++}:

$$D_1^{++}\frac{\delta^2\Delta\Gamma}{\delta V_1^{++}\delta V_2^{++}} = 0, \tag{46}$$

where the superfield arguments have been set equal to zero at the end.

Similarly, differentiating Equation (45) with respect to q_2^+ and $\widetilde{q}_3^+$ and again setting the superfields equal to zero afterwards, we obtain a Ward identity that relates three- and two-point Green functions,

$$D_1^{++}\frac{\delta^3\Delta\Gamma}{\delta V_1^{++}\delta q_2^+\delta\widetilde{q}_3^+} = -i(D_1^+)^4\delta^{14}(z_1-z_2)\delta^{(-3,3)}(u_1,u_2)\frac{\delta^2\Delta\Gamma}{\delta q_1^+\delta\widetilde{q}_3^+}$$
$$+i(D_1^+)^4\delta^{14}(z_1-z_3)\delta^{(-3,3)}(u_1,u_3)\frac{\delta^2\Delta\Gamma}{\delta q_2^+\delta\widetilde{q}_1^+}. \tag{47}$$

The Ward identities are a very convenient tool for checking the correctness of various quantum calculations.

3.2. One-Loop Divergences and Their Gauge Dependence

According to the relation (43), divergent diagrams should have either $N_q = 0$ or $N_q = 2$ of the external hypermultiplet lines (evidently, odd values of N_q are forbidden). However, the number of external gauge lines can be arbitrary, and the degree of divergence of the diagram is independent of this number. Nevertheless, the total divergent part of the effective action can be restored by applying to the arguments based on the gauge invariance encoded in the Ward identities. With this in mind, it is actually enough to calculate the lowest divergent Green functions.

For example, the (quadratically divergent) two-point function of the gauge superfield V^{++} in the one-loop order is determined by the only supergraph presented in Figure 3.

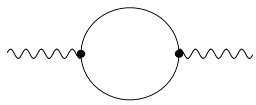

Figure 3. The supergraph giving the one-loop two-point Green function in the Abelian case.

Obviously, the expression for it is gauge independent due to the absence of the gauge propagators. The result obtained in [29] can be presented in the form:

$$\int\frac{d^6p}{(2\pi)^6}\int d^8\theta\, du_1\, du_2\, V^{++}(p,\theta,u_1)V^{++}(-p,\theta,u_2)\frac{1}{(u_1^+u_2^+)^2}\Big[\frac{1}{4f_0^2}-\frac{i}{2}\int\frac{d^6k}{(2\pi)^6}\frac{1}{k^2(k+p)^2}\Big]. \tag{48}$$

When using the dimensional reduction [51] to regularize the theory, the divergent part of this expression is:

$$-\frac{1}{6\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\, du\, (F^{++})^2, \tag{49}$$

where $\varepsilon = 6 - D$. However, the regularization by dimensional reduction allows calculating only the logarithmic divergences, while the considered supergraph diverges quadratically. For finding these quadratic divergences one needs to use another type of regularization. For example, one could use special modifications of the Slavnov higher covariant derivative regularization [52,53] (its harmonic superspace version for $4D$, $\mathcal{N} = 2$ supersymmetric theories was worked out in [54]). In the one-loop approximation, it suffices to use the simplest ultraviolet cut-off procedure. If the loop momentum is cut at the scale Λ, the divergence of the considered contribution to the effective action can be written as [31]:

$$\int d^{14}z\, du_1\, du_2\, V^{++}(z,u_1)V^{++}(z,u_2)\frac{1}{(u_1^+u_2^+)^2}\frac{\Lambda^2}{4(4\pi)^3} - \ln\Lambda\frac{1}{6(4\pi)^3}\int d\zeta^{(-4)}\, du\,(F^{++})^2. \tag{50}$$

This expression is gauge invariant, so there do not appear any further divergent contributions coming from the diagrams with larger numbers of external gauge lines. Indeed, it is easy to see that the gauge-invariant structures proportional to $(F^{++})^n$ with $n \geq 3$ correspond to the finite part of the effective action.

Next, let us consider the divergent part of the Green functions with $N_q = 2$. The simplest one is the two-point Green function of the hypermultiplet. In the one-loop order, it is given by the logarithmically-divergent supergraph presented in Figure 4. The result calculated in [33] is given by the gauge-dependent expression:

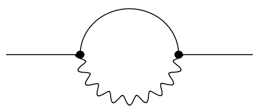

Figure 4. The supergraph defining the one-loop two-point hypermultiplet Green function.

$$-2if_0^2\int\frac{d^6p}{(2\pi)^6}\frac{d^6k}{(2\pi)^6}\frac{1}{k^4(k+p)^2}\int d^8\theta\, du_1\, du_2\,\frac{(\xi_0-1)}{(u_1^+u_2^+)}\widetilde{q}^+(p,\theta,u_1)q^+(-p,\theta\, u_2), \tag{51}$$

which is logarithmically divergent in agreement with Equation (43). The corresponding divergent part (calculated using the regularization by dimensional reduction) is written as:

$$-\frac{2f_0^2}{\varepsilon(4\pi)^3}\int d^{14}z\, du_1\, du_2\,\frac{(\xi_0-1)}{(u_1^+u_2^+)}\widetilde{q}^+(z,u_1)q^+(z,u_2). \tag{52}$$

If applying the cut-off regularization, it is necessary to replace $1/\varepsilon$ by $\ln\Lambda$. We see that the divergence disappears only in the Feynman gauge $\xi_0 = 1$.

Surely, the expression (52) is not gauge invariant. To obtain the gauge-invariant answer, it is necessary to take into account divergent contributions corresponding to Green functions with $N_q = 2$ and an arbitrary number of the external gauge superfield lines. If the number of the external V^{++} lines is equal to one, then the corresponding Green function in the one-loop order is contributed to by the only superdiagram presented in Figure 5. The relevant expression was calculated in [33], and it has the form:

$$\begin{aligned}
&2f_0^2\int\frac{d^6p}{(2\pi)^6}\frac{d^6q}{(2\pi)^6}\frac{d^6k}{(2\pi)^6}\,d^8\theta\Bigg\{-\int du_1\, du_2\,\widetilde{q}^+(q+p,\theta,u_1)V^{++}(-p,\theta,u_2)q^+(-q,\theta,u_1)\\
&\times\frac{\xi_0}{k^2(q+k)^2(q+k+p)^2}\frac{1}{(u_1^+u_2^+)^2} + \int du_1\, du_2\, du_3\Bigg[(D_2^+)^4\,\widetilde{q}^+(q+p,\theta,u_1)\,V^{++}(-p,\theta,u_2)\\
&\times q^+(-q,\theta,u_3)\,\frac{(\xi_0-1)}{k^4(q+k)^2(q+k+p)^2}\,\frac{(u_1^+u_3^+)^2}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3} - \widetilde{q}^+(q+p,\theta,u_1)\,V^{++}(-p,\theta,u_2)\\
&\times q^+(-q,\theta,u_3)\frac{(\xi_0-1)}{k^2(q+k)^2(q+k+p)^2}\,\frac{1}{(u_1^+u_2^+)(u_2^+u_3^+)} - D_{2a}^+D_{2b}^+\,\widetilde{q}^+(q+p,\theta,u_1)\\
&\times V^{++}(-p,\theta,u_2)\,q^+(-q,\theta,u_3)\,\frac{(\xi_0-1)(\widetilde{\gamma}^M)^{ab}k_M}{2k^4(q+k)^2(q+k+p)^2}\,\frac{(u_1^+u_3^+)}{(u_1^+u_2^+)^2(u_2^+u_3^+)^2}\Bigg]\Bigg\},
\end{aligned} \tag{53}$$

where $(\widetilde{\gamma}^M)^{ab} = \varepsilon^{abcd}(\gamma^M)_{cd}/2$. It is logarithmically divergent. The divergent part calculated within the dimensional reduction technique reads [33]:

$$\frac{2if_0^2}{\varepsilon(4\pi)^3}\int d^{14}z\left\{\int du_1\,du_2\,\widetilde{q}_1^+V_2^{++}q_1^+\frac{\xi_0}{(u_1^+u_2^+)^2}+\int du_1\,du_2\,du_3\,\widetilde{q}_1^+\,V_2^{++}q_3^+\frac{(\xi_0-1)}{(u_1^+u_2^+)(u_2^+u_3^+)}\right\}, \quad (54)$$

where the subscripts denote the harmonic arguments.

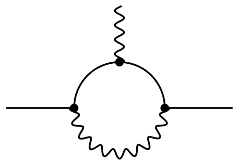

Figure 5. The harmonic supergraph representing the one-loop contribution to the three-point gauge-hypermultiplet function.

To verify the results presented above, it is possible

1. To verify the Ward identity (47);
2. To check that the gauge-dependent terms vanish on shell according to the general theorem of [38,55–59].

Both of these checks have been done in [33], thereby confirming the correctness of the calculations.

However, so far, we have not yet considered all the divergent one-loop diagrams. Even the sum of the expressions (49), (52), and (54),

$$\begin{aligned}\Gamma^{(1)}_\infty &= -\frac{1}{6\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\,du\,(F^{++})^2-\frac{2f_0^2}{\varepsilon(4\pi)^3}\int d^{14}z\,du_1\,du_2\,\frac{(\xi_0-1)}{(u_1^+u_2^+)}\widetilde{q}_1^+q_2^+ + \frac{2if_0^2}{\varepsilon(4\pi)^3}\\ &\times\int d^{14}z\left\{\int du_1\,du_2\,\widetilde{q}_1^+V_2^{++}q_1^+\frac{\xi_0}{(u_1^+u_2^+)^2}+\int du_1\,du_2\,du_3\,\widetilde{q}_1^+\,V_2^{++}q_3^+\frac{(\xi_0-1)}{(u_1^+u_2^+)(u_2^+u_3^+)}\right\}\\ &+O\Big(\widetilde{q}^+(V^{++})^2q^+\Big), \end{aligned}\quad (55)$$

is not gauge invariant. The full gauge-invariant result can be restored, without further calculations, solely on the grounds of gauge invariance considerations. Below, we will show that in the hypermultiplet sector, the gauge invariant result is given by an infinite series in V^{++}. The expression (55) is merely a sum of the lowest terms in the V^{++} expansion of the full gauge-invariant expression.

In order to construct the gauge invariant expression for the one-loop divergences, we recall the V^{++} series representation (56) for the non-analytic superfield q^- defined in (23). The first terms of this series read:

$$q^- = \int\frac{du_1}{(u^+u_1^+)}q_1^+ - i\int\frac{du_1\,du_2}{(u^+u_1^+)(u_1^+u_2^+)}V_1^{++}q_2^+ - \ldots. \quad (56)$$

This representation implies that the total one-loop divergences for $6D, \mathcal{N}=(1,1)$ supersymmetric electrodynamics in the general ξ_0-gauge are written in the form:

$$\begin{aligned}\Gamma^{(1)}_\infty &= -\frac{1}{6\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\,du\,(F^{++})^2+\frac{2if_0^2\xi_0}{\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\,du\,\widetilde{q}^+F^{++}q^+\\ &-\frac{2f_0^2(\xi_0-1)}{\varepsilon(4\pi)^3}\int d^{14}z\,du\,\widetilde{q}^+\,q^-,\end{aligned}\quad (57)$$

where we also made use of the definition (21) and the precise form (28) of V^{--} in the Abelian case.

Note that (in agreement with the general theorems [38,55–59]) the effective action appears to be gauge independent on shell. To demonstrate this fact, we make use of the on-shell property:

$$q^- = \nabla^{--} q^+, \tag{58}$$

whence:

$$\int d^{14}z\, du\, \tilde{q}^+ q^- = \int d\zeta^{(-4)}\, du\, (D^+)^4 \left(\tilde{q}^+ \nabla^{--} q^+\right) = i \int d\zeta^{(-4)}\, du\, \tilde{q}^+ F^{++} q^+. \tag{59}$$

Using this relation, we conclude that all ξ_0-dependent terms in the expression (57) disappear,

$$\Gamma^{(1)}_\infty \Big|_{\text{on shell}} = -\frac{1}{6\varepsilon(4\pi)^3} \int d\zeta^{(-4)}\, du\, (F^{++})^2 + \frac{2if_0^2}{\varepsilon(4\pi)^3} \int d\zeta^{(-4)}\, du\, \tilde{q}^+ F^{++} q^+. \tag{60}$$

4. Quantum Corrections in Non-Abelian 6D, $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1)$ Supersymmetric Theories

4.1. Quantization of Non-Abelian 6D Gauge Theories in the Harmonic Superspace by the Background Field Method

Let us proceed to investigating the non-Abelian case. There are two main differences of the quantization procedure in this case as compared to the Abelian one:

1. It is convenient to use the background (super)field method to construct the manifestly gauge-invariant effective action;

2. The gauge-fixing procedure requires adding ghosts.

According to the background field method, we split the gauge (super)field into the background and quantum parts, so that the theory becomes invariant under two types of gauge transformations. Namely, the background gauge invariance remains unbroken and so is still a manifest symmetry of the effective action. On the contrary, the quantum gauge invariance is broken by gauge fixing, although its remnant, the so-called BRSTsymmetry [60,61], survives as a symmetry of the total gauge-fixed action.

Within the harmonic superspace formalism, the background-quantum splitting is linear. The original superfield V^{++} is presented as a sum of the background gauge superfield $\boldsymbol{V}^{++}$ and the quantum gauge superfield v^{++},

$$V^{++} = \boldsymbol{V}^{++} + v^{++}. \tag{61}$$

The background gauge superfield is treated as an external superfield, for which reason it can appear only on the external legs. We denote the external legs corresponding to $\boldsymbol{V}^{++}$ by the bold wavy lines. The internal and external legs of the quantum gauge superfield will be denoted by the standard wavy lines.

The background-quantum splitting for the hypermultiplets is also possible, but not necessary. The point is that the gauge-fixing term is chosen to be independent of the hypermultiplet superfields, so the effective action depends only on a sum of the quantum and background hypermultiplet superfields. For this reason, here, we do not split the hypermultiplets into the background and quantum parts.

After the background-quantum splitting (61), the gauge invariance (18) produces the background gauge invariance:

$$\boldsymbol{V}^{++} \to e^{i\lambda} \boldsymbol{V}^{++} e^{-i\lambda} - ie^{i\lambda} D^{++} e^{-i\lambda}; \qquad v^{++} \to e^{i\lambda} v^{++} e^{-i\lambda} \qquad q^+ \to e^{i\lambda} q^+ \tag{62}$$

and the quantum gauge invariance:

$$\boldsymbol{V}^{++} \to e^{i\lambda}\boldsymbol{V}^{++}e^{-i\lambda}; \qquad v^{++} \to e^{i\lambda}v^{++}e^{-i\lambda} - ie^{i\lambda}D^{++}e^{-i\lambda}; \qquad q^{+} \to e^{i\lambda}q^{+}. \tag{63}$$

Clearly, if we wish to preserve the background gauge invariance as a manifest symmetry of the effective action, it is necessary to arrange the gauge-fixing term to be invariant under the background transformations. To construct such a term, we introduce the background bridge superfield related to the superfields $\boldsymbol{V}^{++}$ and $\boldsymbol{V}^{--}$ as:

$$\boldsymbol{V}^{++} = -ie^{i\boldsymbol{b}}D^{++}e^{-i\boldsymbol{b}}; \qquad \boldsymbol{V}^{--} = -ie^{i\boldsymbol{b}}D^{--}e^{-i\boldsymbol{b}}. \tag{64}$$

Then, the background gauge transformations (62) should be supplemented by the transformation of the bridge superfield:

$$e^{i\boldsymbol{b}} \to e^{i\lambda}e^{i\boldsymbol{b}}e^{i\tau}, \tag{65}$$

where a new gauge parameter $\tau = \tau(x,\theta)$ does not depend on the harmonic variables. With the help of the bridge superfield, the background gauge-invariant gauge-fixing term is constructed as:

$$\begin{aligned} S_{\text{gf}} = &-\frac{1}{2f_0^2\xi_0}\text{tr}\int d^{14}z\, du_1 du_2 \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} D_1^{++}\Big[e^{-i\boldsymbol{b}(z,u_1)}v^{++}(z,u_1)e^{i\boldsymbol{b}(z,u_1)}\Big] \\ &\times D_2^{++}\Big[e^{-i\boldsymbol{b}(z,u_2)}v^{++}(z,u_2)e^{i\boldsymbol{b}(z,u_2)}\Big]. \end{aligned} \tag{66}$$

It is analogous to the usual ξ-gauge fixing term for non-supersymmetric Yang–Mills theory, the Feynman (minimal) gauge corresponding to the choice $\xi_0 = 1$. Note that in the Abelian case, the dependence on the bridge superfield in (66) is canceled out, and for $6D$, $\mathcal{N} = (1,0)$ electrodynamics, we recover the expression (29).

As is well known, for quantizing non-Abelian theories, one should introduce the Faddeev–Popov ghosts. In the background superfield method, the Nielsen–Kallosh ghosts are also needed. In the harmonic superspace language, the Faddeev–Popov ghost action is written as:

$$S_{\text{FP}} = \text{tr}\int d\zeta^{(-4)}\, du\, b\boldsymbol{\nabla}^{++}\Big(\boldsymbol{\nabla}^{++}c + i[v^{++},c]\Big). \tag{67}$$

Here, the ghosts c and the antighosts b are the Grassmann analytic superfields in the adjoint representation of the gauge group. Correspondingly, the background covariant derivative of the ghost superfield takes the form $\boldsymbol{\nabla}^{++}c = D^{++}c + i[\boldsymbol{V}^{++},c]$.

In the background superfield method, the functional integral after quantization includes determinants, which are usually written as functional integrals over the Nielsen–Kallosh ghosts. Within the harmonic superspace approach, such determinants are given by the expression:

$$\Delta_{NK} \equiv \text{Det}^{1/2}\,\widehat{\Box}\int D\varphi \exp\big(iS_{\text{NK}}\big). \tag{68}$$

Here, we introduced the notation $\widehat{\Box} \equiv \frac{1}{2}(D^+)^4(\boldsymbol{\nabla}^{--})^2$ and:

$$S_{\text{NK}} = -\frac{1}{2}\text{tr}\int d\zeta^{(-4)}\, du\,(\boldsymbol{\nabla}^{++}\varphi)^2, \tag{69}$$

where φ are the commuting Nielsen–Kallosh ghosts, analytic Grassmann-even superfields in the adjoint representation. The determinant Det $\widehat{\Box}$ in (68) can also be cast in the form of a functional integral by introducing the Grassmann-odd analytic superfields $\xi^{(+4)}$ and σ in the adjoint representation,

$$\text{Det}\,\widehat{\Box} = \int D\xi^{(+4)}D\sigma \exp\Big(i\,\text{tr}\int d\zeta^{(-4)}\, du\,\xi^{(+4)}\,\widehat{\Box}\,\sigma\Big). \tag{70}$$

Finally, the total generating functional of the theory under consideration takes the form:

$$Z = e^{iW} = \int Dv^{++}\, D\tilde{q}^{+}\, Dq^{+}\, Db\, Dc\, D\varphi\, \mathrm{Det}^{1/2}\, \widehat{\square}\, \exp\Big[i(S + S_{\mathrm{gf}} + S_{\mathrm{FP}} + S_{\mathrm{NK}} + S_{\mathrm{sources}})\Big]. \tag{71}$$

The sources for the gauge and hypermultiplet superfields differ from the Abelian case basically by the presence of the internal symmetry indices,

$$S_{\mathrm{sources}} = \int d\zeta^{(-4)}\, du\, \Big[v^{++A} J^{(+2)A} + j^{(+3)i}(q^{+})_i + \tilde{j}_i^{(+3)}(\tilde{q}^{+})^i\Big]. \tag{72}$$

It is necessary to take into account that only the quantum gauge superfield v^{++} is present in the term (72). In principle, if necessary, it is also possible to introduce sources for ghosts.

The propagators of the quantum gauge superfield and those of the hypermultiplet are similar to those in the Abelian case:

$$\begin{aligned}(G_V^{(2,2)})^{AB}(z_1,u_1;z_2,u_2) = -2f_0^2\Big(&\frac{\xi_0}{\partial^2}(D_1^{+})^4\delta^{(2,-2)}(u_2,u_1)\\ &-\frac{\xi_0-1}{\partial^4}(D_1^{+})^4(D_2^{+})^4\frac{1}{(u_1^{+}u_2^{+})^2}\Big)\delta^6(x_1-x_2)\delta^8(\theta_1-\theta_2)\delta^{AB};\end{aligned} \tag{73}$$

$$(G_q^{(1,1)})_i{}^j(z_1,u_1;z_2,u_2) = (D_1^{+})^4(D_2^{+})^4\frac{1}{\partial^2}\delta^{14}(z_1-z_2)\frac{1}{(u_1^{+}u_2^{+})^3}\delta_i{}^j. \tag{74}$$

In the explicit calculations in the non-Abelian case, we will use only the Feynman gauge $\xi_0 = 1$, because under this choice, the gauge propagator (73) has the simplest form. The propagators (73) and (74) will be graphically denoted, as in the Abelian case, by the wavy and solid lines (see Figure 6). Furthermore, we will need the ghost propagators. They have the same form for both the Faddeev–Popov and the Nielsen–Kallosh ghosts,

$$\frac{(D_1^{+})^4(D_2^{+})^4}{2\partial^2}\delta^{14}(z_1-z_2)\frac{(u_1^{-}u_2^{-})}{(u_1^{+}u_2^{+})^3}\delta^{AB} \tag{75}$$

and will be depicted by the dashed and dotted lines, respectively.

Figure 6. The lines (1), (2), (3), and (4) denote the propagators of the gauge, hypermultiplet, Faddeev–Popov, and Nielsen–Kallosh ghost superfields.

Finally, the propagator of the superfields $\xi^{(+4)}$ and σ introduced in (70) has the form:

$$-\frac{(D_1^{+})^4}{2\partial^2}\delta^{14}(z_1-z_2)\delta^{(0,0)}(u_1,u_2)\delta^{AB}. \tag{76}$$

The interaction vertices can be easily read off from the interaction terms in the action. It is important that in the non-Abelian case on the external legs, there can appear the background gauge superfield. Such legs will be denoted by the bold wavy lines. Due to the linear background-quantum splitting (61), all vertices can contain both quantum and background wavy lines. Precisely as in the $\mathcal{N} = (1,0)$ supersymmetric electrodynamics, in the non-Abelian theory, only the triple vertex describing the interaction of the hypermultiplet with the gauge superfield is present (the gauge superfield can be either background or quantum).

From the action (10), we observe that there are infinitely many vertices with the number $n \geq 3$ of the gauge superfield lines (and with no lines of any other superfields). Note that the gauge-fixing term (66) also contributes to these vertices (in this case, the legs of the background gauge superfield come from the bridge).

Due to the presence of two super-background covariant derivatives in the ghost action (67), there are triple and quartic vertices containing two ghost lines. These vertices can have no more than one line of the quantum gauge superfield v^{++} and no more than two lines of the background gauge superfield $\boldsymbol{V}^{++}$.

The superfields φ, $\xi^{(+4)}$, and σ interact with the background gauge superfield only. For the superfield φ, only the triple and quartic vertices are possible, while the vertices involving $\xi^{(+4)}$ and σ can also contain an arbitrary number of the background gauge superfields coming from the superfield $\boldsymbol{V}^{--}$ concealed in the operator $\widehat{\Box}$.

4.2. One-Loop Divergences in Harmonic Superspace

In order to calculate the divergent part of the one-loop effective action, we again start from calculating the divergences of the lowest order Green functions and then restore the full result by the reasoning based on the unbroken background gauge invariance. This can be done as follows. According to [43], on shell, the one-loop logarithmic divergences have the structure:

$$\Gamma^{(1)}_{\infty,\ln} = \int d\zeta^{(-4)}\, du\, \Big[c_1(F^{++A})^2 + ic_2 F^{++A}(\widetilde{q}^{+})^i (T^A)_i{}^j (q^+)_j + c_3\Big((\widetilde{q}^{+})^i(q^+)_i\Big)^2\Big], \tag{77}$$

where c_i with $i = 1,2,3$ are real numerical coefficients and the regularization by dimensional reduction is assumed. The coefficients c_i can be obtained by calculating the divergences of the two-point function of the background gauge superfield (c_1) and of the three-point gauge-hypermultiplet function (c_2). The coefficient c_3 vanishes,

$$c_3 = 0, \tag{78}$$

because the corresponding four-point hypermultiplet Green function is finite. Actually, in the non-Abelian case, the degree of divergence for diagrams without external ghost legs is also given by the expression (43). In the case of $L = 1$, $N_q = 4$, $N_D = 0$, we obtain $\omega = -2$, for which reason the one-loop four-point hypermultiplet Green function is given by the convergent integrals.

For calculating the coefficient c_1 in the expression (77), we consider the two-point function of the background gauge superfield. In the one-loop order, it is contributed to by the superdiagrams presented in Figure 7, in which the external bold wavy lines correspond to the background gauge superfield. They were calculated in [31]. The following result for the sum of the corresponding contribution to the effective action has been obtained there:

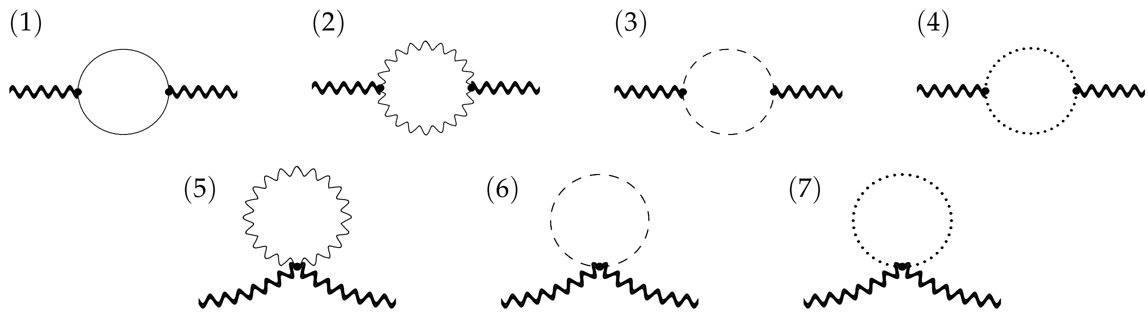

Figure 7. Harmonic supergraphs representing the one-loop two-point Green function of the background gauge superfield.

$$\frac{i}{2}\Big[C_2 - T(R)\Big] \int \frac{d^6p}{(2\pi)^6} \int d^8\theta\, du_1\, du_2\, \boldsymbol{V}^{++A}(p,\theta,u_1)\boldsymbol{V}^{++A}(-p,\theta,u_2)\frac{1}{(u_1^+u_2^+)^2} \int \frac{d^6k}{(2\pi)^6}\frac{1}{k^2(k+p)^2}. \quad (79)$$

This expression is divergent, the leading divergence being quadratic. However, the dimensional reduction can catch only the logarithmic divergences, which can be written as:

$$\frac{1}{6\varepsilon(4\pi)^3}\Big[C_2 - T(R)\Big] \int d^{14}z\, du_1\, du_2\, \boldsymbol{V}^{++A}(z,u_1)\partial^2\boldsymbol{V}^{++A}(z,u_2)\frac{1}{(u_1^+u_2^+)^2}. \quad (80)$$

To calculate the quadratic divergences, one is led to use a regularization with an ultraviolet cut-off Λ. Then, the leading quadratically-divergent terms are represented by the expression:

$$-\frac{\Lambda^2}{4(4\pi)^3}\Big[C_2 - T(R)\Big] \int d^{14}z\, du_1\, du_2\, \boldsymbol{V}^{++A}(z,u_1)\boldsymbol{V}^{++A}(z,u_2)\frac{1}{(u_1^+u_2^+)^2}, \quad (81)$$

while the logarithmic ones are obtained from (80) via the substitution $1/\varepsilon \to \ln\Lambda$.

It is worth noting that the gauge-invariant result in the non-Abelian case also contain higher degrees of $\boldsymbol{V}^{++}$, which are encoded in (77). Comparing the expression (80) with:

$$\int d\zeta^{(-4)}\, du\, (\boldsymbol{F}^{++A})^2 = \int d^{14}z\, du_1\, du_2\, \frac{1}{(u_1^+u_2^+)^2}\boldsymbol{V}^{++A}(z,u_1)\partial^2\boldsymbol{V}^{++A}(z,u_2) + O\Big((\boldsymbol{V}^{++})^3\Big), \quad (82)$$

we obtain:

$$c_1 = \frac{C_2 - T(R)}{6\varepsilon(4\pi)^3}, \quad (83)$$

which implies that, in the case of employing the dimensional reduction regularization, the divergent part of the one-loop effective action is written in the form:

$$\frac{C_2 - T(R)}{3\varepsilon(4\pi)^3}\,\mathrm{tr}\int d\zeta^{(-4)}\, du\, (\boldsymbol{F}^{++})^2 + \text{terms containing hypermultiplets.} \quad (84)$$

As for the quadratic divergences (81), they correspond to the lowest term in the power expansion of the gauge-invariant object:

$$-\Big[C_2 - T(R)\Big]\frac{f_0^2\Lambda^2}{(4\pi)^3}\, S_{\mathrm{SYM}}[\boldsymbol{V}^{++}], \quad (85)$$

where S_{SYM} is given by (10).

The two-point Green function of the hypermultiplet is calculated similarly to the Abelian case already considered earlier. For non-Abelian theories, it is also determined by a single logarithmically-divergent supergraph presented in Figure 4. The only novelty is the presence of the hypermultiplet indices and the factor $C(R)_i{}^j$. Exactly as in the Abelian case, in the Feynman gauge $\xi_0 = 1$ the two-point Green function of the hypermultiplet vanishes (recall (51)).

The coefficient c_2 in the expression (77) can be found by calculating the one-loop contribution to the three-point gauge-hypermultiplet Green function, which is determined by two harmonic supergraphs presented in Figure 8. The details of the calculation can be found in [31], while here, we provide only the answers:

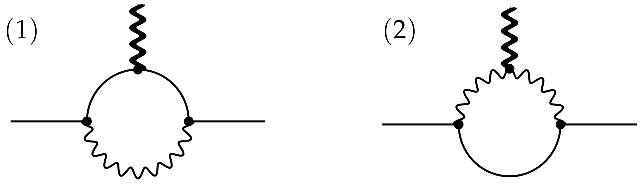

Figure 8. These two harmonic supergraphs determine the three-point gauge-hypermultiplet function in the one-loop approximation.

$$(1) = -2f_0^2 \int \frac{d^6p}{(2\pi)^6}\frac{d^6q}{(2\pi)^6}\, d^8\theta\, du_1\, du_2\, (\tilde{q}^+)^i(q+p,\theta,u_1)\Big[C(R)_i{}^k - \frac{1}{2}C_2\delta_i^k\Big]\boldsymbol{V}^{++}(-p,\theta,u_2)_k{}^j$$
$$\times (q^+)_j(-q,\theta,u_1)\frac{1}{(u_1^+u_2^+)^2}\int \frac{d^6k}{(2\pi)^6}\frac{1}{k^2(q+k)^2(q+k+p)^2}; \qquad (86)$$
$$(2) = f_0^2 C_2 \int \frac{d^6p}{(2\pi)^6}\frac{d^6q}{(2\pi)^6}\, d^8\theta\, du_1\, du_2\, \tilde{q}^+(q+p,\theta,u_1)^i\boldsymbol{V}^{++}(-p,\theta,u_2)_i{}^j q^+(-q,\theta,u_1)_j$$
$$\times \frac{1}{(u_1^+u_2^+)^2}\int \frac{d^6k}{(2\pi)^6}\frac{1}{k^2(k+p)^2(k+p+q)^2}. \qquad (87)$$

Obviously, both of these expressions are logarithmically divergent. When using the regularization by dimensional reduction [51], the divergent part of their sum is written as:

$$\frac{2if_0^2}{\varepsilon(4\pi)^3}\int \frac{d^6p}{(2\pi)^6}\frac{d^6q}{(2\pi)^6}\, d^8\theta\, du\, (\tilde{q}^+)^i(q+p,\theta,u)\Big[C(R)_i{}^k - C_2\delta_i^k\Big]\boldsymbol{V}^{--}_{\text{linear}}(-p,\theta,u)_k{}^j(q^+)_j(-q,\theta,u), \quad (88)$$

where:

$$\boldsymbol{V}^{--}_{\text{linear}} \equiv \int du_1 \frac{\boldsymbol{V}^{++}(z,u_1)}{(u^+u_1^+)^2} \qquad (89)$$

is the lowest (linear) term in the expansion of $\boldsymbol{V}^{--}$ in powers of $\boldsymbol{V}^{++}$.

Rewriting the expression (88) in the coordinate representation, we can cast it in the form:

$$\frac{2if_0^2}{\varepsilon(4\pi)^3}\int d^{14}z\, du\, (\tilde{q}^+)^i\Big[C(R)_i{}^k - C_2\delta_i^k\Big](\boldsymbol{V}^{--}_{\text{linear}})_k{}^j(q^+)_j$$
$$= \frac{2if_0^2}{\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\, du\, (\tilde{q}^+)^i\Big[C(R)_i{}^k - C_2\delta_i^k\Big](\boldsymbol{F}^{++}_{\text{linear}})_k{}^j(q^+)_j, \qquad (90)$$

where the linear part of $\boldsymbol{F}^{++}$ is denoted by:

$$\boldsymbol{F}^{++}_{\text{linear}} \equiv (D^+)^4\boldsymbol{V}^{--}_{\text{linear}}. \qquad (91)$$

The expression (90) is the lowest term in the expansion of the gauge-invariant expression:

$$\frac{2if_0^2}{\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\, du\, (\tilde{q}^+)^i\Big[C(R)_i{}^k - C_2\delta_i^k\Big]\boldsymbol{F}^{++})_k{}^j(q^+)_j \qquad (92)$$

in powers of $\boldsymbol{V}^{++}$. Comparing it with (77), we conclude that:

$$c_2 = 2f_0^2\,\frac{C(R)-C_2}{(4\pi)^3\varepsilon}. \qquad (93)$$

Thus, when using the regularization by dimensional reduction, the total divergent part of the one-loop effective action for an arbitrary $6D$, $\mathcal{N}=(1,0)$ gauge theory can be written as:

$$(\Gamma^{(1)}_{\infty})_{\rm DRED} = \frac{C_2 - T(R)}{3\varepsilon(4\pi)^3}{\rm tr}\int d\zeta^{(-4)}\,du\,(\boldsymbol{F}^{++})^2 - 2if_0^2\,\frac{1}{\varepsilon(4\pi)^3}\int d\zeta^{(-4)}\,du\,\widetilde{q}^{+}\big[C_2 - C(R)\big]\boldsymbol{F}^{++}q^{+}. \qquad (94)$$

This is a final result for one-loop divergences. We see that in the $\mathcal{N}=(1,1)$ theory, where $T(Adj)=C_2$ and $C(Adj)_i{}^j = C_2\delta_i{}^j$, the all one-loop divergences are absent off shell. This result was obtained in the framework of the supersymmetric dimensional regularization.

However, it is interesting to understand how such a result depends on the regularization. This is the reason why it is instructive to study the one-loop divergences in the framework of some another regularization. Here, we present the corresponding result in the regularization by an ultraviolet cut-off Λ. In this case, it is possible to calculate both quadratic and logarithmic one-loop divergences,

$$(\Gamma^{(1)}_{\infty})_{\rm UV\ cut\text{-}off} = -\big[C_2 - T(R)\big]\frac{f_0^2\Lambda^2}{(4\pi)^3}\,S_{SYM}[\boldsymbol{V}^{++}] + \ln\Lambda\Big[\frac{C_2 - T(R)}{3(4\pi)^3}\,{\rm tr}\int d\zeta^{(-4)}\,du \times(\boldsymbol{F}^{++})^2 - 2if_0^2\frac{1}{(4\pi)^3}\int d\zeta^{(-4)}\,du\,\widetilde{q}^{+}\big[C_2 - C(R)\big]\boldsymbol{F}^{++}q^{+}\Big]. \qquad (95)$$

Now, we get the additional divergent term $S_{SYM}[\boldsymbol{V}^{++}]$ in comparison with divergences within the dimensional regularization. Nevertheless, in the $\mathcal{N}=(1,1)$ theory, this divergent term also vanishes. Note that using the cut-off regularization can lead to some problems in higher loops. Actually, because of a possible violation of the BRST invariance, the Slavnov–Taylor identities [49,50] can be broken at the quantum level (see, e.g., the calculation for supersymmetric theories in [62]). However, these identities can be restored with the help of a special subtraction scheme, similar to the one constructed in [63,64]. Moreover, the BRST symmetry guarantees the stability of the background-quantum splitting (61). For non-invariant regularizations, this equation can receive some quantum corrections. Nevertheless, in the one-loop approximation for the considered part of the effective action, not all of these problems are essential. To overcome them in higher loops, it is necessary to use an invariant regularization, e.g., some versions of the higher covariant derivative regularization [52,53] in the harmonic superspace (see [54]).

As we already pointed out, with taking into account the relations (17), we obtain that in the $6D$, $\mathcal{N}=(1,1)$ SYM theory, all the divergences (including the quadratic ones) vanish (the cancellation of quadratic divergences is also suggested by their relationship with the (vanishing) divergences of $4D$, $\mathcal{N}=4$ theory). In the gauge sector, this occurs because both quadratic and logarithmic divergences are proportional to $C_2 - T(R)$. This result agrees with the calculation made earlier in [65,66], where the divergences in the gauge sector have been found using the component formulation of the theory. However, we also demonstrated that the divergences in the hypermultiplet sector vanish, as well, if the theory is quantized in the manifestly $\mathcal{N}=(1,0)$ supersymmetric and gauge-invariant way, and the Feynman gauge condition is used.

4.3. Two-Loop Divergent Part of the Hypermultiplet Two-Point Green Function of 6D SYM Theories

The calculation of quantum corrections in the two-loop approximation is a much more complicated problem. To date, the two-loop divergences in the harmonic superspace formalism have been found only for the two-point Green function of the hypermultiplet. It is determined by the diagrams depicted in Figure 9. In the diagram (5) in Figure 9, the gray disk corresponds to the insertion of the one-loop polarization operator of the quantum gauge superfield. It is given by the sum of the one-loop superdiagrams presented in Figure 10. The details of the two-loop calculations can be found in [32]. The formal result for the Green function under consideration (without a regularization) is given by the expression (written in the Minkowski space before the Wick rotation)

$$
\begin{aligned}
&4f_0^4 \int \frac{d^6p}{(2\pi)^6} d^8\theta \int \frac{du_1\,du_2}{(u_1^+ u_2^+)} \Big[\widetilde{q}^+(p,\theta,u_1)^i \Big(-C(R)^2 + C_2 C(R) \Big)_i{}^j q^+(-p,\theta,u_2)_j \\
&\times \int \frac{d^6k}{(2\pi)^6} \frac{d^6l}{(2\pi)^6} \frac{1}{k^2 l^2 (k+l)^2 (k+l+p)^2 (k+p)^2} + \Big(C_2 - T(R) \Big) \widetilde{q}^+(p,\theta,u_1)^i C(R)_i{}^j \\
&\times q^+(-p,\theta,u_2)_j \int \frac{d^6k}{(2\pi)^6} \frac{d^6l}{(2\pi)^6} \frac{1}{k^4 (k+p)^2 l^2 (k+l)^2} \Big]. \qquad (96)
\end{aligned}
$$

In agreement with Equation (43), this Green function is quadratically divergent. The regularization by dimensional reduction cannot be used for calculating the quadratic divergences, so it is necessary to use different regularization schemes. However, let us consider $\mathcal{N} = (1,1)$ SYM theory, with the hypermultiplet in the adjoint representation, $R = Adj$. Using Equation (17), we observe that the expression (96) for this theory vanishes identically. This implies that the leading quadratic divergences are canceled out and the total divergences can be calculated, based on the dimensional reduction. However, even after the replacement $6 \to D$, the expression (96) vanishes. Therefore, the considered Green function for $\mathcal{N} = (1,1)$ SYM theory vanishes identically. Taking into account that $\mathcal{N} = (1,1)$ supersymmetry intertwines the gauge and hypermultiplet superfields, it is reasonable to suggest that all two-point Green functions of this theory also vanish identically.

Nevertheless, two-loop off-shell divergences may arise in the four-point Green functions. To see this, it is sufficient to calculate the four-point Green function of the hypermultiplet. This work is in progress now.

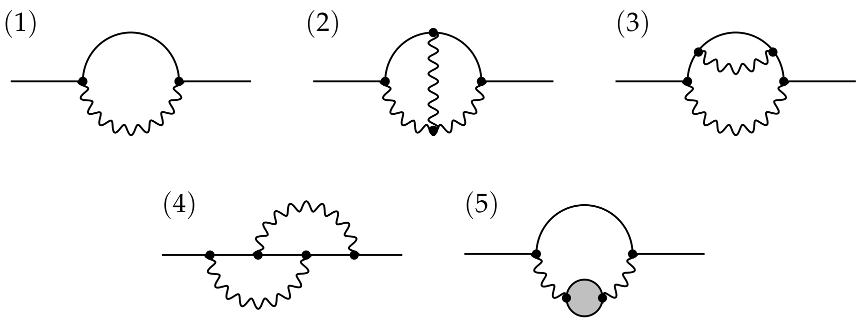

Figure 9. Supergraphs representing the two-point hypermultiplet Green function in the two-loop approximation.

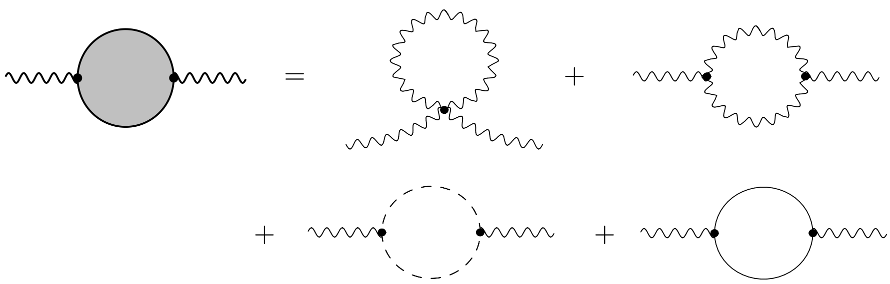

Figure 10. In Figure 9, the gray circle corresponds to the one-loop polarization operator, which is given by the sum of the harmonic supergraphs depicted here.

4.4. Manifestly Gauge Covariant Analysis

In this section, we briefly discuss how the proper-time method can be used for the analysis of divergent contributions in $6D$ $\mathcal{N}=(1,0)$ SYM theory (13). After splitting the superfields V^{++}, q^{+} into the sum of the background parts $\boldsymbol{V}^{++}, \boldsymbol{Q}^{+}$ and the quantum parts v^{++}, q^{+},

$$V^{++} \to \boldsymbol{V}^{++} + v^{++}, \qquad q^{+} \to \boldsymbol{Q}^{+} + q^{+}, \tag{97}$$

we expand the full action in a power series in quantum superfields. In the one-loop order, the first quantum correction to the classical action, $\Gamma^{(1)}[\boldsymbol{V}^{++}, \boldsymbol{Q}^{+}]$, is determined by the following functional integral [16,67]:

$$e^{i\Gamma^{(1)}[\boldsymbol{V}^{++},\boldsymbol{Q}^{+}]} = \mathrm{Det}^{1/2}\,\widehat{\Box} \int \mathcal{D}v^{++}\,\mathcal{D}q^{+}\,\mathcal{D}\mathbf{b}\,\mathcal{D}\mathbf{c}\,\mathcal{D}\varphi\;\; e^{iS_2[v^{++},q^{+},\mathbf{b},\mathbf{c},\varphi,\boldsymbol{V}^{++},\boldsymbol{Q}^{+}]}. \tag{98}$$

In this expression, the full quadratic (with respect to the quantum superfields) action S_2 is the sum of three terms, namely the classical action (13) in which the background-quantum splitting was performed, the gauge-fixing term (66), and the ghost actions (67) and (69). The action S_2 contains the mixed term of quantum vector multiplet and hypermultiplet. After diagonalization, we obtain the following one-loop contribution to the effective action:

$$\begin{aligned}\Gamma[\boldsymbol{V}^{++}, \boldsymbol{Q}^{+}] \;&=\; \frac{i}{2}\mathrm{Tr}\ln\Big\{\,\widehat{\Box}^{AB} - 2f_0^2\,\widetilde{\boldsymbol{Q}}^{+i}(T^A G_q^{(1,1)} T^B)_i{}^j \boldsymbol{Q}^{+}_j\Big\} - \frac{i}{2}\mathrm{Tr}\ln\widehat{\Box} \\ &\quad -i\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathrm{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\mathrm{Tr}\ln\nabla^{++}_{\mathrm{R}},\end{aligned} \tag{99}$$

where $G_q^{(1,1)}(1|2)$ is the background-dependent hypermultiplet Green function (74). Furthermore, we introduce the covariant d'Alembertian $\widehat{\Box} = \frac{1}{2}(D^{+})^4(\nabla^{--})^2$. On the analytic superfields, $\widehat{\Box}$ is reduced to:

$$\widehat{\Box} = \eta^{MN}\boldsymbol{\nabla}_M\boldsymbol{\nabla}_N + \boldsymbol{W}^{+a}\boldsymbol{\nabla}^{-}_a + \boldsymbol{F}^{++}\boldsymbol{\nabla}^{--} - \frac{1}{2}(\boldsymbol{\nabla}^{--}\boldsymbol{F}^{++}), \tag{100}$$

where $\eta^{MN} = \mathrm{diag}(1,-1,-1,-1,-1,-1)$ denotes the six-dimensional Minkowski metric and $\boldsymbol{\nabla}_M = \partial_M + i\boldsymbol{A}_M$ is the vector supercovariant derivative.

The $(\boldsymbol{F}^{++})^2$ part of the effective action depends only on the background gauge superfield $\boldsymbol{V}^{++}$ and is given by the last three terms in Equation (99). More precisely,

$$\begin{aligned}\Gamma^{(1)}_{\boldsymbol{F}^2}[\boldsymbol{V}^{++}] \;&=\; -i\mathrm{Tr}\ln(\boldsymbol{\nabla}^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathrm{Tr}\ln(\boldsymbol{\nabla}^{++})^2_{\mathrm{Adj}} + i\mathrm{Tr}\ln\boldsymbol{\nabla}^{++}_{\mathrm{R}} \\ &=\; -i\mathrm{Tr}\ln\boldsymbol{\nabla}^{++}_{\mathrm{Adj}} + i\mathrm{Tr}\ln\boldsymbol{\nabla}^{++}_{\mathrm{R}}.\end{aligned} \tag{101}$$

Here, the index "R" refers to the representation of the hypermultiplet. Keeping in mind the explicit expressions for the covariant harmonic derivatives, $(\boldsymbol{\nabla}^{++}_{\mathrm{R}})^i_j = D^{++}\delta^i_j + i(\boldsymbol{V}^{++})^C(T^C)_i{}^j$ and $(\boldsymbol{\nabla}^{++}_{\mathrm{Adj}})^{AB} = D^{++}\delta^{AB} - f^{ACB}(\boldsymbol{V}^{++})^C$, we vary the expression (101) with respect to the background gauge superfield $(\boldsymbol{V}^{++})^A$

$$\delta\Gamma^{(1)}_{\boldsymbol{F}^2}[\boldsymbol{V}^{++}] = i\mathrm{Tr}\; f^{ACB}\,\delta(\boldsymbol{V}^{++})^C\,(G^{(1,1)})^{BA} - \mathrm{Tr}\;(T^C)_j{}^i\,\delta(\boldsymbol{V}^{++})^C\,(G_q^{(1,1)})_i{}^j. \tag{102}$$

Here, $(G_q^{(1,1)})_i{}^j$ is the superfield Green function (74) for the operator $(\boldsymbol{\nabla}^{++})_i{}^j$ acting on the superfields in the representation R to which the hypermultiplet belongs. Furthermore, we denoted the Green function for the operator $(\boldsymbol{\nabla}^{++})^{BA}$, which acts on superfields in adjoint representation, by $(G^{(1,1)})^{BA}$. The structure of the function $(G^{(1,1)})^{BA}$ is similar to (74).

The background-dependent Green function $G_q^{(1,1)}(1|2)$ (74) can be written as the following proper-time integral:

$$G_q^{(1,1)}(1|2) = -\int_0^\infty d(is)(is\mu^2)^{\frac{\varepsilon}{2}} e^{is\widehat{\Box}} (D_1^+)^4 (D_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}. \tag{103}$$

Here, s is the proper-time parameter and μ denotes an arbitrary regularization parameter with the dimension of mass. Our purpose is to find the divergent part of the effective action (101). In the proper-time regularization scheme (see, e.g., [10]), the divergences correspond to the pole terms of the form $1/\varepsilon$, $\varepsilon \to 0$, with $D = 6 - \varepsilon$. Then, calculating the divergences according to the standard technique, after some (rather non-trivial) transformations, we obtain:

$$\Gamma^{(1)}_{F^2} = \frac{C_2 - T(R)}{6(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, (F^{++A})^2 = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} \,\mathrm{tr} \int d\zeta^{(-4)} du\, (F^{++})^2, \tag{104}$$

where $F^{++} = F^{++A} t^A$, with t^A being the fundamental representation generators.

The hypermultiplet-dependent part $\widetilde{Q}^+ F^{++} Q^+$ of the one-loop counterterm comes from the first term in (99). In order to find this contribution, firstly, we rewrite it as a sum of two terms,

$$\begin{aligned} \frac{i}{2}\mathrm{Tr}\ln\left\{ \widehat{\Box}^{AB} - 2f_0^2 \widetilde{Q}^{+i} (T^A G_q^{(1,1)} T^B)_i{}^j Q_j^+ \right\} &= \frac{i}{2}\mathrm{Tr}\ln\widehat{\Box} \\ &+ \frac{i}{2}\mathrm{Tr}\ln\left\{ \delta^{AB} - 2f_0^2 (\widehat{\Box}^{-1})^{AC} \widetilde{Q}^{+i} (T^C G_q^{(1,1)} T^B)_i{}^j Q_j^+ \right\}. \end{aligned} \tag{105}$$

Then, following [29], we decompose the second logarithm up to the first order and calculate the functional trace:

$$\begin{aligned} \Gamma^{(1)}_{QFQ} &= -if_0^2 \int d\zeta^{(-4)} du\, \widetilde{Q}^{+j} Q_i^+ \,(\widehat{\Box}^{-1})^{AB} (T^B G^{(1,1)q} T^A)_j{}^i \Big|^{2=1}_{\mathrm{div}} \\ &= -if_0^2 \int d\zeta^{(-4)} du\, \widetilde{Q}^{+i} Q_j^+ \\ &\quad \times (\widehat{\Box}^{-1})^{AB} (T^B \,\widehat{\Box}^{-1}\, T^A)_i{}^j (u_1^+ u_2^+)\, \delta^6(x_1 - x_2)\Big|_{2=1}. \end{aligned} \tag{106}$$

Here, we use the explicit form of the Green function $(G_q^{(1,1)})_i{}^j$ (74) to extract the divergent contribution to the effective action. After this, we decompose the inverse $\widehat{\Box}^{-1}$ of the covariant operator $\widehat{\Box}$ (100) up to the second order and obtain:

$$\Gamma^{(1)}_{QFQ}[V^{++}, Q^+] = -\frac{2if_0^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, \widetilde{Q}^{+i} (C_2 \delta_i^l - C(R)_i{}^l) (F^{++})^A \,(T^A)_l{}^j\, Q_j^+. \tag{107}$$

Summing up the contributions (104) and (107), we obtain the final result for the total divergent contribution:

$$\begin{aligned} \Gamma^{(1)}_{div}[V^{++}, Q^+] &= \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} \,\mathrm{tr} \int d\zeta^{(-4)} du\, (F^{++})^2 \\ &- \frac{2if_0^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du\, \widetilde{Q}^+ (C_2 - C(R)) F^{++} Q^+. \end{aligned} \tag{108}$$

We see that the result (108) derived by the manifestly gauge-invariant method coincides with the previous result (94) based on supergraph calculations.

4.5. Low-Energy Effective Action

The background field method developed in the previous sections is a powerful tool for calculation of the finite contributions to the effective action in a manifestly gauge-invariant way (for the background field method in $4D$ harmonic superspace and its application to the problem of effective action in $\mathcal{N}=2,4$ SYM theories, see papers [68–73] and the references therein). In this section, we evaluate the finite one-loop leading low-energy contribution to the effective action of $6D$, $\mathcal{N}=(1,1)$ SYM theory in the $\mathcal{N}=(1,0)$ harmonic superspace formulation. An important aspect of the consideration is the use of the omega-hypermultiplet.

First, we formulate $6D$, $\mathcal{N}=(1,1)$ SYM theory in terms of $6D$, $\mathcal{N}=(1,0)$ analytic harmonic superfields V^{++} and ω, which are the gauge supermultiplet and the hypermultiplet, respectively. The action of $\mathcal{N}=(1,1)$ SYM theory in this case reads:

$$
\begin{aligned}
S[V^{++},q^{+}] &= \frac{1}{f_0^2}\Big\{\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int d^{14}z\,du_1\ldots du_n\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)} \\
&\quad -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\,\nabla^{++}\omega\nabla^{++}\omega\Big\},
\end{aligned}
\tag{109}
$$

where:

$$
\nabla^{++}\omega = D^{++}\omega + i[V^{++},\omega].
$$

Here, both V^{++} and ω take values in the adjoint representation. The action (109) is invariant under the infinitesimal gauge transformations:

$$
\delta V^{++} = -\nabla^{++}\Lambda, \quad \delta\omega = i[\Lambda,\omega],
\tag{110}
$$

where $\Lambda(\zeta,u)=\widetilde{\Lambda}(\zeta,u)$ is an analytic real gauge parameter.

The action (109) was written in terms of $\mathcal{N}=(1,0)$ harmonic superfields. However, this action possesses an additional hidden $\mathcal{N}=(0,1)$ supersymmetry realized by the transformations:

$$
\delta V^{++} = 2(\epsilon^{+A}u_A^+)\omega - \nabla^{++}\big((\epsilon^{+A}u_A^-)\omega\big),
\tag{111}
$$

$$
\delta\omega = i(\epsilon^{-A}u_A^-)F^{++} - i(\epsilon_a^A u_A^-)\,W^{+a},
\tag{112}
$$

where $A=1,2$ is the Pauli–Gürsey $SU(2)$ index and $W^{+a}=-\frac{i}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+V^{--}$, $D_a^+W^{+a}=4F^{++}$. As a result, this action describes $\mathcal{N}=(1,1)$ SYM theory.

Our further consideration is based on the background field method in six-dimensional $\mathcal{N}=(1,0)$ harmonic superspace, which was developed in the previous subsection. Here, we focus only on aspects related to omega-hypermultiplet. As in the previous sections, we present the original superfields V^{++} and ω as a sum of the background superfields $\boldsymbol{V}^{++}$, $\boldsymbol{\Omega}$, and the quantum superfields v^{++}, ω. In the present case, it is convenient to append the coupling constant f_0 in front of quantum fields:

$$
V^{++}\to \boldsymbol{V}^{++}+f_0v^{++}, \qquad \omega\to\boldsymbol{\Omega}+f_0\omega.
\tag{113}
$$

Then, we expand the action in powers of the quantum fields. The one-loop effective action $\Gamma^{(1)}$ for the model (109) is defined by the quadratic part of quantum action S_2,

$$
\begin{aligned}
S_2 &= S_{\mathrm{gh}}+\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\,v^{++}\,\widehat{\Box}\,v^{++}-\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\,(\nabla^{++}\omega)^2 \\
&\quad -i\mathrm{tr}\int d\zeta^{(-4)}\Big\{\nabla^{++}\omega[v^{++},\boldsymbol{\Omega}]+\nabla^{++}\boldsymbol{\Omega}[v^{++},\omega]+\frac{i}{2}[v^{++},\boldsymbol{\Omega}]^2\Big\}.
\end{aligned}
\tag{114}
$$

The action $S_{\rm gh}$ in (114) is a sum of the action for Faddeev–Popov ghosts b and c (67) and the action for Nielsen–Kallosh ghost φ (69). The covariantly-analytic operator $\widehat{\Box}$ (100) depends on the background gauge superfield.

The action (114) includes the background superfields $\boldsymbol{V}^{++}$ and $\boldsymbol{\Omega}$, which belong to the Lie algebra of the gauge group. Let us suppose that the gauge group of the theory (109) is $SU(N)$. For simplicity, we will also assume that the background superfields $\boldsymbol{V}^{++}$ and $\boldsymbol{\Omega}$ align in a fixed direction in the Cartan subalgebra of $su(N)$:

$$\boldsymbol{V}^{++} = V^{++}(\zeta,u)H, \qquad \boldsymbol{\Omega} = \Omega(\zeta,u)\,H. \tag{115}$$

Here, H is a fixed generator of the Cartan subalgebra corresponding to some Abelian subgroup $U(1)$. For our choice of the background superfields, the symmetry group of classical action $SU(N)$ is broken down to $SU(N-1)\otimes U(1)$. It is worth noting that the pair of the background Abelian superfields (V^{++},Ω) forms the Abelian gauge $\mathcal{N}=(1,1)$ multiplet. In the bosonic sector, it contains the only real gauge vector field $A_M(x)$ and four real scalar fields $\phi(x)$ and $\phi^{(ij)}(x), i,j=1,2$. The fields ϕ and $\phi^{(ij)}$ are scalar components of the hypermultiplet Ω [15]. It is known that the Abelian vector field and four scalars describe the bosonic world-volume degrees of freedom of a single D5-brane in six-dimensional space-time [74,75].

According to the definition (115), the classical motion equations for the background superfields V^{++} and Ω are reduced to the free equations:

$$F^{++}=0, \qquad (D^{++})^2\Omega = 0. \tag{116}$$

In our further consideration, we assume that the background superfields satisfy the classical equation of motion (116) and also are slowly varying in space-time:

$$\partial_M W^{+a} = 0, \qquad \partial_M \Omega = 0. \tag{117}$$

Since we assume that the background vector multiplet solves the free equation of motion, $F^{++}=0$, the gauge superfield strength W^{+a} becomes an analytic superfield on shell. In the general case of unconstrained background, $F^{++}\neq 0$, the superfield W^{+a} is non-analytic.

The transformations of the hidden $\mathcal{N}=(0,1)$ supersymmetry for the gauge superfield strength W^{+a} and Ω (112), in accordance with the conditions (116) and (117), have the simple form:

$$\delta\Omega = -i(\epsilon^A_a u^-_A)\,W^{+a} \qquad \delta W^{+a} = 0. \tag{118}$$

Using (118), one can try to investigate the simplest $\mathcal{N}=(1,1)$ invariants, which can be obtained from the Abelian analytic superfields W^{+a} and Ω under the assumptions (116) and (117). It is easy to check that the following gauge-invariant action,

$$I = f_0^2 \int d\zeta^{(-4)} (W^+)^4 \mathcal{F}(f_0\Omega), \tag{119}$$

is invariant under the transformation (118). Here, we introduced the fourth power of gauge superfield strength $(W^+)^4 = -\frac{1}{24}\varepsilon_{abcd}W^{+a}W^{+b}W^{+c}W^{+d}$. The function $\mathcal{F}(f_0\Omega)$ can in principle be arbitrary. The simplest choice, when the coupling constant f_0 is absent in the invariant, is $\mathcal{F}=\frac{1}{f_0^2\Omega^2}$ in (119), which yields:

$$I_1 = c\int d\zeta^{(-4)}\,\frac{(W^+)^4}{\Omega^2}. \tag{120}$$

The numerical coefficient c in (120) cannot be fixed only by the symmetry considerations and should be obtained using the methods of quantum field theory.

Therefore, our next step is to find the constant c by calculating the leading low-energy contribution to the effective action of the theory (109). To perform the calculation, we choose the Cartan–Weyl basis for the $SU(N)$ generators. In this basis, the quantum superfield v^{++} is decomposed as:

$$v^{++} = v_{\rm i}^{++} H_{\rm i} + v_{\alpha}^{++} E_{\alpha}, \qquad {\rm i} = 1,..,N-1, \quad \alpha = 1,..,N(N-1). \tag{121}$$

For the generators E_α corresponding to the root α, we use the normalization tr $(E_\alpha E_{-\beta}) = \delta_{\alpha\beta}$. The Cartan subalgebra generators $H_{\rm i}$ satisfy the relations $[H_{\rm i}, E_\alpha] = \alpha_{H_i} E_\alpha$. The integration over quantum superfields v^{++} and ω in (98) produces the one-loop effective action for the background superfields V^{++} and Ω,

$$\begin{aligned}\Gamma^{(1)}[V^{++},\Omega] \;\; = \;\; & \frac{i}{2}{\rm Tr}_{\,(2,2)} \ln\left(\widehat{\Box}_H - \alpha_H^2\Omega^2\right) + \frac{i}{2}{\rm Tr}\,\ln\left[(\nabla_H^{++})^2 + A_{(+)}\frac{\alpha_H^2}{\widehat{\Box}_H - \alpha_H^2\Omega^2}A_{(-)}\right] \\ & -\frac{i}{2}{\rm Tr}_{\,(4,0)}\ln\widehat{\Box}_H - i{\rm Tr}\,\ln(\nabla_H^{++})^2 + \frac{i}{2}{\rm Tr}\,\ln(\nabla_H^{++})^2,\end{aligned} \tag{122}$$

where the harmonic covariant derivative $\nabla_H^{++} = D^{++} + \alpha_H V^{++}$ depends on the root α_H and $\widehat{\Box}_H := \Box + \alpha_H W^{+a} D_a^-$. We also introduced the operators $A_{(\pm)}(\Omega) = \Omega\nabla_H^{++} \pm \frac{3}{2}(D^{++}\Omega)$.

The first two terms in the first line of (122) are the contribution from the gauge multiplet and the total contribution from the hypermultiplet, respectively. The factor ${\rm Det}^{1/2}\,\widehat{\Box}$ in (98) produces the first term in the second line of (122). The last two terms in the second line come from the ghosts actions.

We divide the one-loop contribution to the effective action (122) into the two terms:

$$\Gamma^{(1)} = \Gamma^{(1)}_{\rm lead} + \Gamma^{(1)}_{\rm high}. \tag{123}$$

We will see that the first one is responsible for the leading low-energy contribution:

$$\Gamma^{(1)}_{\rm lead} = \frac{i}{2}{\rm Tr}_{\,(2,2)}\ln\left(\widehat{\Box}_H - \alpha_H^2\Omega^2\right) - \frac{i}{2}{\rm Tr}_{\,(4,0)}\ln\left(\widehat{\Box}_H - \alpha_H^2\Omega^2\right). \tag{124}$$

As for the second term $\Gamma^{(1)}_{\rm high}$ in (123), we will show that it corresponds to the next-to-leading approximation. Further, we demonstrate that the $\mathcal{N} = (1,1)$ invariant action (120) can be found as a leading contribution to the one-loop effective action $\Gamma^{(1)}_{\rm lead}$ (124). The action (120) includes only the gauge superfield strength W^{+a}, and superfield Ω and does not contain terms with $D^{++}\Omega$, $D_a^-\Omega$, and $D_a^- W^{+b}$. That is why we will systematically neglect such terms in our calculation. The contribution $\Gamma^{(1)}_{\rm high}$ collects terms with $D^{++}\Omega$ and spinorial derivatives of the background superfields only. Thus, below, the contribution $\Gamma^{(1)}_{\rm high}$ can be ignored.

The scheme of calculation of the contribution (124) is quite similar to the analogous one in the four-dimensional case [76]. First of all, we notice that on shell, the harmonic derivative ∇_H^{++} commutes with the covariant d'Alembertian. However, it is not true for the operator $\widehat{\Box}_H - \alpha_H^2\Omega^2$, since $[\widehat{\Box}_H - \alpha_H^2\Omega^2, \nabla_H^{++}] \sim D^{++}\Omega$. However, all such terms are beyond the scope of our consideration. Thus, in accordance with the method of [76], the well-defined expression for the contribution $\Gamma^{(1)}_{\rm lead}$ to the one-loop effective action reads:

$$\Gamma^{(1)}_{\rm lead} = -\frac{i}{2}{\rm Tr}\int_0^\infty \frac{d(is)}{(is)} e^{is(\widehat{\Box}_H - \alpha_H^2\Omega^2)}\Pi_{\rm T}^{(2,2)}. \tag{125}$$

Here, we have introduced the projection operator on the space of transverse covariantly analytic superfields, $\Pi_{\rm T}^{(2,2)}(\zeta_1, u_1; \zeta_2, u_2)$. One can show [76] that:

$$\Pi_{\mathrm{T}}^{(2,2)}=-\frac{(D_1^+)^4}{\widehat{\square}_1}\Big\{(\nabla_1^-)^4(u_1^+u_2^+)^2-\Delta_1^{--}(u_1^-u_2^+)(u_1^+u_2^+)+\widehat{\square}_1\,(u_1^-u_2^+)^2\Big\}\delta^{14}(z_1-z_2), \tag{126}$$

where we have introduced the notation $\Delta^{--}=i\nabla^{ab}\nabla_a^-\nabla_b^- - W^{-a}\nabla_a^- + \frac{1}{4}(\nabla_a^- W^{-a})$. Then, we substitute (126) in the one-loop contribution $\Gamma^{(1)}_{\mathrm{lead}}$ (125) and take the coincident-harmonic points limit $u_2\to u_1$. It is easy to see that only the third term in (126) survives. As the next steps, we collect the terms quartic in the derivative D_a^- from the exponential in (125) and use the equality $(D^+)^4(D^-)^4\delta^8(\theta_1-\theta_2)\big|_{2=1}=1$. Passing to the momentum representation and calculating the integral over proper-time s, we obtain:

$$\Gamma^{(1)}_{\mathrm{lead}}=\frac{N-1}{(4\pi)^3}\int d\zeta^{(-4)}\,\frac{(W^+)^4}{\Omega^2}. \tag{127}$$

The matrix trace in (127) is calculated as a sum over non-zero roots α_H, with $H=\frac{1}{\sqrt{N(N-1)}}\mathrm{diag}(1,\ldots,1,1-N)$.

As was expected, the $\mathcal{N}=(1,1)$ invariant I_1 (120) comes out as the leading low-energy contribution (127) to the effective action for the theory (109). The coefficient c was calculated, and it is equal to:

$$c=\frac{N-1}{(4\pi)^3}. \tag{128}$$

It is interesting to note that the same expression for the coefficient c was obtained in the $4D$, $\mathcal{N}=4$ SYM theory (see, e.g., [77] and the references therein). The bosonic part of the effective action (127) is:

$$\Gamma^{(1)}_{\mathrm{bos}}\sim\int d^6x\,\frac{F^4}{\phi^2}\left(1+\frac{\phi^{(ij)}\phi_{(ij)}}{\phi^2}+\ldots\right), \tag{129}$$

where $F^4=3F_{MN}F^{MN}F_{PQ}F^{PQ}-4F^{NM}F_{MR}F^{RS}F_{SN}$ and F_{MN} is the Abelian gauge field strength.

5. Conclusions

Harmonic superspace is a very convenient powerful tool for investigating quantum properties of $6D$ $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ theories, because it allows one to keep $\mathcal{N}=(1,0)$ supersymmetry manifest at all steps of calculating quantum corrections. Moreover, this technique considerably simplifies the calculations, because a huge amount of usual Feynman diagrams appear to be included in an essentially smaller number of superdiagrams. Surely, most of the statements and methods related to $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ SYM theories can be reformulated within the harmonic formalism. The results obtained in the harmonic superspace approach in the lowest loops agree with those found with the help of other techniques, say within the component approach. However, the harmonic superspace technique looks certainly more preferable for calculations in the higher loops, where the advantages of the manifestly supersymmetric quantization method are especially essential.

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