symmetry

MDPI

Article

Positive Solutions of One-Dimensional p-Laplacian Problems with Superlinearity

G.C. Yang * and Z.Y. Li

College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, Sichuan, China; lizuyan817@163.com

* Correspondence: gcyang@cuit.edu.cn

Received: 27 July 2018; Accepted: 23 August 2018; Published: 26 August 2018

check for updates

Abstract: We study one-dimensional p-Laplacian problems and answer the unsolved problem. Our method is to study the property of the operator, the concavity of the solutions and the continuity of the first eigenvalues. By the above study, the main difficulty is overcome and the fixed point theorem can be applied for the corresponding compact maps. An affirmative answer is given to the unsolved problem with superlinearity. A global growth condition is not imposed on the nonlinear term f. The assumptions of this paper are more general than the usual, thus the existing results cannot be utilized. Some recent results are improved from weak solutions to classical solutions and from $p \geq 2$ to $p \in (1, \infty)$.

Keywords: one-dimensional p-Laplacian problems; positive solutions; superlinearity; first eigenvalues; fixed point theorem

---

1. Introduction

It is well-known that one-dimensional p-Laplacian problems

$$\begin{cases} -\Delta_p z(x) = f(x, z(x)) \text{ for almost every (a.e.) } x \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{1}$$

are of great importance in the fields of Newtonian fluids ($p = 2$) and non-Newtonian fluids ($p \neq 2$); Dilatant fluids and pseudoplastic fluids may be characterized by $p > 2$ and $1 < p < 2$, respectively (e.g., see [1]), where $\Delta_p z(x) = (\phi_p(z'(x)))'$, $z'(x) = \dfrac{dz}{dx}$ denotes the usual derivative, $\phi_p(s) = |s|^{p-2}s, s \in \mathbb{R}$ and $p \in (1, \infty)$,

The existence of positive solutions of Equation (1) has been widely investigated via various methods and a lot of results have been proved under various assumptions. Let us mention just a few. Using the fixed point index, Wang [2] and Webb and Lan [3] studied Equation (1). In [2], $f(x, u) = g(x) f_0(u)$ and f_0 was assumed to satisfy

$$\lim_{u \to \infty} \frac{f_0(u)}{u^{p-1}} = 0 \quad \text{and} \quad \lim_{u \to 0+} \frac{f_0(u)}{u^{p-1}} = \infty \tag{2}$$

and in [3], $p = 2$ and

$$0 \leq \lim_{u \to \infty} \frac{f(u)}{u} < \pi^2 < \lim_{u \to 0+} \frac{f(u)}{u} \leq \infty$$

was imposed on f. Rynne [4] and Dai and Ma [5] investigated Equation (1) with suitable boundary conditions using bifurcation theory. When $p \geq 2$, the existence of *positive weak solutions* was studied by Ćwiszewski and Maciejewski [6] under the sublinear conditions:

$$0 \leq \lim_{u \to \infty} \frac{f(u)}{u^{p-1}} < \mu_p < \lim_{u \to 0+} \frac{f(u)}{u^{p-1}} \leq \infty, \tag{3}$$

or under the superlinear conditions:

$$0 \leq \lim_{u \to 0+} \frac{f(u)}{u^{p-1}} < \mu_p < \lim_{u \to \infty} \frac{f(u)}{u^{p-1}} \leq \infty, \tag{4}$$

where μ_p is the first eigenvalue of the corresponding homogeneous Dirichlet boundary value problem and $\mu_2 = \pi^2$. Actually, Ćwiszewski et al. [6] covered PDE cases, where f was not required to be nonnegative, but Ćwiszewski et al. [6] only studied weak solutions and requires both a global growth condition on f and $p \geq 2$. Hence, they [6] obtained less restrictive solution under stronger assumptions.

In 2015, Lan et al. [7] proved the existence of positive (classical) solutions for Equation (1)under the general conditions (see (H_1) and (H_2) in Theorem 2.11 [7], which cover Equation (3)) involving the first eigenvalues of the corresponding problems. However, the problem in Equation (1) under the superlinear case is left unsolved [7], that is, whether Equation (1) has positive (classical) solutions under the superlinear conditions in Equation (4).

In [8–10], the existence of solutions for high-dimensional cases was studied, where topological degree theory, bifurcation theory and the variational approach were employed, respectively. One may refer to [5–10] and the references therein for more related study of p-Laplacian problems.

The core of this paper is to give an affirmative answer to the unsolved problem with superlinearity [7]. Our method is to study the property of the operator (see Lemma 4), the concavity of the solutions (see Lemma 5) and the continuity of the first eigenvalues (see Lemma 12). By the study in the above aspects, the difficulty such as of lacking linearity of the operator is overcome, the fixed point theorem can be applied for the corresponding compact maps and new results are obtained. Since we do not assume that f satisfies a global growth condition (see, for example, [6,8,9]) and the assumptions of this paper are more general than the usual that (see, for example, [6]), the existing results cannot be utilized in this paper. In addition, some recent results are improved from weak solutions to classical solutions and from $p \geq 2$ to $p \in (1, \infty)$.

2. Preliminaries

Let $AC[0,1]$ denote the space of all the absolutely continuous functions defined on $[0,1]$. Let function $z : [0,1] \to \mathbb{R}$ with $z(x) > 0$ for $x \in (0,1)$ satisfy $z \in C^1[0,1]$, $\phi_p(z') \in AC[0,1]$. If z satisfies Equation (1) [11], then we call z being a positive (classical) solution of Equation (1).

Let $W_0^{1,p}(0,1)$ denote the standard Sobolev space with norm

$$\|u\|_{W_0^{1,p}} = \left(\int_0^1 |u'(x)|^p \, dx \right)^{1/p} := \|u'\|_{L^p},$$

and P denote the positive cone in $W_0^{1,p}(0,1)$, that is,

$$P = \{u \in W_0^{1,p}(0,1) : u(x) \geq 0 \quad \text{for } x \in [0,1]\}.$$

We recall some facts (see, for example, [7]) and establish several Lemmas. The first fact

$$W_0^{1,p}(0,1) \subseteq C[0,1] \text{ and } \|u\|_{C[0,1]} \leq c_0 \|u\|_{W_0^{1,p}} \text{ for } u \in W_0^{1,p} \tag{5}$$

(Lemma 2.2 in [7]) is used to prove the limit property of the first eigenvalue μ_g (Lemma 8) and the main result, where $c_0 > 0$ is a constant.

The following two Lemmas are the maximum principle and the weak comparison principle.

Lemma 1. *Assume that a function $u \in C[0,1]$ satisfies the following conditions [7]:*

1. *$u'(x)$ exists for $x \in (0,1)$ and $\phi_p(u') \in AC(0,1)$.*
2. *$-\Delta_p u(x) \geq 0$ for a.e. $x \in (0,1)$, and $u(0) = u(1) = 0$.*

Then, $u(x) \geq 0$ for $x \in [0,1]$. If $u \not\equiv 0$ on $(0,1)$, then $u(x) > 0$ for $x \in (0,1)$.

Lemma 2. *Assume that $u, w \in W_0^{1,p}(0,1)$ satisfy [7],*

$$(-\Delta_p u(x), v(x)) \leq (-\Delta_p w(x), v(x)) \quad \textit{for } v \in P,$$

where $(-\Delta_p u(x), v(x)) = \int_0^1 (-\Delta_p u(x)) v(x)\, dx$.

Then, $u(x) \leq w(x)$ a.e. on $(0,1)$.

Let

$$D(\Delta_p) = \{u \in C_0^1[0,1] : \phi_p(u') \in AC[0,1]\}$$

and $C_0^1[0,1] = \{u \in C^1[0,1] : u(0) = u(1) = 0\}$ denote a Banach space with the norm

$$\|u\|_{C^1[0,1]} = \|u\|_{C[0,1]} + \|u'\|_{C[0,1]}.$$

Lemma 3. *For every $v \in L^1(0,1)$, there exists a unique function u in $D(\Delta_p)$ satisfying the quasilinear boundary value problem [7],*

$$\begin{cases} -\Delta_p u(x) = v(x) \ \textit{for a.e. } x \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{6}$$

We denote by T the inverse of $-\Delta_p$. Then, $T : L^1(0,1) \to D(\Delta_p)$ is defined by

$$Tv = u, \tag{7}$$

where u is in Equation (6).

It is easy to verify that T satisfies

$$T(\lambda v) = \lambda^{\frac{1}{p-1}} T(v) \text{ for } v \in L^1(0,1) \text{ and } \lambda \geq 0. \tag{8}$$

Lemma 4. *The map $T : L^1(0,1) \to D(\Delta_p)$ is increasing, that is, $w_1, w_2 \in L^1(0,1)$, $w_1 \leq w_2$ implies $Tw_1 \leq Tw_2$.*

Proof. We may assume that, by Lemma 3, $u_i \in D(\Delta_p)(i = 1,2)$ satisfying $Tw_i = u_i (i = 1,2)$. Then, $-\Delta_p u_i(x) = w_i(x)(i = 1,2)$. $w_1 \leq w_2$ implies

$$(-\Delta_p u_1(x), v(x)) \leq (-\Delta_p u_2(x), v(x)) \quad \text{for } v \in P.$$

By Lemma 2, we have that $u_1 \leq u_2$ and $Tw_1 \leq Tw_2$. □

In [12], the following fact was proved (Proposition 2.1, [12]): Assume that $u : [0,1] \to \mathbb{R}$ is continuous, $u'(x)$ exists for $x \in (0,1)$ and is decreasing on $(0,1)$. Then, u is concave down on $[0,1]$. Utilizing this fact, we prove

Lemma 5. *Let $w \in L^1(0,1)$ with $w(x) \geq 0$ a.e. on $[0,1]$ and $u \in D(\Delta_p)$ such that $-\Delta_p u(x) = w(x)$. Then, u is concave down on $[0,1]$.*

Proof. Let $\eta \in (0,1)$ such that $u'(\eta) = 0$. Then,

$$u'(x) = \begin{cases} \phi_p^{-1}(\int_x^\eta w(s)ds) & \text{if } 0 \leq x \leq \eta, \\ \phi_p^{-1}(-\int_\eta^x w(s)ds) & \text{if } \eta \leq x \leq 1, \end{cases}$$

where $\phi_p^{-1}(s) = \mathrm{sgn}(s)|s|^{\frac{1}{p-1}}$ denotes the inverse function of ϕ_p. Since $\phi_p^{-1}(s)$ is increasing, $u'(x)$ is decreasing on $[0,1]$, thus $u(x)$ is concave down on $[0,1]$. □

We need some assumptions on the nonlinear term f [7].

(C_1) Assume that $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is the Carathéodory function, that is, $f(\cdot, u)$ is measurable for $u \in \mathbb{R}_+$ and $f(x, \cdot)$ is continuous for a.e. $x \in [0,1]$.

(C_2) For each $r > 0$, there exists $g_r \in L^1_+(0,1)$ such that

$$f(x,u) \leq g_r(x) \quad \text{for a.e. } x \in [0,1] \text{ and all } u \in [0,r].$$

Define a map A from P to $D(\Delta_p)$ by

$$Az(x) = (TFz)(x), \tag{9}$$

where $Fz(x) = f(x, z(x))$ is the well-known Nemytskii operator $F : C_+[0,1] \to L^1_+(0,1)$ and T is in Equation (7).

By Theorem 2.8 in [7] and Lemma 1, we have

Lemma 6. *Under the assumption (C_1) and (C_2), the following conclusions hold.*

(i) $A(P) \subseteq P$ and A is compact, where A defined in Equation (9) and $A(P) = \{Ax : x \in P\}$.
(ii) $z \in P \setminus \{0\}$ satisfying $z = Az$ is equivalent to z being a positive solution of Equation (1).

Lemma 7. *For each $g \in L^1_+(0,1)$ with $\int_0^1 g(x)\,dx > 0$, there exist $\mu_g > 0$ and $\varphi_g \in C_0^1[0,1] \cap (P \setminus \{0\})$ satisfying [7],*

$$\begin{cases} -\Delta_p \varphi_g(x) = \mu_g g(x) \varphi_g^{p-1}(x) & \text{for a.e. } x \in (0,1), \\ \varphi_g(0) = 0 = \varphi_g(1). \end{cases} \tag{10}$$

The positive value μ_g is called to be the first eigenvalue of (10), φ_g is called to be the eigenfunction for μ_g. Moreover, we know that, for each $g \in L^1_+(0,1) \setminus \{0\}$,

$$\mu_g = \inf\Big\{ \frac{\int_0^1 |v'(x)|^p\,dx}{\int_0^1 g(x)|v(x)|^p\,dx} : v \in W_0^{1,p}(0,1) \setminus \{0\} \Big\}, \tag{11}$$

where $\dfrac{\int_0^1 |v'(x)|^p\,dx}{\int_0^1 g(x)|v(x)|^p\,dx} = \infty$ if $\int_0^1 g(x)|v(x)|^p\,dx = 0$. For $g \equiv 1$, μ_g is given in ([11] (3.8)) by

$$\mu_1(p) := \Big\{ 2 \int_0^{(p-1)^{\frac{1}{p}}} \big[1 - s^p (p-1)^{-1}\big]^{-\frac{1}{p}}\,ds \Big\}^p. \tag{12}$$

Lemma 8. *Let $n > 1$ be a natural number, $g \in L^1_+(0,1)$ with $\int_0^1 g(x)\,dx > 0$ and*

$$g_n(x) = \begin{cases} g(x) & \text{if } \frac{1}{n} \le x \le 1 - \frac{1}{n}, \\ 0 & \text{if } x \in [0, \frac{1}{n}) \cup (1 - \frac{1}{n}, 1]. \end{cases}$$

Then, $\lim_{n\to\infty} \mu_{g_n} = \mu_g$.

Proof. For $v \in W_0^{1,p}(0,1) \setminus \{0\}$, we have

$$\mu_g \le \int_0^1 |v'(x)|^p\,dx \Big/ \int_0^1 g(x)|v(x)|^p\,dx \le \int_0^1 |v'(x)|^p\,dx \Big/ \int_0^1 g_n(x)|v(x)|^p\,dx$$

and $\mu_g \le \mu_{g_n}$ for any n.

For any $\epsilon > 0$, by Equation (11), there exists $v \in W_0^{1,p}(0,1) \setminus \{0\}$ such that

$$\int_0^1 |v'(x)|^p\,dx \Big/ \int_0^1 g(x)|v(x)|^p\,dx < \mu_g + \frac{\epsilon}{2}.$$

By Equation (5), $v \in C[0,1]$ and $\lim_{n\to\infty} \int_{\frac{1}{n}}^{1-\frac{1}{n}} g(x)|v(x)|^p\,dx = \int_0^1 g(x)|v(x)|^p\,dx$. This, together with $\int_0^1 g(x)\,ds > 0$, shows that there exists $n_0 > 0$ such that

$$\int_0^1 |v'(x)|^p\,dx \Big/ \int_0^1 g_n(x)|v(x)|^p\,dx = \int_0^1 |v'(x)|^p\,dx \Big/ \int_{\frac{1}{n}}^{1-\frac{1}{n}} g(x)|v(x)|^p\,dx < \mu_g + \epsilon$$

for $n \ge n_0$, that is, $\mu_{g_n} < \mu_g + \epsilon$. The result follows. □

Lemma 9. *Let $z_n, e \in P \setminus \{0\}$ with $Te \in P \setminus \{0\}$ and $t_n > 0$ such that $z_n = T(Fz_n + t_n e)$. If $\lim_{n\to\infty} \|z_n\|_{W_0^{1,p}} = \infty$, then $\lim_{n\to\infty} \|z_n\|_{C[0,1]} = \infty$.*

Proof. In fact, if it is false, then we have a constant $r_1 > 0$ and a subset $\{n_i\} \subseteq N$ (N is the natural number set) satisfying $0 \le z_{n_i}(x) \le r_1$ for all i and $x \in [0,1]$. Obviously, we may assume $\{n_i\} = N$.

By Lemma 4, $z_n \ge T(t_n e) = t_n^{\frac{1}{p-1}} Te$, we see that $\{t_n\}$ is bounded. Let $r_2 > 0$ be a constant such that $0 \le t_n \le r_2$ for all n. Let $\xi_n \in (0,1)$ such that $z_n'(\xi_n) = 0$ and

$$z_n'(x) = \begin{cases} \phi_p^{-1}\left(\int_x^{\xi_n} (f(s, z_n(s)) + t_n e(s))ds\right) & \text{if } 0 \le x \le \xi_n, \\ \phi_p^{-1}\left(-\int_{\xi_n}^{x} (f(s, z_n(s)) + t_n e(s))ds\right) & \text{if } \xi_n \le x \le 1. \end{cases}$$

By (C_2), let $g_{r_1} \in L^1_+(0,1)$ satisfy $|f(x,z)| \le g_{r_1}(x)$ for a.e $x \in [0,1]$ and all $0 \le z \le r_1$. From $f(x, z_n(x)) + t_n e(x) \le g_{r_1}(x) + r_2 e(x)$, we have that $\{z_n'(x)\}$ is bounded, which contradicts $\lim_{n\to\infty} \|z_n\|_{W_0^{1,p}} = \infty$. Hence $\lim_{n\to\infty} \|z_n\|_{C[0,1]} = \infty$. □

Let $r > 0$ and let $P_r = \{z \in P : \|z\| < r\}$, $\partial P_r = \{z \in P : \|z\| = r\}$ and $\overline{P}_r = \{z \in P : \|z\| \le r\}$.

Lemma 10. *(i) If $A : \overline{P}_r \to P$ is compact and satisfies $z \ne tAz$ for $z \in \partial P_r$ and $t \in (0,1]$, then $i_P(A, P_r) = 1$ [7,13].*

(ii) If $A : \overline{P}_r \to P$ is compact and $z \ne Az$ for $z \in \overline{P}_r$, then $i_P(A, P_r) = 0$.

(iii) Assume that $h : [0,1] \times \overline{P}_r \to P$ is compact and satisfies $z \ne h(t,z)$ for $(t,z) \in [0,1] \times \partial P_r$. Then $i_P(h(0,\cdot), P_r) = i_P(h(1,\cdot), P_r)$.

(iv) If $i_P(A, P_r) = 1$ and $i_P(A, P_\rho) = 0$ for some $\rho \in (r, \infty)$, then A has a fixed point in $P_\rho \setminus \overline{P}_r$.

3. Main Result and Proof

Now, we state and prove our main result.

Theorem 1. *Assume that* (C_1), (C_2) *and the following conditions hold.*

- (H_1) *There exist* $r_0 > 0$, $\epsilon_1 \in (0, \mu_{\kappa_{r_0}})$ *and* $\kappa_{r_0} \in L^1_+(0,1) \setminus \{0\}$ *satisfying*

$$f(x,u) \le (\mu_{\kappa_{r_0}} - \epsilon_1)\kappa_{r_0}(x)u^{p-1} \quad \text{for a.e. } x \in [0,1] \text{ and each } u \in [0, r_0].$$

- (H_2) *There exist* $\rho_0 > 0$, $\epsilon_2 > 0$ *and* $\psi_{\rho_0} \in L^1_+(0,1) \setminus \{0\}$ *satisfying*

$$f(x,u) \ge (\mu_{\psi_{\rho_0}} + \epsilon_2)\psi_{\rho_0}(x)u^{p-1} \quad \text{for a.e. } x \in [0,1] \text{ and each } u \in [\rho_0, \infty).$$

Then, Equation (1) *has a positive solution* z *in* $C^1_0[0,1]$.

Proof. Let $r = c_0^{-1} r_0$. We prove that

$$z \ne tAz \quad \text{for } z \in \partial P_r \text{ and } t \in [0,1]. \tag{13}$$

In fact, if it is false, let $z \in \partial P_r$ and $t \in (0,1]$ satisfy $z = tAz$. By Equation (8), $z(x) = T(t^{p-1}Fz)(x)$ for $x \in [0,1]$. It follows from Equation (7) that

$$-\Delta_p z(x) = t^{p-1} f(x, z(x)) \quad \text{for a.e. } x \in [0,1]. \tag{14}$$

By Equation (5), $\|z\|_{C[0,1]} \le c_0\|z\|_{W_0^{1,p}} = c_0 r = r_0$ and by (H_1) $f(x,z(x)) \le (\mu_{\kappa_{r_0}} - \epsilon_1)\kappa_{r_0}(x)z^{p-1}(x)$ for a.e. $x \in [0,1]$. By Equations (14) and (11) with $g = \kappa_{r_0}$, we have

$$\begin{aligned} \|z\|^p_{W_0^{1,p}} &= (-\Delta_p z, z) = t^{p-1}\int_0^1 f(x,z(x))z(x)\,dx \le \int_0^1 f(x,z(x))z(x)\,dx \\ &\le \int_0^1 [(\mu_{\kappa_{r_0}} - \epsilon_1)\kappa_{r_0}(x)z^{p-1}(x)]z(x)\,dx \le (\mu_{\kappa_{r_0}} - \epsilon_1)\mu^{-1}_{\kappa_{r_0}}\|z\|^p_{W_0^{1,p}} < \|z\|^p_{W_0^{1,p}}. \end{aligned}$$

It is a contradiction. By Lemma 10 (i), we have $i_P(A, P_r) = 1$.

If there is $z \in \partial P_\rho$ satisfying $z = T(Fz)$, then the result of Theorem 1 holds. Let $g = \psi_{\rho_0}$. By Lemma 8, there is $n_0 > 0$ satisfying $0 < \mu_{g_{n_0}} < \mu_{\psi_{\rho_0}} + \epsilon_2$. Let e denote the eigenfunction corresponding to the eigenvalue $\mu_{g_{n_0}}$, that is,

$$\begin{cases} -\Delta_p e(x) = \mu_{g_{n_0}} g_{n_0}(x) e^{p-1}(x) & \text{for a.e. } x \in (0,1), \\ e(0) = 0 = e(1). \end{cases} \tag{15}$$

We assume $z \ne T(Fz)$ for $z \in \partial P_\rho$ and prove that there exists $\rho > r$ such that

$$z \ne T(Fz + \nu(-\Delta_p e)) \quad \text{for } z \in \partial P_\rho \text{ and } \nu > 0. \tag{16}$$

In fact, if it is false, there are $z_n \in \partial P_{\rho_n}$ with $\rho_n \to \infty$ and $\nu_n > 0$ such that $z_n \ne T(Fz_n)$ and

$$z_n = T(Fz_n + \nu_n(-\Delta_p e)). \tag{17}$$

By Lemma 4, we see $z_n \ge T(\nu_n(-\Delta_p e)) = \nu_n^{\frac{1}{p-1}} e$. Let

$$\tau_n = \sup\{\zeta > 0 : z_n(x) \ge \zeta^{\frac{1}{p-1}} e(x) \quad \text{for } x \in (0,1)\}. \tag{18}$$

Then, $0 < \nu_n \leq \tau_n < \infty$ and

$$z_n(x) \geq \tau_n^{\frac{1}{p-1}} e(x) \quad \text{for } x \in (0,1). \tag{19}$$

By $T(-\Delta_p e) = e \in P \setminus \{0\}$, Equation (17) and Lemma 9, we see $\|z_n\|_{C[0,1]} \to \infty$. Hence, there exists $n = \tilde{n}$ satisfying $\rho_{\tilde{n}} > r$ and $\|z_{\tilde{n}}\|_{C[0,1]} \geq n_0\rho_0$.

Since $-\Delta_p z_{\tilde{n}}(x) = f(x, z_{\tilde{n}}(x)) + v_{\tilde{n}} e(x) \geq 0$ for $x \in [0,1]$, Lemma 5 shows that $z_{\tilde{n}}(x)$ is concave down on $[0,1]$. Let $\xi \in (0,1)$ such that $z_{\tilde{n}}(\xi) = \|z_{\tilde{n}}\|_{C[0,1]} \geq n_0\rho_0$. Then

$$z_{\tilde{n}}(x) \geq \begin{cases} \dfrac{n_0\rho_0}{\xi} x & \text{if } x \in [0,\xi], \\ \dfrac{n_0\rho_0}{1-\xi}(1-x) & \text{if } x \in [\xi, 1]. \end{cases}$$

It is easy to verify $z_{\tilde{n}}(x) \geq \rho_0$ for $x \in [\frac{1}{n_0}, 1 - \frac{1}{n_0}]$.

By (H_2) and Equation (19), we have

$$f(x, z_{\tilde{n}}(x)) \geq (\mu_{\psi_{\rho_0}} + \epsilon_2) g_{n_0}(x) z_{\tilde{n}}^{p-1}(x), x \in [0,1]$$

and

$$f(x, z_{\tilde{n}}(x)) \geq (\mu_{\psi_{\rho_0}} + \epsilon_2) \tau_{\tilde{n}} g_{n_0}(x) e^{p-1}(x), x \in [0,1]. \tag{20}$$

From Equation (20) and Lemma 4 we have

$$z_{\tilde{n}}(x) \geq TF(z_{\tilde{n}}(x)) \geq T((\mu_{\psi_{\rho_0}} + \epsilon_2)\tau_{\tilde{n}} g_{n_0}(x) e^{p-1}(x)) = (\mu_{\psi_{\rho_0}} + \epsilon_2)^{\frac{1}{p-1}} \tau_{\tilde{n}}^{\frac{1}{p-1}} T(g_{n_0}(x) e^{p-1}(x))$$

for $x \in [0,1]$. By Equation (15), we see $\mu_{g_{n_0}}^{-\frac{1}{p-1}} e(x) = T(g_{n_0}(x) e^{p-1}(x))$ for $x \in [0,1]$ and

$$z_{\tilde{n}}(x) \geq TF(z_{\tilde{n}}(x)) \geq (\mu_{\psi_{\rho_0}} + \epsilon_2)^{\frac{1}{p-1}} \mu_{g_{n_0}}^{-\frac{1}{p-1}} \tau_{\tilde{n}}^{\frac{1}{p-1}} e(x), x \in [0,1].$$

This implies $z_{\tilde{n}}(x) \geq \xi_{\tilde{n}}^{\frac{1}{p-1}} e(x)$, where $\xi_{\tilde{n}} = (\mu_{\psi_{\rho_0}} + \epsilon_2)\mu_{g_{n_0}}^{-1} \tau_{\tilde{n}} > \tau_{\tilde{n}}$, which contradicts the definition of $\tau_{\tilde{n}}$ in Equation (18). Hence, there exists $\rho (= \rho_{\tilde{n}}) > r$ such that Equation (16) holds.

Let $\sigma > (\frac{c_0\rho}{\|e\|_{C[0,1]}})^{p-1}$. Then, $z \neq T(Fz + \sigma(-\Delta_p e))$ for $z \in P_\rho$. In fact, if there exists $z \in P_\rho$ such that $z = T(Fz + \sigma(-\Delta_p e))$, then $z \geq T(\sigma(-\Delta_p e)) = \sigma^{\frac{1}{p-1}} e$. By Equation (5), we see $\|z\|_{C[0,1]} \leq c_0\|z\|_{W_0^{1,p}} \leq c_0\rho$ and $\sigma \leq (\frac{c_0\rho}{\|e\|_{C[0,1]}})^{p-1}$. It is a contradiction. Hence, $i_P(T(Fz + \sigma(-\Delta_p e)), P_\rho) = 0$ by Lemma 10 (ii).

We define a map $h : [0,1] \times \overline{P}_\rho \to P$ by

$$h(t,z) = T(Fz + \sigma t(-\Delta_p e)).$$

Then, $h : [0,1] \times \overline{P}_\rho \to P$ is compact and by Equation (16), $z \neq h(t,z)$ for $(t,z) \in [0,1] \times \partial P_\rho$. By Lemma 10 (iii), we obtain

$$i_P(A, P_\rho) = i_P(h(0,\cdot), P_\rho) = i_P(h(1,\cdot), P_\rho) = i_P(T(Fz + \sigma(-\Delta_p e)), P_\rho) = 0.$$

By Lemma 10 (iv), there exists $z \in P_\rho \setminus \overline{P}_r$ satisfying $z = Az$ and thus, by Lemma 6 (ii), z is a positive solution of Equation (1). $\square$

4. Conclusions

First, we give an affirmative answer to the unsolved problem [7].

Let $E \subseteq [0,1]$ with $m(E) = 0$ and

$$\overline{f}(u) = \sup_{x\in[0,1]\setminus E} f(x,u), \quad \underline{f}(u) = \inf_{x\in[0,1]\setminus E} f(x,u).$$

Notation 1.

$$f^0 = \limsup_{u\to 0+} \overline{f}(u)/u^{p-1}, \quad f_\infty = \liminf_{u\to\infty} \underline{f}(u)/u^{p-1}.$$

Corollary 1. *Assume that* (C_1), (C_2) *and the following condition(superlinear conditions) hold.*

$$0 \le f^0 < \mu_1(p) < f_\infty \le \infty, \tag{21}$$

where $\mu_1(p)$ *is given by Equation* (12).

Then, Equation (1) *possesses a positive solution z in* $C_0^1[0,1]$.

Proof. By Equation (21), (H_1) with $\kappa_{r_0} \equiv 1$ and (H_2) with $\psi_{\rho_0} \equiv 1$ hold for some $\epsilon_i > 0 (i = 1,2)$ and ρ_0, r_0 with $0 < r_0 < \rho_0 < \infty$. By Theorem 1, we know that the result holds. □

By Corollary 1, an affirmative answer is given to the unsolved problem [7]:

Corollary 2. *Assume that* (C_1), (C_2) *hold and*

$$0 \le \lim_{u\to 0+} \frac{f(u)}{u^{p-1}} < \mu_1(p) < \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} \le \infty. \tag{22}$$

Then, Equation (1) *possesses a positive solution z in* $C_0^1[0,1]$.

Next, some results are improved and the existing results cannot be used in this paper.

In [6], Ćwiszewski and Maciejewski studied *positive weak solutions* under the superlinear conditions in Equation (4) or (21), where a global growth condition on f and $p \ge 2$ were required. Corollary 2 improves Ćwiszewski and Maciejewski's results (Theorem 1.1 with $n = 1$, [6]) from $p \ge 2$ to $p \in (1,\infty)$ and from weak solutions to classical solutions under the superlinear conditions.

The following example shows that the assumptions (H_1) and (H_2) of this paper are more general than the usual superlinear conditions in Equation (21).

Example 1. *Let* $f(x,u) = cx^{-\frac{1}{2}}u^{\sigma-1}$, $\sigma > p$ *and* $c > 0$ *be a constant. Then, f satisfies* (C_1)*-*(C_2)*. Let* $\kappa(x) = x^{-\frac{1}{2}}$ *and* $c > \mu_\kappa$*. Choosing* $\varepsilon_1 = \frac{\mu_\kappa}{2}$, $\varepsilon_2 = \frac{c-\mu_\kappa}{2} > 0$*, then*

$$f(x,u) \le (\mu_{\kappa_{r_0}} - \epsilon_1)\kappa_{r_0}(x)u^{p-1} \text{ for } x \in [0,1], u \in [0, (\frac{\mu_\kappa}{2c})^{\frac{1}{\sigma-p}}] = [0, r_0],$$

$$f(x,u) \ge (\mu_{\psi_{\rho_0}} + \epsilon_2)\psi_{\rho_0}(x)u^{p-1} \text{ for } x \in [0,1], u \in [1,\infty) = [\rho_0,\infty),$$

where $\kappa_{r_0}(x) = \psi_{\rho_0}(x) = x^{-\frac{1}{2}}$*. This shows that* (H_1) *and* (H_2) *in Theorem* 1 *hold and Equation* (1) *possesses one positive solution for any* $0 < \mu_\kappa < c$*.*

However,

$$\overline{f}(u) = \sup_{x\in[0,1]\setminus E} f(x,u) = \infty \text{ for } u > 0 \text{ and } f^0 = \infty,$$

$$\underline{f}(u) = \inf_{x\in[0,1]\setminus E} f(x,u) = cu^{\sigma-1} \text{ for } u > 0 \text{ and } f_\infty = \infty,$$

the usual superlinear conditions(see, for example, [6]) $f^0 < \mu_1(p) < f_\infty$ are not true. The key inequality [5]

$$0 \leq \lim_{|s|\to 0} \frac{f(t,s)}{s} := c(t) \leq \lambda_k(p)$$

does not hold and the global growth condition (see, see for example, [6,8,9])

$$0 \leq |f(x,s)| \leq C_0(1+s^{q-1}) \text{ for all } x \in \Omega \text{ and } s \in [0,\infty)$$

is not imposed on f. Hence the existing results such as [5,6,8–10] can not be used to treat this case.

Finally, in the study of boundary value problems, the linearity of the corresponding operators was applied in an essential way in [3,12]. However, when $p \neq 2$, the corresponding operators of Equation (1) is nonlinear, which is the main difficulty we encounter in this paper. We expect the results obtained in this paper to be applied to other areas and, under (H_1) and (H_2) ($p = 2$, see [12]), Equation (1) to be studied further for the case of f taking negative values.

Author Contributions: All authors contributed to the main results. G.C.Y. drafted the manuscript. Z.Y.L. improved the final version. All authors approved the final manuscript.

Funding: This work was supported in part by the National Natural Science Foundation of China (No. 11171046) and Applied Basic Research Project of Sichuan Province (No. 2018JY0169).

Acknowledgments: We would like to thank the anonymous referees and editors for their careful reading on the manuscript and providing constructive comments.

Conflicts of Interest: The authors declare they have no conflict of interests.

References

1. Lindqvist, P. On nonlinear Rayleigh quotients. *Potential Anal.* **1993**, *2*, 199–218. [CrossRef]
2. Wang, J. The existence of positive solutions for the one-dimensional p-Laplacian. *Proc. Am. Math. Soc.* **1997**, *125*, 2275–2283. [CrossRef]
3. Webb, J.R.L.; Lan, K.Q. Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type. *Topol. Methods Nonlinear Anal.* **2006**, *27*, 91–116.
4. Rynne, B.P. Eigenvalue criteria for existence of positive solutions of second-order, multi-point, p-Laplacian boundary value problems. *Topol. Methods Nonlinear Anal.* **2010**, *36*, 311–326.
5. Dai, G.; Ma, R. Unilateral global bifurcation phenomena and nodal solutions for p-Laplacian. *J. Differ. Equ.* **2012**, *252*, 2448–2468. [CrossRef]
6. Ćwiszewski, A.; Maciejewski, M. Positive stationary solutions for p-Laplacian problems with nonpositive perturbation. *J. Differ. Equ.* **2013**, *254*, 1120–1136. [CrossRef]
7. Lan, K.Q.; Yang, X.J.; Yang, G.C. Positive solutions of one-dimensional p-Laplacian equations and applications to population models of one species. *Topol. Methods Nonlinear Anal.* **2015**, *46*, 431–445.
8. Guo, Y.; Nie, J. Existence and multiplicity of nontrivial solutions for p-Laplacian Schrödinger-Kirchhoff-type equations. *J. Math. Anal. Appl.* **2015**, *428*, 1054–1069. [CrossRef]
9. An, Y.; Kim, C.G.; Shi, J. Exact multiplicity of positive solutions for a p-Laplacian equation with positive convex nonlinearity. *J. Differ. Equ.* **2016**, *260*, 2091–2118. [CrossRef]
10. Figueiredo, D.G.D.; Gossez, J.P.; Ubilla, P. Nonhomogeneous Dirichlet problems for the p-Laplacian. *Calc. Var. Part. Differ. Equ.* **2017**, *56*, 32. [CrossRef]
11. Del Pino, M.; Elgueta, M.; Manásevich, R. A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'(x)|^{p-2}u'(x))' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. *J. Differ. Equ.* **1989**, *2*, 1–13. [CrossRef]

12. Yang, G.C.; Feng, H.B. New results of positive solutions for the Sturm-Liouville problem. *Bound. Value Probl.* **2016**, *2016*, 64. [CrossRef]
13. Amann, H. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **1976**, *18*, 620–709. [CrossRef]