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On Degree-Based Topological Indices of Symmetric Chemical Structures

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Abstract: A Topological index also known as connectivity index is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound. Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. In QSAR/QSPR study, physico-chemical properties and topological indices such as Randić, atom-bond connectivity (ABC) and geometric-arithmetic (GA) index are used to predict the bioactivity of chemical compounds. Graph theory has found a considerable use in this area of research. In this paper, we study HDCN1(m,n) and HDCN2(m,n) of dimension m, n and derive analytical closed results of general Randić index $R_\alpha(\mathcal{G})$ for different values of α. We also compute the general first Zagreb, ABC, GA, ABC_4 and GA_5 indices for these Hex derived cage networks for the first time and give closed formulas of these degree-based indices.

Keywords: general randić index; atom-bond connectivity (ABC) index; geometric-arithmetic (GA) index; Hex-Derived Cage networks; $HDCN1(m,n)$, $HDCN2(m,n)$

1. Introduction

A graph is formed by vertices and edges connecting the vertices. A network is a connected simple graph having no multiple edges and loops. A topological index is a function Top : $\sum \rightarrow \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers and $\sum$ is the finite simple graph with property that $\text{Top}(G_1) = Top(G_2)$ if G_1 and G_2 are isomorphic. A topological index is a numerical value associated with chemical constitution for correlation of chemical structure with various physical properties, chemical reactivity or biological activity. Many tools, such as topological indices has provided by graph theory to the chemists. *Cheminformatics* is new subject which is a combination of chemistry, mathematics and information science. It studies Quantitative structure-activity (QSAR) and structure-property (QSPR) relationships that are used to predict the biological activities and properties of chemical compounds. In the QSAR /QSPR study, physico-chemical properties and topological indices such as Wiener index, Szeged index, Randić index, Zagreb indices and ABC index are used to predict bioactivity of the chemical compounds. "In terms of graph theory, the structural formula of a chemical compound represents the molecular graph, in which vertices are represents to atoms and edges as chemical bonds". A molecular descriptor is a numeric number, which represents the properties of a chemical graph. Basically, a molecular descriptor and topological descriptor are different from each other. A molecular descriptor represents the underlying chemical graph but a topological descriptor are the representation of physico-chemical properties of underlying chemical graph in addition to show the whole structure. Topological indices have many applications in the field of nanobiotechnology and QSAR/QSPR study. Topological indices were firstly introduced by Wiener [1], he named the resulting index as path number while he was working on boiling point of Paraffin. Later on, it renamed as Wiener index [2]. Consider "n" Hex-Derived networks $(HDN1(1,1))$,$(HDN1(2,2))$ and so on to $(HDN1(m,n))$. Connect every boundary vertices of $(HDN1(1,1))$ to its mirror image vertices in $(HDN1(2,2))$ by an edge and so on to $(HDN1(m,n))$. As a result, we found a graph, which is called Hex-Derived Cage networks with "n" layers. In this article, the notations which we used take from the books [3,4].

In this article, Graph $(\mathcal{G})$ is considered to be a graph with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$, the $d(a)$ is the degree of vertex $a \in V(\mathcal{G})$ and $S(a) = \sum_{b \in N_{\mathcal{G}}(a)} d(b)$ where $N_{\mathcal{G}}(a) = \{b \in V(\mathcal{G}) \mid ab \in E(\mathcal{G})\}$.

Let $\mathcal{G}$ be a graph. Then the Wiener index is written as

$$W(\mathcal{G}) = \frac{1}{2} \sum_{(a,b)} d(a,b) \tag{1}$$

The Randić index [5] is the oldest degree-based topological index invented by Milan Randić, denoted as $R_{-\frac{1}{2}}(\mathcal{G})$ and defined as

$$R_{-\frac{1}{2}}(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} \frac{1}{\sqrt{d(a)d(b)}} \tag{2}$$

$R_{\alpha}(\mathcal{G})$ is a general Randić index and it is defined as

$$R_{\alpha}(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} (d(a)d(b))^{\alpha} \quad \text{for} \quad \alpha \in \mathbb{R} \tag{3}$$

A topological index which has a great importance was introduced by Ivan Gutman and *Trinajstić* is Zagreb index and defined as

$$M_1(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} (d(a) + d(b)) \tag{4}$$

Estrada et al. in [6] invented a very famous degree-based topological index ABC and defined as

$$ABC(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} \sqrt{\frac{d(a) + d(b) - 2}{d(a)d(b)}} \tag{5}$$

GA index is also a very famous connectivity topological descriptor, which invented by Vukičević et al. [7] and denoted as

$$GA(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} \frac{2\sqrt{d(a)d(b)}}{(d(a) + d(b))} \tag{6}$$

ABC_4 and GA_5 indices find only if we find the edge partition of interconnection networks each edge in the graphs depend on sum of the degrees of end vertices. ABC_4 index invented by Ghorbani et al. [8] and written as

$$ABC_4(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} \sqrt{\frac{S(a) + S(b) - 2}{S(a)S(b)}} \tag{7}$$

The latest version of index is GA_5 invented by Graovac et al. [9] and defined as

$$GA_5(\mathcal{G}) = \sum_{ab \in E(\mathcal{G})} \frac{2\sqrt{S(a)S(b)}}{(S(a) + S(b))} \tag{8}$$

For any graph G for $\alpha = 1$, the general Randić index is second Zagreb index.

2. Main Results

Hex-Derived Cage networks $HDCN1(m,n)$ (show in Figure 1) and $HDCN2(m,n)$ (show in Figure 2) give closed formulas of that indices, we study the general Randić, first Zagreb, ABC, GA, ABC_4 and GA_5 indices of certain graphs in [10]. These days there is a broad research activity on ABC and GA indices and their variants, for additionally investigation of topological indices of different families see, [1,11–23].

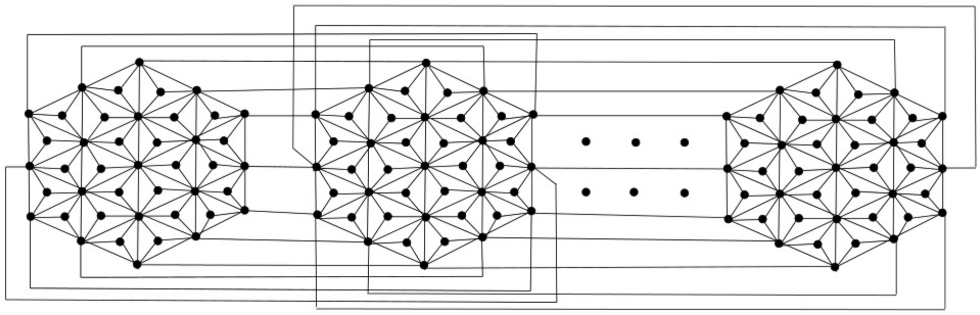

Figure 1. Hex-Derived Network ($HDCN1(3,n)$).

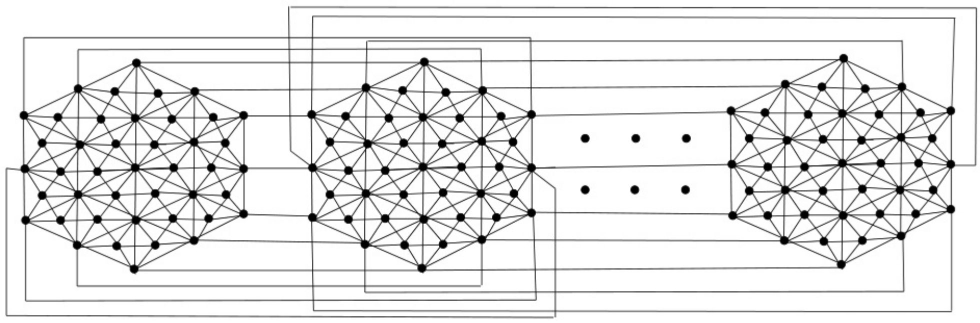

Figure 2. Hex-Derived Network ($HDCN2(3,n)$).

2.1. Results for Hex-Derived Cage Networks

We compute specific degree-based topological indices of Hex-Derived Cage networks. In this paper, we calculate Randić index $R_\alpha(\mathcal{G})$ with $\alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}$, M_1, ABC, GA, ABC_4 and GA_5 for Hex-Derived Cage networks $HDCN1(m,n)$ and $HDCN2(m,n)$.

Theorem 1. *Let $\mathcal{G}_1 \cong HDCN1(m,n)$ be the Hex-Derived Cage network, then its general Randić index is equal to*

$$R_\alpha(\mathcal{G}_1) = \begin{cases} 18(108n^3 - 219n^2 + 25n + 91), & \alpha = 1; \\ 6(36n^3 + 3(7\sqrt{3} - 34)n^2 + (4\sqrt{21} + 6\sqrt{7} + & \\ 28\sqrt{6} - 84\sqrt{3} + 12\sqrt{2} + 35)n + 2\sqrt{42} - & \\ 8\sqrt{21} - 12\sqrt{7} - 56\sqrt{6} + 100\sqrt{3} + 39), & \alpha = \frac{1}{2}; \\ \frac{11907n^3 - 17003n^2 + 12343n - 3051}{21168}, & \alpha = -1; \\ \frac{15n^3}{4} + (\frac{8}{\sqrt{3}} - \frac{125}{12})n^2 + (\frac{4}{\sqrt{7}} + & \\ 4\sqrt{6} - \frac{32}{\sqrt{3}} + \sqrt{2} + 5\sqrt{\frac{3}{7}} + \frac{109}{21})n - & \\ \frac{8}{\sqrt{7}} - 8\sqrt{6} + \frac{38}{\sqrt{3}} + 3\sqrt{2} + 2\sqrt{\frac{6}{7}} - & \\ 10\sqrt{\frac{3}{7}} + \frac{13}{14}, & \alpha = -\frac{1}{2}. \end{cases}$$

Proof. Let $\mathcal{G}_1$ be the Hex-Derived Cage network $(HDCN1(m,n))$ where $m = n \geq 5$. The edge set of $HDCN1(m,n)$ are divided into seventeen partitions based on the degree of end vertices shows in Table 1. Thus from Equation (3), is follows that

$$R_\alpha(\mathcal{G}_1) = \sum_{ab \in E(\mathcal{G})} (d(a)d(b))^\alpha$$

For $\alpha = 1$

$$R_1(\mathcal{G}_1) = \sum_{j=1}^{17} \sum_{ab \in E_j(\mathcal{G})} deg(u) \cdot deg(v)$$

By using the edge partition given in Table 1, we have
$R_1(\mathcal{G}_1)$=$18|E_1(\mathcal{G}_1)| + 21|E_2(\mathcal{G}_1)| + 24|E_3(\mathcal{G}_1)| + 27|E_4(\mathcal{G}_1)| + 36|E_5(\mathcal{G}_1)| + 42|E_6(\mathcal{G}_1)| + 48|E_7(\mathcal{G}_1)| + 72|E_8(\mathcal{G}_1)| + 49|E_9(\mathcal{G}_1)| + 63|E_{10}(\mathcal{G}_1)| + 84|E_{11}(\mathcal{G}_1)| + 64|E_{12}(\mathcal{G}_1)| + 72|E_{13}(\mathcal{G}_1)| + 96|E_{14}(\mathcal{G}_1)| + 81|E_{15}(\mathcal{G}_1)| + 108|E_{16}(\mathcal{G}_1)| + 144|E_{17}(\mathcal{G}_1)|$

After simplification, we have

$$R_1(\mathcal{G}_1) = 18(108n^3 - 219n^2 + 25n + 91)$$

For $\alpha = \frac{1}{2}$

$$R_{\frac{1}{2}}(\mathcal{G}_1) = \sum_{j=1}^{17} \sum_{ab \in E_j(\mathcal{G})} \sqrt{d(a) \cdot d(b)}$$

Using the edge partition from Table 1, we have
$R_{\frac{1}{2}}(\mathcal{G}_1)$=$3\sqrt{2}|E_1(\mathcal{G}_1)| + \sqrt{21}|E_2(\mathcal{G}_1)| + 2\sqrt{6}|E_3(\mathcal{G}_1)| + 3\sqrt{3}|E_4(\mathcal{G}_1)| + 6|E_5(\mathcal{G}_1)| + \sqrt{42}|E_6(\mathcal{G}_1)| + 4\sqrt{3}|E_7(\mathcal{G}_1)| + 6\sqrt{2}|E_8(\mathcal{G}_1)| + 7|E_9(\mathcal{G}_1)| + 3\sqrt{7}|E_{10}(\mathcal{G}_1)| + 2\sqrt{21}|E_{11}(\mathcal{G}_1)| + 8|E_{12}(\mathcal{G}_1)| + 6\sqrt{2}|E_{13}(\mathcal{G}_1)| + 4\sqrt{6}|E_{14}(\mathcal{G}_1)| + 9|E_{15}(\mathcal{G}_1)| + 6\sqrt{3}|E_{16}(\mathcal{G}_1)| + 12|E_{17}(\mathcal{G}_1)|$

After simplification, we have
$R_{\frac{1}{2}}(\mathcal{G}_1)$=$6(36n^3 + 3(7\sqrt{3} - 34)n^2 + (4\sqrt{21} + 6\sqrt{7} + 28\sqrt{6} - 84\sqrt{3} + 12\sqrt{2} + 35)n + 2\sqrt{42} - 8\sqrt{21} - 12\sqrt{7} - 56\sqrt{6} + 100\sqrt{3} + 39)$

For $\alpha = -1$

$$R_{-1}(\mathcal{G}_1) = \sum_{j=1}^{17} \sum_{ab \in E_j(\mathcal{G})} \frac{1}{d(a) \cdot d(b)}$$

$R_{-1}(\mathcal{G}_1)$=$\frac{1}{18}|E_1(\mathcal{G}_1)| + \frac{1}{21}|E_2(\mathcal{G}_1)| + \frac{1}{24}|E_3(\mathcal{G}_1)| + \frac{1}{27}|E_4(\mathcal{G}_1)| + \frac{1}{36}|E_5(\mathcal{G}_1)| + \frac{1}{42}|E_6(\mathcal{G}_1)| + \frac{1}{48}|E_7(\mathcal{G}_1)| + \frac{1}{72}|E_8(\mathcal{G}_1)| + \frac{1}{49}|E_9(\mathcal{G}_1)| + \frac{1}{63}|E_{10}(\mathcal{G}_1)| + \frac{1}{84}|E_{11}(\mathcal{G}_1)| + \frac{1}{64}|E_{12}(\mathcal{G}_1)| + \frac{1}{72}|E_{13}(\mathcal{G}_1)| + \frac{1}{96}|E_{14}(\mathcal{G}_1)| + \frac{1}{81}|E_{15}(\mathcal{G}_1)| + \frac{1}{108}|E_{16}(\mathcal{G}_1)| + \frac{1}{144}|E_{17}(\mathcal{G}_1)|$

After simplification, we have

$$R_{-1}(\mathcal{G}_1) = \frac{11907n^3 - 17003n^2 + 12343n - 3051}{21168}$$

For $\alpha = -\frac{1}{2}$

$$R_{-\frac{1}{2}}(\mathcal{G}_1) = \sum_{j=1}^{17} \sum_{ab \in E_j(\mathcal{G})} \frac{1}{\sqrt{d(a) \cdot d(b)}}$$

$R_{-\frac{1}{2}}(\mathcal{G}_1) = \frac{\sqrt{2}}{6}|E_1(\mathcal{G}_1)| + \frac{\sqrt{21}}{21}|E_2(\mathcal{G}_1)| + \frac{\sqrt{6}}{12}|E_3(\mathcal{G}_1)| + \frac{\sqrt{3}}{9}|E_4(\mathcal{G}_1)| + \frac{1}{6}|E_5(\mathcal{G}_1)| + \frac{\sqrt{42}}{42}|E_6(\mathcal{G}_1)| + \frac{\sqrt{3}}{12}|E_7(\mathcal{G}_1)| + \frac{\sqrt{2}}{12}|E_8(\mathcal{G}_1)| + \frac{1}{7}|E_9(\mathcal{G}_1)| + \frac{\sqrt{7}}{21}|E_{10}(\mathcal{G}_1)| + \frac{\sqrt{21}}{42}|E_{11}(\mathcal{G}_1)| + \frac{1}{8}|E_{12}(\mathcal{G}_1)| + \frac{\sqrt{2}}{12}|E_{13}(\mathcal{G}_1)| + \frac{\sqrt{6}}{24}|E_{14}(\mathcal{G}_1)| + \frac{1}{9}|E_{15}(\mathcal{G}_1)| + \frac{\sqrt{3}}{18}|E_{16}(\mathcal{G}_1)| + \frac{1}{12}|E_{17}(\mathcal{G}_1)|$

After simplification, we have

$R_{-\frac{1}{2}}(\mathcal{G}_1)=\frac{15n^3}{4} + (\frac{8}{\sqrt{3}} - \frac{125}{12})n^2 + (\frac{4}{\sqrt{7}} + 4\sqrt{6} - \frac{32}{\sqrt{3}} + \sqrt{2} + 5\sqrt{\frac{3}{7}} + \frac{109}{21})n - \frac{8}{\sqrt{7}} - 8\sqrt{6} + \frac{38}{\sqrt{3}} + 3\sqrt{2} + 2\sqrt{\frac{6}{7}} - 10\sqrt{\frac{3}{7}} + \frac{13}{14}$ □

In the below theorem, we calculate the Zagreb index of $\mathcal{G}_1$(m,n).

Theorem 2. *The first Zagreb index of hex-derived cage network $HDCN1(m,n)$ is equal to*

$$M_1(\mathcal{G}_1) = 18(27n^3 - 51n^2 + 10n + 14)$$

Proof. With the help of Table 1, we calculate the Zagreb index as

$$M_1(\mathcal{G}_1) = \sum_{ab\in E(\mathcal{G})} (d(a) + d(b)) = \sum_{j=1}^{17} \sum_{ab\in E_j(\mathcal{G})} (d(a) + d(b))$$

$M_1(\mathcal{G}_1) = 9|E_1(\mathcal{G}_1)| + 10|E_2(\mathcal{G}_1)| + 11|E_3(\mathcal{G}_1)| + 12|E_4(\mathcal{G}_1)| + 15|E_5(\mathcal{G}_1)| + 13|E_6(\mathcal{G}_1)| + 14|E_7(\mathcal{G}_1)| + 18|E_8(\mathcal{G}_1)| + 14|E_9(\mathcal{G}_1)| + 16|E_{10}(\mathcal{G}_1)| + 19|E_{11}(\mathcal{G}_1)| + 16|E_{12}(\mathcal{G}_1)| + 17|E_{13}(\mathcal{G}_1)| + 20|E_{14}(\mathcal{G}_1)| + 18|E_{15}(\mathcal{G}_1)| + 21|E_{16}(\mathcal{G}_1)| + 24|E_{17}(\mathcal{G}_1)|$

After some calculations, we get

$$M_1(\mathcal{G}_1) = 18(27n^3 - 51n^2 + 10n + 14)$$

□

Table 1. Edge partition of Hex-Derived Cage network ($HDCN1$) based on degrees of end vertices of each edge.

(d_u, d_v) where $ab \in E(\mathcal{G}_1)$	Number of Edges	(d_u, d_v) where $ab \in E(\mathcal{G}_1)$	Number of Edges
$E_1 = (3,6)$	24	$E_{10} = (7,7)$	$6n - 18$
$E_2 = (3,7)$	$2(6n - 12)$	$E_{11} = (7,9)$	$2(6n - 12)$
$E_3 = (3,8)$	$6(6n - 12)$	$E_{12} = (7,12)$	$6n - 12$
$E_4 = (3,9)$	$18n^2 - 72n + 72$	$E_{13} = (8,8)$	$2(6n - 18)$
$E_5 = (3,12)$	$18n^3 - 54n^2 + 42n$	$E_{14} = (8,9)$	$2(6n - 12)$
$E_6 = (6,7)$	12	$E_{15} = (8,12)$	$4(6n - 12)$
$E_7 = (6,8)$	24	$E_{16} = (9,9)$	$12n^2 - 60n + 72$
$E_8 = (6,12)$	12	$E_{17} = (9,12)$	$12n^2 - 48n + 48$
$E_9 = (12,12)$	$9n^3 - 33n^2 + 30n$		

In the next theorem, we calculate the ABC, GA, ABC_4 and GA_5 indices of Hex-Derived Cage network $HDCN1(m,n)$.

Theorem 3. *Let HDCN1(m,n) be Hex-Derived Cage network, then we have*

- $ABC(\mathcal{G}_1) =\frac{3}{4}(4\sqrt{13} + \sqrt{22})n^3 + \frac{1}{12}(8\sqrt{57} + 24\sqrt{30} - 33\sqrt{22} - 108\sqrt{13} + 64)n^2 + (-\frac{80}{3} + 8\sqrt{\frac{6}{7}} + 4\sqrt{2} + \frac{54\sqrt{3}}{7} + 3\sqrt{\frac{7}{2}} + 5\sqrt{\frac{11}{2}} + 9\sqrt{6} - 8\sqrt{\frac{19}{3}} + \sqrt{\frac{51}{7}} + 7\sqrt{13} - 7\sqrt{30})n + 44 - 16\sqrt{\frac{6}{7}} - 4\sqrt{2} - \frac{120\sqrt{3}}{7} - \sqrt{\frac{7}{2}} - 18\sqrt{6} + 8\sqrt{\frac{19}{3}} - 2\sqrt{\frac{51}{7}} + 2\sqrt{\frac{66}{7}} + 6\sqrt{30}.$

- $GA(\mathcal{G}_1)) = \frac{117n^3}{5} + \frac{3}{35}(185\sqrt{3} - 749)n^2 + (\frac{108}{5} + \frac{144\sqrt{2}}{17} - \frac{444\sqrt{3}}{7} + \frac{1248\sqrt{6}}{55} + \frac{9\sqrt{7}}{2} + \frac{348\sqrt{21}}{95})n + \frac{24\sqrt{42}}{13} - \frac{696\sqrt{21}}{95} - 9\sqrt{7} - \frac{2496\sqrt{6}}{55} + \frac{540\sqrt{3}}{7} + \frac{120\sqrt{2}}{17} + 18.$

- $ABC_4(\mathcal{G}_1) = \frac{6}{5}\sqrt{\frac{38}{7}}(n-4)^2 + 2\sqrt{\frac{69}{79}}(n-2) + \frac{1}{3}\sqrt{\frac{62}{5}}n(3n^2 - 15n + 19) + \frac{1}{15}\sqrt{\frac{89}{2}}n(3n^2 - 17n + 24) + \sqrt{\frac{177}{14}}(n^2 - 5n + 6) + 2\sqrt{\frac{115}{77}}(n^2 - 5n + 6) + 2\sqrt{\frac{86}{105}}(n^2 - 5n + 6) + 2\sqrt{\frac{2}{5}}(n^2 - 5n + 6) + \frac{1}{7}\sqrt{\frac{83}{2}}(n^2 - 6n + 8) + \frac{8}{23}\sqrt{34}(n^2 - 9n + 20) + 2\sqrt{\frac{113}{79}}(n-2) + \sqrt{\frac{334}{395}}(n-2) + 4\sqrt{\frac{30}{79}}(n-2) + 6\sqrt{\frac{15}{79}}(n-2) + 6\sqrt{\frac{58}{41}}(n-3) + 4\sqrt{\frac{34}{41}}(n-3) + 12\sqrt{\frac{7}{41}}(n-3) + 18\sqrt{\frac{6}{287}}(n-3) + \frac{20(n-4)^2}{\sqrt{253}} + 2\sqrt{\frac{194}{115}}(n-4)^2 + 2\sqrt{\frac{151}{161}}(n-4)^2 + \frac{144(n-4)}{\sqrt{5293}} + \frac{48(n-4)}{\sqrt{85}} + \frac{3}{5}\sqrt{\frac{254}{79}}(n-4) + \frac{54}{41}\sqrt{2}(n-4) + 2\sqrt{\frac{114}{67}}(n-4) + 2\sqrt{\frac{447}{469}}(n-4) + 3\sqrt{\frac{22}{29}}(n-4) + 4\sqrt{\frac{6}{23}}(n-4) + 12\sqrt{\frac{6}{29}}(n-4) + 6\sqrt{\frac{93}{469}}(n-4) + 12\sqrt{\frac{138}{1189}}(n-4) + 30\sqrt{\frac{2}{119}}(n-4) + 18\sqrt{\frac{5}{391}}(n-4) + 28\sqrt{\frac{6}{737}}(n-4) + \frac{3}{17}\sqrt{134}(n-5) + \frac{6}{29}\sqrt{114}(n-5) + \frac{24}{67}\sqrt{33}(n-5) + \frac{21}{25}\sqrt{2}(n-5) + \frac{206\sqrt{194}}{385} + 4\frac{\sqrt{\frac{678}{11}}}{7} + \frac{6\sqrt{41}}{7} + \frac{24}{\sqrt{29}} + 2\sqrt{\frac{110}{19}} + 3\sqrt{\frac{43}{14}} + 3\sqrt{\frac{53}{19}} + 4\sqrt{\frac{138}{77}} + 4\sqrt{\frac{94}{55}} + 4\sqrt{\frac{786}{737}} + 4\sqrt{\frac{78}{79}} + 4\sqrt{\frac{74}{77}} + 2\sqrt{\frac{82}{95}} + 6\sqrt{\frac{190}{287}} + \frac{24\sqrt{\frac{22}{41}}}{7} + 6\sqrt{\frac{65}{133}} + \frac{60\sqrt{\frac{3}{7}}}{7} + 12\sqrt{\frac{10}{41}} + 6\sqrt{\frac{115}{779}} + 12\sqrt{\frac{10}{77}} + 12\sqrt{\frac{34}{287}} + 12\sqrt{\frac{2}{17}} + 12\sqrt{\frac{78}{779}} + 24\sqrt{\frac{5}{91}} + 60\sqrt{\frac{2}{247}} + 36\sqrt{\frac{2}{553}}.$

- $GA_5(\mathcal{G}_1) = \frac{40}{13}\sqrt{14}(n-4)^2 + \frac{48}{163}\sqrt{1659}(n-2) + \frac{12}{7}\sqrt{10}n(3n^2 - 15n + 19) + \frac{24}{29}\sqrt{210}(n^2 - 5n + 6) + \frac{16}{13}\sqrt{77}(n^2 - 5n + 6) + \frac{12}{19}\sqrt{70}(n^2 - 5n + 6) + \frac{18}{5}\sqrt{21}(n^2 - 5n + 6) + 9n^3 - 33n^2 - 66n + \frac{3}{14}\sqrt{2607}(n-2) + \frac{36}{169}\sqrt{790}(n-2) + \frac{48}{107}\sqrt{553}(n-2) + \frac{144}{115}\sqrt{79}(n-2) + \frac{24}{55}\sqrt{574}(n-3) + \frac{72}{43}\sqrt{205}(n-3) + \frac{216}{59}\sqrt{82}(n-3) + \frac{64}{19}\sqrt{41}(n-3) + \frac{6}{17}\sqrt{253}(n-4)^2 + \frac{8}{11}\sqrt{230}(n-4)^2 + \frac{16}{17}\sqrt{161}(n-4)^2 + \frac{12}{73}\sqrt{5293}(n-4) + \frac{3}{17}\sqrt{4623}(n-4) + \frac{6}{25}\sqrt{2211}(n-4) + \frac{24}{97}\sqrt{2010}(n-4) + \frac{48}{151}\sqrt{1407}(n-4) + \frac{24}{35}\sqrt{1189}(n-4) + \frac{48}{137}\sqrt{1173}(n-4) + \frac{8}{21}\sqrt{986}(n-4) + \frac{48}{101}\sqrt{561}(n-4) + \frac{48}{49}\sqrt{510}(n-4) + \frac{48}{95}\sqrt{469}(n-4) + \frac{48}{43}\sqrt{406}(n-4) + \frac{24}{19}\sqrt{357}(n-4) + \frac{20}{43}\sqrt{158}(n-4) + \frac{40}{39}\sqrt{134}(n-4) + \frac{32}{15}\sqrt{29}(n-4) + 12(n-4) + 36(n-5) + \frac{48\sqrt{5214}}{145} + \frac{48\sqrt{4422}}{133} + \frac{48\sqrt{1558}}{79} + \frac{48\sqrt{1122}}{67} + \frac{48\sqrt{1066}}{67} + \frac{16\sqrt{779}}{39} + \frac{96\sqrt{574}}{97} + \frac{16\sqrt{494}}{17} + \frac{48\sqrt{462}}{47} + \frac{32\sqrt{287}}{23} + \frac{16\sqrt{266}}{11} + \frac{96\sqrt{231}}{61} + \frac{32\sqrt{203}}{19} + \frac{72\sqrt{190}}{83} + \frac{48\sqrt{154}}{25} + \frac{96\sqrt{91}}{41} + \frac{21\sqrt{79}}{16} + \frac{336\sqrt{66}}{115} + 3\sqrt{55} + \frac{28\sqrt{41}}{15} + \frac{32\sqrt{38}}{9} + \frac{36\sqrt{19}}{7} + \frac{192\sqrt{7}}{11} + \frac{1336\sqrt{2}}{33} + 258.$

Proof. From Table 1 we calculate the $ABC(\mathcal{G}_1)$ as

$$ABC(\mathcal{G}_1) = \sum_{ab\in E(\mathcal{G})}\sqrt{\frac{d(a)+d(b)-2}{d(a)\cdot d(b)}} = \sum_{j=1}^{17}\sum_{ab\in E_j(\mathcal{G})}\sqrt{\frac{d(a)+d(b)-2}{d(a)\cdot d(b)}}$$

$ABC(\mathcal{G}_1) = \frac{\sqrt{14}}{6}|E_1(\mathcal{G}_1)| + \frac{2\sqrt{42}}{21}|E_2(\mathcal{G}_1)| + \frac{\sqrt{6}}{4}|E_3(\mathcal{G}_1)| + \frac{\sqrt{30}}{9}|E_4(\mathcal{G}_1)| + \frac{\sqrt{13}}{6}|E_5(\mathcal{G}_1)| + \frac{\sqrt{462}}{42}|E_6(\mathcal{G}_1)| + \frac{1}{2}|E_7(\mathcal{G}_1)| + \frac{\sqrt{2}}{3}|E_8(\mathcal{G}_1)| + \frac{2\sqrt{3}}{7}|E_9(\mathcal{G}_1)| + \frac{\sqrt{2}}{3}|E_{10}(\mathcal{G}_1)| + \frac{\sqrt{357}}{42}|E_{11}(\mathcal{G}_1)| + \frac{\sqrt{14}}{8}|E_{12}(\mathcal{G}_1)| + \frac{\sqrt{30}}{12}|E_{13}(\mathcal{G}_1)| + \frac{\sqrt{3}}{4}|E_{14}(\mathcal{G}_1)| + \frac{4}{9}|E_{15}(\mathcal{G}_1)| + \frac{\sqrt{57}}{18}|E_{16}(\mathcal{G}_1)| + \frac{\sqrt{22}}{12}|E_{17}(\mathcal{G}_1)|.$

After simplification, we have

$ABC(\mathcal{G}_1) = \frac{3}{4}(4\sqrt{13} + \sqrt{22})n^3 + \frac{1}{12}(8\sqrt{57} + 24\sqrt{30} - 33\sqrt{22} - 108\sqrt{13} + 64)n^2 + (-\frac{80}{3} + 8\sqrt{\frac{6}{7}} + 4\sqrt{2} + \frac{54\sqrt{3}}{7} + 3\sqrt{\frac{7}{2}} + 5\sqrt{\frac{11}{2}} + 9\sqrt{6} - 8\sqrt{\frac{19}{3}} + \sqrt{\frac{51}{7}} + 7\sqrt{13} - 7\sqrt{30})n + 44 - 16\sqrt{\frac{6}{7}} - 4\sqrt{2} - \frac{120\sqrt{3}}{7} - \sqrt{\frac{7}{2}} - 18\sqrt{6} + 8\sqrt{\frac{19}{3}} - 2\sqrt{\frac{51}{7}} + 2\sqrt{\frac{66}{7}} + 6\sqrt{30}.$

Now we calculate GA from Equation (6) as

$$GA(\mathcal{G}_1) = \sum_{ab\in E(\mathcal{G})}\frac{2\sqrt{d(a)d(b)}}{(d(a)+d(b))} = \sum_{j=1}^{17}\sum_{ab\in E_j(\mathcal{G})}\frac{2\sqrt{d(a)d(b)}}{(d(a)+d(b))}$$

From Table 1 calculate $GA(\mathcal{G}_1)$ as

$GA(\mathcal{G}_1)$=$\frac{2\sqrt{2}}{3}|E_1(\mathcal{G}_1)| + \frac{\sqrt{21}}{5}|E_2(\mathcal{G}_1)| + \frac{4\sqrt{6}}{11}|E_3(\mathcal{G}_1)| + \frac{\sqrt{3}}{2}|E_4(\mathcal{G}_1)| + \frac{4}{5}|E_5(\mathcal{G}_1)| + \frac{2\sqrt{42}}{13}|E_6(\mathcal{G}_1)| + \frac{4\sqrt{3}}{7}|E_7(\mathcal{G}_1)| + \frac{2\sqrt{2}}{3}|E_8(\mathcal{G}_1)| + 1|E_9(\mathcal{G}_1)| + \frac{3\sqrt{7}}{8}|E_{10}(\mathcal{G}_1)| + \frac{4\sqrt{21}}{19}|E_{11}(\mathcal{G}_1)| + 1|E_{12}(\mathcal{G}_1)| + \frac{12\sqrt{2}}{17}|E_{13}(\mathcal{G}_1)| + \frac{2\sqrt{6}}{5}|E_{14}(\mathcal{G}_1)| + 1|E_{15}(\mathcal{G}_1)| + \frac{4\sqrt{3}}{7}|E_{16}(\mathcal{G}_1)| + 1|E_{17}(\mathcal{G}_1)|$.

After simplification, we have
$GA(\mathcal{G}_1) = \frac{117n^3}{5} + \frac{3}{35}(185\sqrt{3} - 749)n^2 + (\frac{108}{5} + \frac{144\sqrt{2}}{17} - \frac{444\sqrt{3}}{7} + \frac{1248\sqrt{6}}{55} + \frac{9\sqrt{7}}{2} + \frac{348\sqrt{21}}{95})n + \frac{24\sqrt{42}}{13} - \frac{696\sqrt{21}}{95} - 9\sqrt{7} - \frac{2496\sqrt{6}}{55} + \frac{540\sqrt{3}}{7} + \frac{120\sqrt{2}}{17} + 18.$

If we consider an edge partition based on degree sum of neighbors of end vertices; then the edge set $E(HDCN1(m,n))$ are divided into sixtynine edge partition $E_j(HDCN1(m,n)), 18 \leq j \leq 86$ shows in Table 2.

From Equation (7), we have

$$ABC_4(\mathcal{G}_1) = \sum_{ab \in E(\mathcal{G})} \sqrt{\frac{S(a)+S(b)-2}{S(a)S(b)}} = \sum_{j=18}^{86} \sum_{ab \in E_j(\mathcal{G})} \sqrt{\frac{S(a)+S(b)-2}{S(a)S(b)}}.$$

From Table 2 we use edge partition, we get
$ABC_4(\mathcal{G}_1) = \frac{2\sqrt{1066}}{67}|E_{18}(\mathcal{G}_1)| + \frac{\sqrt{1456}}{41}|E_{19}(\mathcal{G}_1)| + \frac{\sqrt{1976}}{51}|E_{20}(\mathcal{G}_1)| + \frac{2\sqrt{1372}}{77}|E_{21}(\mathcal{G}_1)| + \frac{\sqrt{1400}}{39}|E_{22}(\mathcal{G}_1)| + \frac{\sqrt{1568}}{42}|E_{23}(\mathcal{G}_1)| + \frac{\sqrt{1624}}{43}|E_{24}(\mathcal{G}_1)| + \frac{\sqrt{1848}}{47}|E_{25}(\mathcal{G}_1)| + \frac{2\sqrt{1876}}{95}|E_{26}(\mathcal{G}_1)| + \frac{2\sqrt{2212}}{107}|E_{27}(\mathcal{G}_1)| + \frac{\sqrt{2296}}{55}|E_{28}(\mathcal{G}_1)| + \frac{\sqrt{1980}}{48}|E_{29}(\mathcal{G}_1)| + \frac{2\sqrt{2010}}{97}|E_{30}(\mathcal{G}_1)| + \frac{\sqrt{2040}}{\sqrt{49}}|E_{31}(\mathcal{G}_1)| + \frac{2\sqrt{2070}}{99}|E_{32}(\mathcal{G}_1)| + \frac{\sqrt{2520}}{57}|E_{33}(\mathcal{G}_1)| + \frac{\sqrt{1792}}{44}|E_{34}(\mathcal{G}_1)| + \frac{\sqrt{1856}}{45}|E_{35}(\mathcal{G}_1)| + \frac{\sqrt{2432}}{54}|E_{36}(\mathcal{G}_1)| + \frac{\sqrt{2624}}{57}|E_{37}(\mathcal{G}_1)| + \frac{2\sqrt{2178}}{99}|E_{38}(\mathcal{G}_1)| + \frac{\sqrt{2211}}{50}|E_{39}(\mathcal{G}_1)| + \frac{2\sqrt{2244}}{101}|E_{40}(\mathcal{G}_1)| + \frac{\sqrt{2277}}{51}|E_{41}(\mathcal{G}_1)| + \frac{\sqrt{2607}}{3}|E_{56}(\mathcal{G}_1)| + \frac{2\sqrt{2772}}{117}|E_{43}(\mathcal{G}_1)| + \frac{\sqrt{2736}}{56}|E_{44}(\mathcal{G}_1)| + \frac{2\sqrt{2844}}{115}|E_{45}(\mathcal{G}_1)| + \frac{\sqrt{2952}}{59}|E_{46}(\mathcal{G}_1)| + \frac{\sqrt{3024}}{60}|E_{47}(\mathcal{G}_1)| + \frac{\sqrt{3240}}{63}|E_{48}(\mathcal{G}_1)| + \frac{\sqrt{2009}}{45}|E_{49}(\mathcal{G}_1)| + \frac{2\sqrt{2296}}{97}|E_{50}(\mathcal{G}_1)| + \frac{2\sqrt{3116}}{117}|E_{51}(\mathcal{G}_1)| + \frac{2\sqrt{2450}}{99}|E_{52}(\mathcal{G}_1)| + \frac{2\sqrt{3234}}{115}|E_{53}(\mathcal{G}_1)| + \frac{\sqrt{3871}}{64}|E_{54}(\mathcal{G}_1)| + 1|E_{55}(\mathcal{G}_1)| + \frac{2\sqrt{3350}}{117}|E_{56}(\mathcal{G}_1)| + \frac{2\sqrt{3950}}{129}|E_{57}(\mathcal{G}_1)| + \frac{\sqrt{3248}}{57}|E_{58}(\mathcal{G}_1)| + \frac{\sqrt{3696}}{56}|E_{59}(\mathcal{G}_1)| + \frac{\sqrt{4256}}{66}|E_{60}(\mathcal{G}_1)| + \frac{\sqrt{4292}}{69}|E_{61}(\mathcal{G}_1)| + \frac{\sqrt{3364}}{58}|E_{62}(\mathcal{G}_1)| + \frac{\sqrt{3944}}{63}|E_{63}(\mathcal{G}_1)| + \frac{\sqrt{4756}}{70}|E_{64}(\mathcal{G}_1)| + \frac{2\sqrt{4422}}{133}|E_{65}(\mathcal{G}_1)| + \frac{\sqrt{4488}}{67}|E_{66}(\mathcal{G}_1)| + \frac{2\sqrt{5214}}{145}|E_{67}(\mathcal{G}_1)| + \frac{\sqrt{5544}}{75}|E_{68}(\mathcal{G}_1)| + \frac{\sqrt{4489}}{67}|E_{69}(\mathcal{G}_1)| + \frac{\sqrt{4623}}{68}|E_{70}(\mathcal{G}_1)| + \frac{\sqrt{5293}}{73}|E_{71}(\mathcal{G}_1)| + \frac{2\sqrt{5628}}{151}|E_{72}(\mathcal{G}_1)| + \frac{\sqrt{4624}}{68}|E_{73}(\mathcal{G}_1)| + \frac{2\sqrt{4692}}{137}|E_{74}(\mathcal{G}_1)| + \frac{\sqrt{5712}}{76}|E_{75}(\mathcal{G}_1)| + \frac{\sqrt{4761}}{69}|E_{76}(\mathcal{G}_1)| + \frac{2\sqrt{5796}}{153}|E_{77}(\mathcal{G}_1)| + \frac{\sqrt{6232}}{79}|E_{78}(\mathcal{G}_1)| + \frac{\sqrt{6840}}{83}|E_{79}|(\mathcal{G}_1) + \frac{2\sqrt{6636}}{163}|E_{80}(\mathcal{G}_1)| + \frac{2\sqrt{7110}}{169}|E_{81}(\mathcal{G}_1)| + \frac{\sqrt{4724}}{82}|E_{82}(\mathcal{G}_1)| + \frac{\sqrt{7380}}{86}|E_{83}(\mathcal{G}_1)| + \frac{\sqrt{7056}}{84}|E_{84}(\mathcal{G}_1)| + \frac{\sqrt{7560}}{87}|E_{85}(\mathcal{G}_1)| + 1|E_{86}(\mathcal{G}_1)|$.

After simplification, we get
$ABC_4(\mathcal{G}_1) = \frac{6}{5}\sqrt{\frac{38}{7}}(n-4)^2 + 2\sqrt{\frac{69}{79}}(n-2) + \frac{1}{3}\sqrt{\frac{62}{5}}n(3n^2 - 15n + 19) + \frac{1}{15}\sqrt{\frac{89}{2}}n(3n^2 - 17n + 24) + \sqrt{\frac{177}{14}}(n^2 - 5n + 6) + 2\sqrt{\frac{115}{77}}(n^2 - 5n + 6) + 2\sqrt{\frac{86}{105}}(n^2 - 5n + 6) + 2\sqrt{\frac{2}{5}}(n^2 - 5n + 6) + \frac{1}{7}\sqrt{\frac{83}{2}}(n^2 - 6n + 8) + \frac{8}{23}\sqrt{34}(n^2 - 9n + 20) + 2\sqrt{\frac{113}{79}}(n-2) + \sqrt{\frac{334}{395}}(n-2) + 4\sqrt{\frac{30}{79}}(n-2) + 6\sqrt{\frac{15}{79}}(n-2) + 6\sqrt{\frac{58}{41}}(n-3) + 4\sqrt{\frac{34}{41}}(n-3) + 12\sqrt{\frac{7}{41}}(n-3) + 18\sqrt{\frac{6}{287}}(n-3) + \frac{20(n-4)^2}{\sqrt{253}} + 2\sqrt{\frac{194}{115}}(n-4)^2 + 2\sqrt{\frac{151}{161}}(n-4)^2 + \frac{144(n-4)}{\sqrt{5293}} + \frac{48(n-4)}{\sqrt{85}} + \frac{3}{5}\sqrt{\frac{254}{79}}(n-4) + \frac{54}{41}\sqrt{2}(n-4) + 2\sqrt{\frac{114}{67}}(n-4) + 2\sqrt{\frac{447}{469}}(n-4) + 3\sqrt{\frac{22}{29}}(n-4) + 4\sqrt{\frac{6}{23}}(n-4) + 12\sqrt{\frac{6}{29}}(n-4) + 6\sqrt{\frac{93}{469}}(n-4) + 12\sqrt{\frac{138}{1189}}(n-4) + 30\sqrt{\frac{2}{119}}(n-4) + 18\sqrt{\frac{5}{391}}(n-4) + 28\sqrt{\frac{6}{737}}(n-4) + \frac{3}{17}\sqrt{134}(n-5) + \frac{6}{29}\sqrt{114}(n-5) + \frac{24}{67}\sqrt{33}(n-5) + \frac{21}{25}\sqrt{2}(n-5) + \frac{206\sqrt{194}}{385} + 4\frac{\sqrt{\frac{678}{11}}}{7} + \frac{6\sqrt{41}}{7} + \frac{24}{\sqrt{29}} + 2\sqrt{\frac{110}{19}} + 3\sqrt{\frac{43}{14}} + 3\sqrt{\frac{53}{19}} + 4\sqrt{\frac{138}{77}} + 4\sqrt{\frac{94}{55}} + 4\sqrt{\frac{786}{737}} + 4\sqrt{\frac{78}{79}} + 4\sqrt{\frac{74}{77}} + 2\sqrt{\frac{82}{95}} + 6\sqrt{\frac{190}{287}} + \frac{24\sqrt{\frac{22}{41}}}{7} + 6\sqrt{\frac{65}{133}} + \frac{60\sqrt{\frac{3}{7}}}{7} + 12\sqrt{\frac{10}{41}} + 6\sqrt{\frac{115}{779}} + 12\sqrt{\frac{10}{77}} + 12\sqrt{\frac{34}{287}} + 12\sqrt{\frac{2}{17}} + 12\sqrt{\frac{78}{779}} + 24\sqrt{\frac{5}{91}} + 60\sqrt{\frac{2}{247}} + 36\sqrt{\frac{2}{553}}$

Now we find $GA_5(\mathcal{G}_1)$ as

$$GA_5(\mathcal{G}_1) = \sum_{ab\in E(\mathcal{G})} \frac{2\sqrt{S(a)S(b)}}{(S(a)+S(b))} = \sum_{j=18}^{86} \sum_{ab\in E_j(\mathcal{G})} \frac{2\sqrt{S(a)S(b)}}{(S(a)+S(b))}.$$

Using the edge partition from Table 2, we get

$GA_5(\mathcal{G}_1) = \sqrt{\frac{65}{1066}}|E_{18}(\mathcal{G}_1)| + \sqrt{\frac{5}{91}}|E_{19}(\mathcal{G}_1)| + \frac{5}{\sqrt{494}}|E_{20}(\mathcal{G}_1)| + \sqrt{\frac{75}{1372}}|E_{21}(\mathcal{G}_1)| + \sqrt{\frac{19}{350}}|E_{22}(\mathcal{G}_1)| + \sqrt{\frac{41}{784}}|E_{23}(\mathcal{G}_1)| + \sqrt{\frac{3}{58}}|E_{24}(\mathcal{G}_1)| + \sqrt{\frac{23}{462}}|E_{25}(\mathcal{G}_1)| + \sqrt{\frac{93}{1876}}|E_{26}(\mathcal{G}_1)| + \sqrt{\frac{105}{2212}}|E_{27}(\mathcal{G}_1)| + \sqrt{\frac{27}{574}}|E_{28}(\mathcal{G}_1)| + \sqrt{\frac{47}{990}}|E_{29}(\mathcal{G}_1)| + \sqrt{\frac{19}{402}}|E_{30}(\mathcal{G}_1)| + \sqrt{\frac{4}{85}}|E_{31}(\mathcal{G}_1)| + \sqrt{\frac{97}{2070}}|E_{32}(\mathcal{G}_1)| + \sqrt{\frac{14}{315}}|E_{33}(\mathcal{G}_1)| + \sqrt{\frac{43}{896}}|E_{34}(\mathcal{G}_1)| + \sqrt{\frac{11}{232}}|E_{35}(\mathcal{G}_1)| + \sqrt{\frac{53}{1216}}|E_{36}(\mathcal{G}_1)| + \sqrt{\frac{7}{164}}|E_{37}(\mathcal{G}_1)| + \sqrt{\frac{97}{2178}}|E_{38}(\mathcal{G}_1)| + \sqrt{\frac{98}{2211}}|E_{39}(\mathcal{G}_1)| + \sqrt{\frac{9}{204}}|E_{40}(\mathcal{G}_1)| + \sqrt{\frac{10}{2277}}|E_{41}(\mathcal{G}_1)| + \sqrt{\frac{110}{2607}}|E_{56}(\mathcal{G}_1)| + \sqrt{\frac{115}{2772}}|E_{43}(\mathcal{G}_1)| + \sqrt{\frac{55}{1368}}|E_{44}(\mathcal{G}_1)| + \sqrt{\frac{113}{2844}}|E_{45}(\mathcal{G}_1)| + \sqrt{\frac{29}{738}}|E_{46}(\mathcal{G}_1)| + \sqrt{\frac{59}{1512}}|E_{47}(\mathcal{G}_1)| + \sqrt{\frac{31}{810}}|E_{48}(\mathcal{G}_1)| + \sqrt{\frac{88}{2009}}|E_{49}(\mathcal{G}_1)| + \sqrt{\frac{95}{2296}}|E_{50}(\mathcal{G}_1)| + \sqrt{\frac{115}{3116}}|E_{51}(\mathcal{G}_1)| + \sqrt{\frac{97}{2450}}|E_{52}(\mathcal{G}_1)| + \sqrt{\frac{113}{3234}}|E_{53}(\mathcal{G}_1)| + \sqrt{\frac{126}{3871}}|E_{54}(\mathcal{G}_1)| + \frac{\sqrt{98}}{50}|E_{55}(\mathcal{G}_1)| + \sqrt{\frac{23}{670}}|E_{56}(\mathcal{G}_1)| + \sqrt{\frac{127}{3950}}|E_{57}(\mathcal{G}_1)| + \sqrt{\frac{7}{203}}|E_{58}(\mathcal{G}_1)| + \sqrt{\frac{5}{154}}|E_{59}(\mathcal{G}_1)| + \sqrt{\frac{65}{2128}}|E_{60}(\mathcal{G}_1)| + \sqrt{\frac{17}{574}}|E_{61}(\mathcal{G}_1)| + \sqrt{\frac{57}{1682}}|E_{62}(\mathcal{G}_1)| + \sqrt{\frac{31}{952}}|E_{63}(\mathcal{G}_1)| + \sqrt{\frac{69}{2378}}|E_{64}(\mathcal{G}_1)| + \sqrt{\frac{131}{4422}}|E_{65}(\mathcal{G}_1)| + \sqrt{\frac{33}{1122}}|E_{66}(\mathcal{G}_1)| + \sqrt{\frac{143}{5214}}|E_{67}(\mathcal{G}_1)| + \sqrt{\frac{37}{1344}}|E_{68}(\mathcal{G}_1)| + \sqrt{\frac{132}{4489}}|E_{69}(\mathcal{G}_1)| + \sqrt{\frac{134}{4623}}|E_{70}(\mathcal{G}_1)| + \sqrt{\frac{144}{5293}}|E_{71}(\mathcal{G}_1)| + \sqrt{\frac{149}{5628}}|E_{72}(\mathcal{G}_1)| + \sqrt{\frac{67}{2312}}|E_{73}(\mathcal{G}_1)| + \sqrt{\frac{135}{4692}}|E_{74}(\mathcal{G}_1)| + \sqrt{\frac{75}{4856}}|E_{75}(\mathcal{G}_1)| + \sqrt{\frac{136}{4761}}|E_{76}(\mathcal{G}_1)| + \sqrt{\frac{151}{5796}}|E_{77}(\mathcal{G}_1)| + \sqrt{\frac{159}{6232}}|E_{78}(\mathcal{G}_1)| + \sqrt{\frac{41}{1710}}|E_{79}|(\mathcal{G}_1) + \sqrt{\frac{161}{6384}}|E_{80}(\mathcal{G}_1)| + \sqrt{\frac{167}{7110}}|E_{81}(\mathcal{G}_1)| + \sqrt{\frac{81}{6724}}|E_{82}(\mathcal{G}_1)| + \sqrt{\frac{17}{738}}|E_{83}(\mathcal{G}_1)| + \sqrt{\frac{83}{3528}}|E_{84}(\mathcal{G}_1)| + \sqrt{\frac{86}{3785}}|E_{85}(\mathcal{G}_1)| + \sqrt{\frac{89}{450}}|E_{86}(\mathcal{G}_1)|.$

After simplification, we get

$GA_5(\mathcal{G}_1) = \frac{40}{13}\sqrt{14}(n-4)^2 + \frac{48}{163}\sqrt{1659}(n-2) + \frac{12}{7}\sqrt{10}n(3n^2-15n+19) + \frac{24}{29}\sqrt{210}(n^2-5n+6) + \frac{16}{13}\sqrt{77}(n^2-5n+6) + \frac{12}{19}\sqrt{70}(n^2-5n+6) + \frac{18}{5}\sqrt{21}(n^2-5n+6) + 9n^3 - 33n^2 - 66n + \frac{3}{14}\sqrt{2607}(n-2) + \frac{36}{169}\sqrt{790}(n-2) + \frac{48}{107}\sqrt{553}(n-2) + \frac{144}{115}\sqrt{79}(n-2) + \frac{24}{55}\sqrt{574}(n-3) + \frac{72}{43}\sqrt{205}(n-3) + \frac{216}{59}\sqrt{82}(n-3) + \frac{64}{19}\sqrt{41}(n-3) + \frac{6}{17}\sqrt{253}(n-4)^2 + \frac{8}{11}\sqrt{230}(n-4)^2 + \frac{16}{17}\sqrt{161}(n-4)^2 + \frac{12}{73}\sqrt{5293}(n-4) + \frac{3}{17}\sqrt{4623}(n-4) + \frac{6}{25}\sqrt{2211}(n-4) + \frac{24}{97}\sqrt{2010}(n-4) + \frac{48}{151}\sqrt{1407}(n-4) + \frac{24}{35}\sqrt{1189}(n-4) + \frac{48}{137}\sqrt{1173}(n-4) + \frac{8}{21}\sqrt{986}(n-4) + \frac{48}{101}\sqrt{561}(n-4) + \frac{48}{49}\sqrt{510}(n-4) + \frac{48}{95}\sqrt{469}(n-4) + \frac{48}{43}\sqrt{406}(n-4) + \frac{24}{19}\sqrt{357}(n-4) + \frac{20}{43}\sqrt{158}(n-4) + \frac{40}{39}\sqrt{134}(n-4) + \frac{32}{15}\sqrt{29}(n-4) + 12(n-4) + 36(n-5) + \frac{48\sqrt{5214}}{145} + \frac{48\sqrt{4422}}{133} + \frac{48\sqrt{1558}}{79} + \frac{48\sqrt{1122}}{67} + \frac{48\sqrt{1066}}{67} + \frac{16\sqrt{779}}{39} + \frac{96\sqrt{574}}{97} + \frac{16\sqrt{494}}{17} + \frac{48\sqrt{462}}{47} + \frac{32\sqrt{287}}{23} + \frac{16\sqrt{266}}{11} + \frac{96\sqrt{231}}{61} + \frac{32\sqrt{203}}{19} + \frac{72\sqrt{190}}{83} + \frac{48\sqrt{154}}{25} + \frac{96\sqrt{91}}{41} + \frac{21\sqrt{79}}{16} + \frac{336\sqrt{66}}{115} + 3\sqrt{55} + \frac{28\sqrt{41}}{15} + \frac{32\sqrt{38}}{9} + \frac{36\sqrt{19}}{7} + \frac{192\sqrt{7}}{11} + \frac{1336\sqrt{2}}{33} + 258 \quad \square$

Table 2. Edge partition of Hex-Derived Cage network ($HDCN1$) based on degrees of end vertices of each edge.

(S_u, S_v) where $ab \in E(\mathcal{G}_1)$	Number of Edges	(S_u, S_v) where $ab \in E(\mathcal{G}_1)$	Number of Edges
$E_{18} = (26, 41)$	24	$E_{53} = (49, 66)$	24
$E_{19} = (26, 56)$	24	$E_{54} = (49, 79)$	12
$E_{20} = (26, 76)$	24	$E_{55} = (50, 50)$	$6n - 30$
$E_{21} = (28, 49)$	24	$E_{56} = (50, 67)$	$2(6n - 24)$
$E_{22} = (28, 50)$	$2(6n - 24)$	$E_{57} = (50, 79)$	$6n - 24$
$E_{23} = (28, 56)$	24	$E_{58} = (56, 58)$	24
$E_{24} = (28, 58)$	$4(6n - 24)$	$E_{59} = (56, 66)$	24
$E_{25} = (28, 66)$	24	$E_{60} = (56, 76)$	24
$E_{26} = (28, 67)$	$2(6n - 24)$	$E_{61} = (56, 82)$	24
$E_{27} = (28, 79)$	$2(6n - 12)$	$E_{62} = (58, 58)$	$2(6n - 30)$
$E_{28} = (28, 82)$	$2(6n - 18)$	$E_{63} = (58, 68)$	$2(6n - 24)$
$E_{29} = (30, 66)$	24	$E_{64} = (58, 82)$	$4(6n - 24)$
$E_{30} = (30, 67)$	$2(6n - 24)$	$E_{65} = (66, 67)$	24
$E_{31} = (30, 68)$	$4(6n - 24)$	$E_{66} = (66, 68)$	24
$E_{32} = (30, 69)$	$12n^2 - 96n + 192$	$E_{67} = (66, 79)$	24
$E_{33} = (30, 84)$	$6n^2 - 30n + 36$	$E_{68} = (66, 84)$	24
$E_{34} = (32, 56)$	24	$E_{69} = (67, 67)$	$2(6n - 30)$
$E_{35} = (32, 58)$	$2(6n - 24)$	$E_{70} = (67, 69)$	$2(6n - 24)$
$E_{36} = (32, 76)$	24	$E_{71} = (67, 79)$	$2(6n - 24)$
$E_{37} = (32, 82)$	$4(6n - 18)$	$E_{72} = (67, 84)$	$2(6n - 24)$
$E_{38} = (33, 66)$	24	$E_{73} = (68, 68)$	$2(6n - 30)$
$E_{39} = (33, 67)$	$2(6n - 24)$	$E_{74} = (68, 69)$	$2(6n - 24)$
$E_{40} = (33, 68)$	$2(6n - 24)$	$E_{75} = (68, 84)$	$4(6n - 24)$
$E_{41} = (33, 69)$	$6n^2 - 48n + 96$	$E_{76} = (69, 69)$	$12n^2 - 108n + 240$
$E_{42} = (33, 79)$	$2(6n - 12)$	$E_{77} = (69, 84)$	$12n^2 - 96n + 192$
$E_{43} = (33, 84)$	$12n^2 - 60n + 72$	$E_{78} = (76, 82)$	24
$E_{44} = (36, 76)$	24	$E_{79} = (76, 90)$	12
$E_{45} = (36, 79)$	$2(6n - 12)$	$E_{80} = (79, 84)$	$2(6n - 12)$
$E_{46} = (36, 82)$	$6(6n - 18)$	$E_{81} = (79, 90)$	$6n - 12$
$E_{47} = (36, 84)$	$18n^2 - 90n + 108$	$E_{82} = (82, 82)$	$2(6n - 24)$
$E_{48} = (36, 90)$	$18n^3 - 90n^2 + 114n$	$E_{83} = (82, 90)$	$4(6n - 18)$
$E_{49} = (41, 49)$	12	$E_{84} = (84, 84)$	$6n^2 - 36n + 48$
$E_{50} = (41, 56)$	24	$E_{85} = (84, 90)$	$12n^2 - 60n + 72$
$E_{51} = (41, 76)$	12	$E_{86} = (90, 90)$	$9n^3 - 51n^2 + 72n$
$E_{52} = (49, 50)$	12		

2.2. *Results for Hex-Derived Cage Network (HDCN2(m,n))*

In this portion, we find some degree-based topological indices for Hex-Derived Cage network ($HDCN2(m,n)$). We calculate the general Randić index $R_\alpha(\mathcal{G})$ with $\alpha = \{1, -1, \frac{1}{2}, -\frac{1}{2}\}$, ABC, GA, ABC_4 and GA_5 in the the below theorems for ($HDCN2(m,n)$).

Theorem 4. *Let $G_2 \cong HDCN2(m,n)$ be the Hex-Derived Cage network, then its general Randić index is equal to*

$$R_\alpha(\mathcal{G}_2) = \begin{cases} 6(486n^3 - 1068n^2 + 312n + 293), & \alpha = 1; \\ 6(9(2\sqrt{2}+3)n^3 + (2\sqrt{30} + 2\sqrt{15} + 3\sqrt{6} + 6\sqrt{5} + \\ 12\sqrt{3} - 60\sqrt{2} - 81)n^2 + 2(\sqrt{35} - 2\sqrt{30} + \sqrt{21} - \\ \sqrt{15} + 4\sqrt{10} + 3\sqrt{7} + 2\sqrt{6} - 12\sqrt{5} - 20\sqrt{3} \\ +30\sqrt{2} + 14)n + 2\sqrt{42} - 4\sqrt{35} + 4\sqrt{30} - 4\sqrt{21} - \\ 16\sqrt{10} - 12\sqrt{7} - 20\sqrt{6} + 24\sqrt{5} + 48\sqrt{3} - \\ 12\sqrt{2} + 39), & \alpha = \frac{1}{2}; \\ \frac{297675n^3 - 445655n^2 + 282283n - 40155}{529200}, & \alpha = -1; \\ \frac{3}{4}(2\sqrt{2}+3)n^3 + (\frac{4}{\sqrt{5}} + \frac{2}{\sqrt{3}} - 5\sqrt{2} + 2\sqrt{\frac{6}{5}} + \sqrt{\frac{2}{3}} + \\ \sqrt{\frac{3}{5}} - \frac{83}{12})n^2 + (\frac{461}{105}) + 6\sqrt{\frac{2}{5}} + \sqrt{\frac{3}{7}} - \sqrt{\frac{3}{5}} - \sqrt{\frac{2}{3}} - \\ 4\sqrt{\frac{6}{5}} + 5\sqrt{2} - \frac{5}{\sqrt{3}} - \frac{16}{\sqrt{5}} + \frac{4}{\sqrt{7}} + \frac{12}{\sqrt{35}})n - \frac{24}{\sqrt{35}} - \\ \frac{8}{\sqrt{7}} + \frac{16}{\sqrt{5}} + \frac{8}{\sqrt{3}} - \sqrt{2} + 4\sqrt{\frac{6}{5}} + 2\sqrt{\frac{6}{7}} - 2\sqrt{\frac{2}{3}} - \\ 2\sqrt{\frac{3}{7}} - 12\sqrt{\frac{2}{5}} + \frac{13}{14}, & \alpha = -\frac{1}{2}. \end{cases}$$

Proof. Let $\mathcal{G}_2$ be the Hex-Derived Cage network $(HDCN2(m,n))$ where $m = n \geq 5$. The edge set of $HDCN2(m,n)$ is divided into twenty partitions based on the degree of end vertices. Table 3 shows these edge partition of $HDCN2(m,n)$.

$$R_\alpha(\mathcal{G}_2) = \sum_{ab \in E(\mathcal{G})} (d(a)d(b))^\alpha$$

For $\alpha = 1$

$$R_1(\mathcal{G}_2) = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} deg(u) \cdot deg(v)$$

Using the edge partition from Table 3, we get
$R_1(\mathcal{G}_2) = 25|E_1(\mathcal{G}_2)| + 30|E_2(\mathcal{G}_2)| + 35|E_3(\mathcal{G}_2)| + 40|E_4(\mathcal{G}_2)| + 45|E_5(\mathcal{G}_2)| + 60|E_6(\mathcal{G}_2)| + 36|E_7(\mathcal{G}_2)| + 42|E_8(\mathcal{G}_2)| + 48|E_9(\mathcal{G}_2)| + 54|E_{10}(\mathcal{G}_2)| + 72|E_{11}(\mathcal{G}_2)| + 49|E_{12}(\mathcal{G}_2)| + 63|E_{13}(\mathcal{G}_2)| + 84|E_{14}(\mathcal{G}_2)| + 64|E_{15}(\mathcal{G}_2)| + 72|E_{16}(\mathcal{G}_2)| + 96|E_{17}(\mathcal{G}_2)| + 81|E_{18}(\mathcal{G}_2)| + 108|E_{19}(\mathcal{G}_2)| + 144|E_{20}(\mathcal{G}_2)|$

After simplification, we get

$$R_1(\mathcal{G}_2) = 6(486n^3 - 1068n^2 + 312n + 293)$$

For $\alpha = \frac{1}{2}$

$$R_{\frac{1}{2}}(\mathcal{G}_2) = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} \sqrt{d(a) \cdot d(b)}$$

Using edge partition from Table 3, we get
$R_{\frac{1}{2}}(\mathcal{G}_2) = 5|E_1(\mathcal{G}_2)| + \sqrt{30}|E_2(\mathcal{G}_2)| + \sqrt{35}|E_3(\mathcal{G}_2)| + 2\sqrt{10}|E_4(\mathcal{G}_2)| + 3\sqrt{5}|E_5(\mathcal{G}_2)| + 2\sqrt{15}|E_6(\mathcal{G}_2)| + 6|E_7(\mathcal{G}_2)| + \sqrt{42}|E_8(\mathcal{G}_2)| + 4\sqrt{3}|E_9(\mathcal{G}_2)| + 3\sqrt{6}|E_{10}(\mathcal{G}_2)| + 6\sqrt{2}|E_{11}(\mathcal{G}_2)| + 7|E_{12}(\mathcal{G}_2)| + 3\sqrt{7}|E_{13}(\mathcal{G}_2)| + 2\sqrt{21}|E_{14}(\mathcal{G}_2)| + 8|E_{15}(\mathcal{G}_2)| + 6\sqrt{2}|E_{16}(\mathcal{G}_2)| + 4\sqrt{6}|E_{17}(\mathcal{G}_2)| + 9|E_{18}(\mathcal{G}_2)| + 6\sqrt{3}|E_{19}(\mathcal{G}_2)| + 12|E_{20}(\mathcal{G}_2)|$

After simplification, we get
$R_{\frac{1}{2}}(\mathcal{G}_2) = 6(9(2\sqrt{2}+3)n^3 + (2\sqrt{30} + 2\sqrt{15} + 3\sqrt{6} + 6\sqrt{5} + 12\sqrt{3} - 60\sqrt{2} - 81)n^2 + 2(\sqrt{35} - 2\sqrt{30} + \sqrt{21} - \sqrt{15} + 4\sqrt{10} + 3\sqrt{7} + 2\sqrt{6} - 12\sqrt{5} - 20\sqrt{3} + 30\sqrt{2} + 14)n + 2\sqrt{42} - 4\sqrt{35} + 4\sqrt{30} - 4\sqrt{21} - 16\sqrt{10} - 12\sqrt{7} - 20\sqrt{6} + 24\sqrt{5} + 48\sqrt{3} - 12\sqrt{2} + 39)$

For $\alpha = -1$

$$R_{-1}(\mathcal{G}_2) = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} \frac{1}{d(a) \cdot d(b)}$$

$R_{-1}(\mathcal{G}_2) = \frac{1}{25}|E_1(\mathcal{G}_2)| + \frac{1}{30}|E_2(\mathcal{G}_2)| + \frac{1}{35}|E_3(\mathcal{G}_2)| + \frac{1}{40}|E_4(\mathcal{G}_2)| + \frac{1}{45}|E_5(\mathcal{G}_2)| + \frac{1}{60}|E_6(\mathcal{G}_2)| + \frac{1}{36}|E_7(\mathcal{G}_2)| + \frac{1}{42}|E_8(\mathcal{G}_2)| + \frac{1}{48}|E_9(\mathcal{G}_2)| + \frac{1}{54}|E_{10}(\mathcal{G}_2)| + \frac{1}{72}|E_{11}(\mathcal{G}_2)| + \frac{1}{49}|E_{12}(\mathcal{G}_2)| + \frac{1}{63}|E_{13}(\mathcal{G}_2)| + \frac{1}{84}|E_{14}(\mathcal{G}_2)| + \frac{1}{64}|E_{15}(\mathcal{G}_2)| + \frac{1}{72}|E_{16}(\mathcal{G}_2)| + \frac{1}{96}|E_{17}(\mathcal{G}_2)| + \frac{1}{81}|E_{18}(\mathcal{G}_2)| + \frac{1}{108}|E_{19}(\mathcal{G}_2)| + \frac{1}{144}|E_{20}(\mathcal{G}_2)|$

After simplification, we get

$$R_{-1}(\mathcal{G}_2) = \frac{297675n^3 - 445655n^2 + 282283n - 40155}{529200}$$

For $\alpha = -\frac{1}{2}$

$$R_{-\frac{1}{2}}(\mathcal{G}_2) = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} \frac{1}{\sqrt{d(a) \cdot d(b)}}$$

$R_{-\frac{1}{2}}(\mathcal{G}_2) = \frac{1}{5}|E_1(\mathcal{G}_2)| + \frac{1}{\sqrt{30}}|E_2(\mathcal{G}_2)| + \frac{1}{\sqrt{35}}|E_3(\mathcal{G}_2)| + \frac{1}{2\sqrt{10}}|E_4(\mathcal{G}_2)| + \frac{1}{3\sqrt{5}}|E_5(\mathcal{G}_2)| + \frac{1}{2\sqrt{15}}|E_6(\mathcal{G}_2)| + \frac{1}{6}|E_7(\mathcal{G}_2)| + \frac{1}{\sqrt{42}}|E_8(\mathcal{G}_2)| + \frac{1}{4\sqrt{3}}|E_9(\mathcal{G}_2)| + \frac{1}{3\sqrt{6}}|E_{10}(\mathcal{G}_2)| + \frac{1}{6\sqrt{2}}|E_{11}(\mathcal{G}_2)| + \frac{1}{7}|E_{12}(\mathcal{G}_2)| + \frac{1}{3\sqrt{7}}|E_{13}(\mathcal{G}_2)| + \frac{1}{2\sqrt{21}}|E_{14}(\mathcal{G}_2)| + \frac{1}{8}|E_{15}(\mathcal{G}_2)| + \frac{1}{6\sqrt{2}}|E_{16}(\mathcal{G}_2)| + \frac{1}{4\sqrt{6}}|E_{17}(\mathcal{G}_2)| + \frac{1}{9}|E_{18}(\mathcal{G}_2)| + \frac{1}{6\sqrt{3}}|E_{19}(\mathcal{G}_2)| + \frac{1}{12}|E_{20}(\mathcal{G}_2)|$

After simplification, we get

$R_{-\frac{1}{2}}(\mathcal{G}_2) = \frac{3}{4}(2\sqrt{2}+3)n^3 + (\frac{4}{\sqrt{5}} + \frac{2}{\sqrt{3}} - 5\sqrt{2} + 2\sqrt{\frac{6}{5}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{5}} - \frac{83}{12})n^2 + (\frac{461}{105}) + 6\sqrt{\frac{2}{5}} + \sqrt{\frac{3}{7}} - \sqrt{\frac{3}{5}} - \sqrt{\frac{2}{3}} - 4\sqrt{\frac{6}{5}} + 5\sqrt{2} - \frac{5}{\sqrt{3}} - \frac{16}{\sqrt{5}} + \frac{4}{\sqrt{7}} + \frac{12}{\sqrt{35}})n - \frac{24}{\sqrt{35}} - \frac{8}{\sqrt{7}} + \frac{16}{\sqrt{5}} + \frac{8}{\sqrt{3}} - \sqrt{2} + 4\sqrt{\frac{6}{5}} + 2\sqrt{\frac{6}{7}} - 2\sqrt{\frac{2}{3}} - 2\sqrt{\frac{3}{7}} - 12\sqrt{\frac{2}{5}} + \frac{13}{14}$ □

In this theorem, we find the first Zagreb index for hex-derived cage network $\mathcal{G}_2$.

Theorem 5. *For Hex-Derived Cage Network ($\mathcal{G}_2$), the first Zagreb index is equal to*

$$M_1(\mathcal{G}_2) = 12(54n^3 - 109n^2 + 34n + 21)$$

Proof. Let $\mathcal{G}_2$ be the Hex-Derived Cage Network ($\mathcal{G}_2$). Using the edge partition from Table 3, we have

$$M_1(\mathcal{G}_2) = \sum_{ab \in E(\mathcal{G})} (d(a) + d(b)) = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} (d(a) + d(b))$$

$M_1(\mathcal{G}_2) = 10|E_1(\mathcal{G}_2)| + 11|E_2(\mathcal{G}_2)| + 12|E_3(\mathcal{G}_2)| + 13|E_4(\mathcal{G}_2)| + 14|E_5(\mathcal{G}_2)| + 17|E_6(\mathcal{G}_2)| + 12|E_7(\mathcal{G}_2)| + 13|E_8(\mathcal{G}_2)| + 14|E_9(\mathcal{G}_2)| + 15|E_{10}(\mathcal{G}_2)| + 18|E_{11}(\mathcal{G}_2)| + 14|E_{12}(\mathcal{G}_2)| + 16|E_{13}(\mathcal{G}_2)| + 19|E_{14}(\mathcal{G}_2)| + 16|E_{15}(\mathcal{G}_2)| + 17|E_{16}(\mathcal{G}_2)| + 20|E_{17}(\mathcal{G}_2)| + 18|E_{18}(\mathcal{G}_2)| + 21|E_{19}(\mathcal{G}_2)| + 24|E_{20}(\mathcal{G}_2)|$.

After simplification, we get

$$M_1(\mathcal{G}_2) = 12(54n^3 - 109n^2 + 34n + 21)$$

□

In below theorem, we calculate the ABC, GA, ABC_4 and GA_5 indices of Hex-Derived Cage Network $\mathcal{G}_2$.

Theorem 6. *Let $\mathcal{G}_2$ be the Hex-Derived Cage Network for every positive integer $m = n \geq 5$; then we have*

- $ABC(\mathcal{G}_2) = 2\sqrt{2}(3n^3 - 10n^2 + 8n + 2) + \frac{1}{2}\sqrt{\frac{11}{2}}n(3n^2 - 11n + 10) + \sqrt{\frac{5}{2}}n(3n^2 - 11n + 10) + 6\sqrt{\frac{6}{5}}(n^2 - 2n + 2) + \frac{16}{3}(n^2 - 5n + 6) + 3n(n - 1) + \frac{12\sqrt{2}n}{5} + 6n + \sqrt{\frac{26}{3}}(n - 2)^2 + 2\sqrt{\frac{19}{3}}(n - 2)^2 + 8\sqrt{\frac{3}{5}}(n - 2)^2 + \sqrt{30}(n - 2) + \sqrt{\frac{51}{7}}(n - 2) + 6\sqrt{\frac{22}{5}}(n - 2) + 6\sqrt{3}(n - 2) + 4\sqrt{2}(n - 2) + 12\sqrt{\frac{2}{7}}(n - 2) + 3\sqrt{\frac{7}{2}}(n - 3) + \frac{12}{7}\sqrt{3}(n - 3) + 2\sqrt{\frac{66}{7}}$

- $GA(\mathcal{G}_2) = 4\sqrt{2}(3n^3 - 10n^2 + 8n + 2) + \frac{24}{11}\sqrt{30}(n^2 - 2n + 2) + 18n^3 - 54n^2 + \frac{24}{17}\sqrt{15}(n - 1)n + \frac{48\sqrt{3}n}{7} + 12n + \frac{12}{5}\sqrt{6}(n - 2)^2 + \frac{36}{7}\sqrt{5}(n - 2)^2 + \frac{48}{7\sqrt{3}}(n - 2)^2 + 2\sqrt{35}(n - 2) + \frac{24}{19}\sqrt{21}(n - 2) + \frac{96}{13}\sqrt{10}(n - 2) + \frac{9}{2}\sqrt{7}(n - 2) + \frac{48}{5}\sqrt{6}(n - 2) + \frac{144}{17}\sqrt{2}(n - 2) + 12(n - 3) + \frac{24\sqrt{42}}{13} + 54.$

- $ABC_4(\mathcal{G}_2) = \frac{1}{18}\sqrt{\frac{107}{2}}n(3n^2 - 17n + 24) + \frac{1}{9}\sqrt{\frac{53}{2}}(3n^3 - 13n^2 + 16n - 10) + \frac{4}{9}\sqrt{5}n(3n^2 - 15n + 19) + \frac{2}{7}\sqrt{\frac{534}{7}}(n^2 - 5n + 6) + 3\sqrt{\frac{102}{101}}(n^2 - 5n + 6) + 2\sqrt{\frac{69}{101}}(n^2 - 5n + 6) + \frac{24}{7}\sqrt{\frac{37}{101}}(n^2 - 5n + 6) + 3\sqrt{\frac{94}{707}}(n^2 - 5n + 6) + \frac{60}{101}\sqrt{2}(n^2 - 6n + 8) + \frac{15}{19}\sqrt{6}(n^2 - 9n + 20) + 7\sqrt{\frac{2}{95}}(3n - 8) + \frac{1}{7}\sqrt{\frac{202}{3}}(n - 2)^2 + \frac{4}{7}\sqrt{\frac{258}{13}}(n - 2) + \frac{4}{13}\sqrt{19}(n - 2) + \frac{5}{3}\sqrt{2}(n - 2) + \frac{12}{7}\sqrt{\frac{142}{95}}(n - 2) + \sqrt{\frac{67}{95}}(n - 2) + 12\sqrt{\frac{194}{9595}}(n - 2) + 24\sqrt{\frac{11}{1235}}(n - 2) + \sqrt{\frac{129}{5}}(n - 3) + 2\sqrt{\frac{302}{33}}(n - 3) + \frac{4}{3}\sqrt{\frac{205}{33}}(n - 3) + \sqrt{\frac{274}{55}}(n - 3) + 2\sqrt{\frac{145}{33}}(n - 3) + \frac{3}{7}\sqrt{\frac{123}{19}}(n - 4)^2 + 2\sqrt{\frac{174}{133}}(n - 4)^2 + 30\sqrt{\frac{7}{1919}}(n - 4)^2 + \frac{4}{35}\sqrt{366}(n - 4) + \frac{2}{5}\sqrt{\frac{447}{19}}(n - 4) + \frac{6}{5}\sqrt{\frac{206}{13}}(n - 4) + \frac{2}{3}\sqrt{14}(n - 4) + 4\sqrt{\frac{46}{35}}(n - 4) + 2\sqrt{\frac{42}{37}}(n - 4) + \frac{12}{5}\sqrt{\frac{46}{65}}(n - 4) + \frac{24}{5}\sqrt{\frac{58}{101}}(n - 4) + 3\sqrt{\frac{37}{65}}(n - 4) + 6\sqrt{\frac{2}{13}}(n - 4) + 6\sqrt{\frac{38}{259}}(n - 4) + 6\sqrt{\frac{2}{19}}(n - 4) + 6\sqrt{\frac{334}{3515}}(n - 4) + 6\sqrt{\frac{346}{3737}}(n - 4) + \frac{66}{7}\sqrt{\frac{2}{37}}(n - 4) + 72\sqrt{\frac{2}{715}}(n - 4) + \frac{56}{33}(n - 4) + \frac{6}{37}\sqrt{146}(n - 5) + \frac{8}{25}\sqrt{37}(n - 5) + \frac{96}{65}\sqrt{2}(n - 5) + \frac{32}{\sqrt{37}} + 2\sqrt{\frac{249}{37}} + \frac{8\sqrt{6}}{5} + 2\sqrt{\frac{202}{35}} + 2\sqrt{\frac{109}{21}} + \frac{8\sqrt{\frac{106}{35}}}{3} + \sqrt{\frac{66}{23}} + \frac{183\sqrt{2}}{37} + 4\sqrt{\frac{678}{511}} + 2\sqrt{\frac{70}{53}} + \frac{48\sqrt{\frac{30}{73}}}{7} + 8\sqrt{\frac{14}{37}} + 8\sqrt{\frac{330}{949}} + 8\sqrt{\frac{134}{511}} + 8\sqrt{\frac{6}{23}} + 12\sqrt{\frac{127}{851}} + 6\sqrt{\frac{3}{23}} + \frac{32\sqrt{\frac{10}{77}}}{3} + 12\sqrt{\frac{290}{2701}} + 12\sqrt{\frac{17}{161}} + 12\sqrt{\frac{21}{253}} + 24\sqrt{\frac{30}{689}} + 24\sqrt{\frac{2}{65}} + 12\sqrt{\frac{146}{5035}} + 24\sqrt{\frac{166}{6935}} + 16\sqrt{\frac{6}{265}} + 48\sqrt{\frac{31}{3869}} + 48\sqrt{\frac{43}{7373}}.$

- $GA_5(\mathcal{G}_2) = 3n(3n^2 - 17n + 24) + 4\sqrt{2}n(3n^2 - 15n + 19) + \frac{12}{143}\sqrt{4242}(n^2 - 5n + 6) + \frac{28}{25}\sqrt{101}(n^2 - 5n + 6) + \frac{24}{13}\sqrt{42}(n^2 - 5n + 6) + \frac{2424}{209}(n^2 - 5n + 6) + \frac{36}{149}\sqrt{570}(3n - 8) + 9n^3 - 21n^2 - 90n + \frac{252}{103}\sqrt{6}(n - 2)^2 + \frac{6}{49}\sqrt{9595}(n - 2) + \frac{12}{67}\sqrt{3705}(n - 2) + \frac{72}{203}\sqrt{285}(n - 2) + \frac{7}{6}\sqrt{95}(n - 2) + \frac{21}{11}\sqrt{39}(n - 2) + \frac{144}{17}\sqrt{2}(n - 2) + \frac{144}{139}\sqrt{110}(n - 3) + \frac{72}{17}\sqrt{66}(n - 3) + \frac{9120\sqrt{33}(n-3)}{1127} + \frac{48}{11}\sqrt{30}(n - 3) + \frac{16}{59}\sqrt{1919}(n - 4)^2 + \frac{24}{59}\sqrt{798}(n - 4)^2 + \frac{168}{125}\sqrt{19}(n - 4)^2 + \frac{24}{175}\sqrt{7474}(n - 4) + \frac{24}{169}\sqrt{7030}(n - 4) + \frac{24}{113}\sqrt{2886}(n - 4) + \frac{8}{25}\sqrt{1406}(n - 4) + \frac{12}{29}\sqrt{777}(n - 4) + \frac{36}{41}\sqrt{715}(n - 4) + \frac{15}{11}\sqrt{303}(n - 4) + \frac{1350}{791}\sqrt{195}(n - 4) + \frac{9}{8}\sqrt{111}(n - 4) + \frac{56}{41}\sqrt{74}(n - 4) + \frac{240}{151}\sqrt{57}(n - 4) + \frac{1496}{217}\sqrt{26}(n - 4) + \frac{80}{13}\sqrt{14}(n - 4) + \frac{210}{31}\sqrt{3}(n - 4) + 12(n - 4) + 36(n - 5) + \frac{8\sqrt{7373}}{29} + \frac{2\sqrt{6935}}{7} + \frac{16\sqrt{5402}}{49} + \frac{6\sqrt{5035}}{37} + \frac{8\sqrt{3869}}{21} + \frac{48\sqrt{3066}}{115} + \frac{3\sqrt{2847}}{7} + \frac{12\sqrt{2067}}{23} + \frac{32\sqrt{851}}{43} + \frac{18\sqrt{511}}{17} + \frac{9\sqrt{455}}{8} + \frac{72\sqrt{318}}{107} + \frac{36\sqrt{265}}{49} + \frac{36\sqrt{259}}{25} + \frac{288\sqrt{253}}{191} + \frac{60\sqrt{219}}{37} + \frac{72\sqrt{185}}{41} + \frac{288\sqrt{161}}{155} + \frac{144\sqrt{138}}{73} + \frac{144\sqrt{115}}{137} + \frac{192\sqrt{111}}{85} + \frac{8\sqrt{77}}{3} + \frac{168\sqrt{73}}{61} + \frac{288\sqrt{70}}{103} + \frac{732\sqrt{69}}{175} + 4\sqrt{35} + \frac{192\sqrt{21}}{37} + 258.$

Proof. Using the edge partition from Table 3, we find ABC as

$$ABC(\mathcal{G}_2) = \sum_{ab \in E(\mathcal{G})} \sqrt{\frac{d(a) + d(b) - 2}{d(a) \cdot d(b)}} = \sum_{j=1}^{20} \sum_{ab \in E_j(\mathcal{G})} \sqrt{\frac{d(a) + d(b) - 2}{d(a) \cdot d(b)}}$$

$ABC(\mathcal{G}_2) = \frac{2\sqrt{2}}{5}|E_1(\mathcal{G}_2)| + \sqrt{\frac{3}{10}}|E_2(\mathcal{G}_2)| + \sqrt{\frac{2}{7}}|E_3(\mathcal{G}_2)| + \frac{\sqrt{11}}{2\sqrt{10}}|E_4(\mathcal{G}_2)| + \frac{2\sqrt{3}}{3\sqrt{5}}|E_5(\mathcal{G}_2)| + \frac{1}{2}|E_6(\mathcal{G}_2)| + \sqrt{\frac{5}{18}}|E_7(\mathcal{G}_2)| + \sqrt{\frac{11}{42}}|E_8(\mathcal{G}_2)| + \frac{1}{2}|E_9(\mathcal{G}_2)| + \sqrt{\frac{13}{54}}|E_{10}(\mathcal{G}_2)| + \frac{\sqrt{2}}{3}|E_{11}(\mathcal{G}_2)| + \frac{2\sqrt{3}}{7}|E_{12}(\mathcal{G}_2)| +$

$\sqrt{\frac{14}{63}}|E_{13}(\mathcal{G}_2)| + \sqrt{\frac{17}{84}}|E_{14}(\mathcal{G}_2)| + \sqrt{\frac{7}{32}}|E_{15}(\mathcal{G}_2)| + \sqrt{\frac{5}{24}}|E_{16}(\mathcal{G}_2)| + \sqrt{\frac{3}{16}}|E_{17}(\mathcal{G}_2)| + \sqrt{\frac{16}{81}}|E_{18}(\mathcal{G}_2)| + \sqrt{\frac{19}{108}}|E_{19}(\mathcal{G}_2)| + \sqrt{\frac{11}{72}}|E_{20}(\mathcal{G}_2)|$.

After simplification, we get

$ABC(\mathcal{G}_2)=2\sqrt{2}(3n^3 - 10n^2 + 8n + 2) + \frac{1}{2}\sqrt{\frac{11}{2}}n(3n^2 - 11n + 10) + \sqrt{\frac{5}{2}}n(3n^2 - 11n + 10) + 6\sqrt{\frac{6}{5}}(n^2 - 2n + 2) + \frac{16}{3}(n^2 - 5n + 6) + 3n(n-1) + \frac{12\sqrt{2}n}{5} + 6n + \sqrt{\frac{26}{3}}(n-2)^2 + 2\sqrt{\frac{19}{3}}(n-2)^2 + 8\sqrt{\frac{3}{5}}(n-2)^2 + \sqrt{30}(n-2) + \sqrt{\frac{51}{7}}(n-2) + 6\sqrt{\frac{22}{5}}(n-2) + 6\sqrt{3}(n-2) + 4\sqrt{2}(n-2) + 12\sqrt{\frac{2}{7}}(n-2) + 3\sqrt{\frac{7}{2}}(n-3) + \frac{12}{7}\sqrt{3}(n-3) + 2\sqrt{\frac{66}{7}}$

Using the edge partition from Table 3, we find GA as

$GA(\mathcal{G}_2) = 1|E_1(\mathcal{G}_2)| + \frac{2\sqrt{30}}{11}|E_2(\mathcal{G}_2)| + \frac{\sqrt{35}}{6}|E_3(\mathcal{G}_2)| + \frac{4\sqrt{10}}{13}|E_4(\mathcal{G}_2)| + \frac{3\sqrt{5}}{7}|E_5(\mathcal{G}_2)| + \frac{4\sqrt{15}}{17}|E_6(\mathcal{G}_2)| + 1|E_7(\mathcal{G}_2)| + \frac{2\sqrt{42}}{13}|E_8(\mathcal{G}_2)| + \frac{2\sqrt{12}}{7}|E_9(\mathcal{G}_2)| + \frac{2\sqrt{54}}{15}|E_{10}(\mathcal{G}_2)| + \frac{2\sqrt{2}}{3}|E_{11}(\mathcal{G}_2)| + 1|E_{12}(\mathcal{G}_2)| + \frac{3\sqrt{7}}{8}|E_{13}(\mathcal{G}_2)| + \frac{4\sqrt{21}}{19}|E_{14}(\mathcal{G}_2)| + 1|E_{15}(\mathcal{G}_2)| + \frac{12\sqrt{2}}{17}|E_{16}(\mathcal{G}_2)| + \frac{\sqrt{23}}{5}|E_{17}(\mathcal{G}_2)| + 1|E_{18}(\mathcal{G}_2)| + \frac{2\sqrt{3}}{7}|E_{19}(\mathcal{G}_2)| + 1|E_{20}(\mathcal{G}_2)|$.

After simplification, we get

$GA(\mathcal{G}_2) =4\sqrt{2}(3n^3 - 10n^2 + 8n + 2) + \frac{24}{11}\sqrt{30}(n^2 - 2n + 2) + 18n^3 - 54n^2 + \frac{24}{17}\sqrt{15}(n-1)n + \frac{48\sqrt{3}n}{7} + 12n + \frac{12}{5}\sqrt{6}(n-2)^2 + \frac{36}{7}\sqrt{5}(n-2)^2 + \frac{48}{7\sqrt{3}}(n-2)^2 + 2\sqrt{35}(n-2) + \frac{24}{19}\sqrt{21}(n-2) + \frac{96}{13}\sqrt{10}(n-2) + \frac{9}{2}\sqrt{7}(n-2) + \frac{48}{5}\sqrt{6}(n-2) + \frac{144}{17}\sqrt{2}(n-2) + 12(n-3) + \frac{24\sqrt{42}}{13} + 54.$

If we suppose an edge partition based on degree sum of neighbors of end vertices, then the edge set $E(\mathcal{G}_2)$ can be divided into seventy six edge partition $E_j(\mathcal{G}_2)$, $21 \leq j \leq 96$. Table 4 shows these edge partitions.

From Equation (7), we get

$$ABC_4(\mathcal{G}_2) = \sum_{ab\in E(\mathcal{G})}\sqrt{\frac{S(a)+S(b)-2}{S(a)S(b)}} = \sum_{j=21}^{96}\sum_{ab\in E_j(\mathcal{G})}\sqrt{\frac{S(a)+S(b)-2}{S(a)S(b)}}.$$

Using the edge partition from Table 3, we get

$ABC_4(\mathcal{G}_2) = \sqrt{\frac{72}{1369}}|E_{21}(\mathcal{G}_2)| + \frac{4}{\sqrt{333}}|E_{22}(\mathcal{G}_2)| + \sqrt{\frac{83}{1776}}|E_{23}(\mathcal{G}_2)| + \sqrt{\frac{98}{2321}}|E_{24}(\mathcal{G}_2)| + \sqrt{\frac{127}{3404}}|E_{25}(\mathcal{G}_2)| + \sqrt{\frac{74}{1443}}|E_{26}(\mathcal{G}_2)| + \sqrt{\frac{86}{1911}}|E_{27}(\mathcal{G}_2)| + \sqrt{\frac{90}{2067}}|E_{28}(\mathcal{G}_2)| + \sqrt{\frac{42}{2106}}|E_{29}(\mathcal{G}_2)| + \sqrt{\frac{110}{2847}}|E_{30}(\mathcal{G}_2)| + \sqrt{\frac{111}{2886}}|E_{31}(\mathcal{G}_2)| + \sqrt{\frac{44}{1235}}|E_{32}(\mathcal{G}_2)| + \sqrt{\frac{43}{940}}|E_{33}(\mathcal{G}_2)| + \sqrt{\frac{101}{2520}}|E_{34}(\mathcal{G}_2)| + \sqrt{\frac{103}{2600}}|E_{35}(\mathcal{G}_2)| + \sqrt{\frac{137}{3960}}|E_{36}(\mathcal{G}_2)| + \sqrt{\frac{89}{2058}}|E_{37}(\mathcal{G}_2)| + \sqrt{\frac{113}{3066}}|E_{38}(\mathcal{G}_2)| + \sqrt{\frac{17}{1554}}|E_{39}(\mathcal{G}_2)| + \sqrt{\frac{115}{3150}}|E_{40}(\mathcal{G}_2)| + \sqrt{\frac{29}{798}}|E_{41}(\mathcal{G}_2)| + \sqrt{\frac{141}{4242}}|E_{42}(\mathcal{G}_2)| + \sqrt{\frac{96}{2385}}|E_{43}(\mathcal{G}_2)| + \sqrt{\frac{106}{2835}}|E_{44}(\mathcal{G}_2)| + \sqrt{\frac{27}{558}}|E_{45}(\mathcal{G}_2)| + \frac{5}{\sqrt{648}}|E_{46}(\mathcal{G}_2)| + \sqrt{\frac{109}{3024}}|E_{47}(\mathcal{G}_2)| + \sqrt{\frac{111}{3120}}|E_{48}(\mathcal{G}_2)| + \sqrt{\frac{69}{228}}|E_{49}(\mathcal{G}_2)| + \sqrt{\frac{145}{4752}}|E_{50}(\mathcal{G}_2)| + \sqrt{\frac{101}{2646}}|E_{51}(\mathcal{G}_2)| + \sqrt{\frac{120}{3577}}|E_{52}(\mathcal{G}_2)| + \frac{11}{\sqrt{3626}}|E_{53}(\mathcal{G}_2)| + \sqrt{\frac{122}{3675}}|E_{54}(\mathcal{G}_2)| + \sqrt{\frac{123}{3724}}|E_{55}(\mathcal{G}_2)| + \sqrt{\frac{142}{4655}}|E_{56}(\mathcal{G}_2)| + \sqrt{\frac{148}{4949}}|E_{57}(\mathcal{G}_2)| + \sqrt{\frac{105}{2862}}|E_{58}(\mathcal{G}_2)| + \sqrt{\frac{124}{3869}}|E_{59}(\mathcal{G}_2)| + \sqrt{\frac{146}{5035}}|E_{60}(\mathcal{G}_2)| + \sqrt{\frac{53}{1458}}|E_{61}(\mathcal{G}_2)| + \sqrt{\frac{63}{1998}}|E_{62}(\mathcal{G}_2)| + \sqrt{\frac{105}{621}}|E_{63}(\mathcal{G}_2)| + \sqrt{\frac{147}{5130}}|E_{64}(\mathcal{G}_2)| + \sqrt{\frac{151}{5346}}|E_{65}(\mathcal{G}_2)| + \sqrt{\frac{153}{5454}}|E_{66}(\mathcal{G}_2)| + \sqrt{\frac{20}{729}}|E_{67}(\mathcal{G}_2)| + \sqrt{\frac{126}{4095}}|E_{68}(\mathcal{G}_2)| + \sqrt{\frac{134}{4599}}|E_{69}(\mathcal{G}_2)| + \sqrt{\frac{153}{5796}}|E_{70}(\mathcal{G}_2)| + \sqrt{\frac{160}{6237}}|E_{71}(\mathcal{G}_2)| + \sqrt{\frac{128}{4225}}|E_{72}(\mathcal{G}_2)| + \sqrt{\frac{46}{1625}}|E_{73}(\mathcal{G}_2)| + \sqrt{\frac{54}{2145}}|E_{74}(\mathcal{G}_2)| + \sqrt{\frac{145}{5402}}|E_{75}(\mathcal{G}_2)| + \sqrt{\frac{116}{5475}}|E_{76}(\mathcal{G}_2)| + \sqrt{\frac{166}{6935}}|E_{77}(\mathcal{G}_2)| + \sqrt{\frac{172}{7373}}|E_{78}(\mathcal{G}_2)| + \sqrt{\frac{73}{2738}}|E_{79}(\mathcal{G}_2)| + \sqrt{\frac{37}{1406}}|E_{80}(\mathcal{G}_2)| + \sqrt{\frac{167}{7030}}|E_{81}(\mathcal{G}_2)| + \sqrt{\frac{173}{7474}}|E_{82}(\mathcal{G}_2)| + \sqrt{\frac{148}{5625}}|E_{83}(\mathcal{G}_2)| + \sqrt{\frac{149}{5700}}|E_{84}(\mathcal{G}_2)| + \sqrt{\frac{58}{2525}}|E_{85}(\mathcal{G}_2)| + \sqrt{\frac{75}{2888}}|E_{86}(\mathcal{G}_2)| + \sqrt{\frac{175}{7676}}|E_{87}(\mathcal{G}_2)| + \sqrt{\frac{189}{9108}}|E_{88}(\mathcal{G}_2)| + \sqrt{\frac{99}{4948}}|E_{89}(\mathcal{G}_2)| + \sqrt{\frac{194}{9595}}|E_{90}(\mathcal{G}_2)| +$

$\sqrt{\frac{201}{10260}}|E_{91}(\mathcal{G}_2)| + \frac{14}{\sqrt{9801}}|E_{92}(\mathcal{G}_2)| + \sqrt{\frac{205}{10692}}|E_{93}(\mathcal{G}_2)| + \sqrt{\frac{200}{10201}}|E_{94}(\mathcal{G}_2)| + \sqrt{\frac{207}{10908}}|E_{95}(\mathcal{G}_2)| + \sqrt{\frac{3}{162}}|E_{96}(\mathcal{G}_2)|.$

After simplification, we have

$ABC_4(\mathcal{G}_2) = \frac{1}{18}\sqrt{\frac{107}{2}}n(3n^2 - 17n + 24) + \frac{1}{9}\sqrt{\frac{53}{2}}(3n^3 - 13n^2 + 16n - 10) + \frac{4}{9}\sqrt{5}n(3n^2 - 15n + 19) + \frac{2}{7}\sqrt{\frac{534}{7}}(n^2 - 5n + 6) + 3\sqrt{\frac{102}{101}}(n^2 - 5n + 6) + 2\sqrt{\frac{69}{101}}(n^2 - 5n + 6) + \frac{24}{7}\sqrt{\frac{37}{101}}(n^2 - 5n + 6) + 3\sqrt{\frac{94}{707}}(n^2 - 5n + 6) + \frac{60}{101}\sqrt{2}(n^2 - 6n + 8) + \frac{15}{19}\sqrt{6}(n^2 - 9n + 20) + 7\sqrt{\frac{2}{95}}(3n - 8) + \frac{1}{7}\sqrt{\frac{202}{3}}(n - 2)^2 + \frac{4}{7}\sqrt{\frac{258}{13}}(n - 2) + \frac{4}{13}\sqrt{19}(n - 2) + \frac{5}{3}\sqrt{2}(n - 2) + \frac{12}{7}\sqrt{\frac{142}{95}}(n - 2) + \sqrt{\frac{67}{95}}(n - 2) + 12\sqrt{\frac{194}{9595}}(n - 2) + 24\sqrt{\frac{11}{1235}}(n - 2) + \sqrt{\frac{129}{5}}(n - 3) + 2\sqrt{\frac{302}{33}}(n - 3) + \frac{4}{3}\sqrt{\frac{205}{33}}(n - 3) + \sqrt{\frac{274}{55}}(n - 3) + 2\sqrt{\frac{145}{33}}(n - 3) + \frac{3}{7}\sqrt{\frac{123}{19}}(n - 4)^2 + 2\sqrt{\frac{174}{133}}(n - 4)^2 + 30\sqrt{\frac{7}{1919}}(n - 4)^2 + \frac{4}{35}\sqrt{366}(n - 4) + \frac{2}{5}\sqrt{\frac{447}{19}}(n - 4) + \frac{6}{5}\sqrt{\frac{206}{13}}(n - 4) + \frac{2}{3}\sqrt{14}(n - 4) + 4\sqrt{\frac{46}{35}}(n - 4) + 2\sqrt{\frac{42}{37}}(n - 4) + \frac{12}{5}\sqrt{\frac{46}{65}}(n - 4) + \frac{24}{5}\sqrt{\frac{58}{101}}(n - 4) + 3\sqrt{\frac{37}{65}}(n - 4) + 6\sqrt{\frac{2}{13}}(n - 4) + 6\sqrt{\frac{38}{259}}(n - 4) + 6\sqrt{\frac{2}{19}}(n - 4) + 6\sqrt{\frac{334}{3515}}(n - 4) + 6\sqrt{\frac{346}{3737}}(n - 4) + \frac{66}{7}\sqrt{\frac{2}{37}}(n - 4) + 72\sqrt{\frac{2}{715}}(n - 4) + \frac{56}{33}(n - 4) + \frac{6}{37}\sqrt{146}(n - 5) + \frac{8}{25}\sqrt{37}(n - 5) + \frac{96}{65}\sqrt{2}(n - 5) + \frac{32}{\sqrt{37}} + 2\sqrt{\frac{249}{37}} + \frac{8\sqrt{6}}{5} + 2\sqrt{\frac{202}{35}} + 2\sqrt{\frac{109}{21}} + \frac{8\sqrt{\frac{106}{35}}}{3} + \sqrt{\frac{66}{23}} + \frac{183\sqrt{2}}{37} + 4\sqrt{\frac{678}{511}} + 2\sqrt{\frac{70}{53}} + \frac{48\sqrt{\frac{30}{73}}}{7} + 8\sqrt{\frac{14}{37}} + 8\sqrt{\frac{330}{949}} + 8\sqrt{\frac{134}{511}} + 8\sqrt{\frac{6}{23}} + 12\sqrt{\frac{127}{851}} + 6\sqrt{\frac{3}{23}} + \frac{32\sqrt{\frac{10}{77}}}{3} + 12\sqrt{\frac{290}{2701}} + 12\sqrt{\frac{17}{161}} + 12\sqrt{\frac{21}{253}} + 24\sqrt{\frac{30}{689}} + 24\sqrt{\frac{2}{65}} + 12\sqrt{\frac{146}{5035}} + 24\sqrt{\frac{166}{6935}} + 16\sqrt{\frac{6}{265}} + 48\sqrt{\frac{31}{3869}} + 48\sqrt{\frac{43}{7373}}$

From Equation (8), we get

$$GA_5(\mathcal{G}_2) = \sum_{ab \in E(\mathcal{G})} \frac{2\sqrt{S(a)S(b)}}{(S(a)+S(b))} = \sum_{j=21}^{96} \sum_{ab \in E_j(\mathcal{G})} \frac{2\sqrt{S(a)S(b)}}{(S(a)S(b))}.$$

Table 3. Edge partition of Hex-Derived Cage network ($HDCN2$) based on degrees of end vertices of each edge.

(d_u, d_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges	(d_u, d_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges
$E_1 = (5,5)$	$6n$	$E_{11} = (6,12)$	$18n^3 - 60n^2 + 48n + 12$
$E_2 = (5,6)$	$12n^2 - 24n + 24$	$E_{12} = (7,7)$	$6n - 18$
$E_3 = (5,7)$	$2(6n - 12)$	$E_{13} = (7,9)$	$2(6n - 12)$
$E_4 = (5,8)$	$4(6n - 12)$	$E_{14} = (7,12)$	$6n - 12$
$E_5 = (5,9)$	$12n^2 - 48n + 48$	$E_{15} = (8,8)$	$2(6n - 18)$
$E_6 = (5,12)$	$6n^2 - 6n$	$E_{16} = (8,9)$	$2(6n - 12)$
$E_7 = (6,6)$	$9n^3 - 33n^2 + 30n$	$E_{17} = (8,12)$	$4(6n - 12)$
$E_8 = (6,7)$	12	$E_{18} = (9,9)$	$12n^2 - 60n + 72$
$E_9 = (6,8)$	$12n$	$E_{19} = (9,12)$	$12n^2 - 48n + 48$
$E_{10} = (6,9)$	$6n^2 - 24n + 24$	$E_{20} = (12,12)$	$9n^3 - 33n^2 + 30n$

Using the edge partition from Table 4, we get

$GA_5(\mathcal{G}_2) = 1|E_{21}(\mathcal{G}_2)| + \frac{\sqrt{1665}}{41}|E_{22}(\mathcal{G}_2)| + \frac{2\sqrt{1776}}{85}|E_{23}(\mathcal{G}_2)| + \frac{\sqrt{2331}}{50}|E_{24}(\mathcal{G}_2)| + \frac{2\sqrt{3404}}{129}|E_{25}(\mathcal{G}_2)| + \frac{\sqrt{1443}}{38}|E_{26}(\mathcal{G}_2)| + \frac{\sqrt{1911}}{44}|E_{27}(\mathcal{G}_2)| + \frac{\sqrt{2067}}{46}|E_{28}(\mathcal{G}_2)| + \frac{2\sqrt{2106}}{93}|E_{29}(\mathcal{G}_2)| + \frac{\sqrt{2847}}{56}|E_{30}(\mathcal{G}_2)| + \frac{2\sqrt{2886}}{113}|E_{31}(\mathcal{G}_2)| + \frac{\sqrt{3705}}{67}|E_{32}(\mathcal{G}_2)| + \frac{\sqrt{1920}}{44}|E_{33}(\mathcal{G}_2)| + \frac{2\sqrt{2520}}{103}|E_{34}(\mathcal{G}_2)| + \frac{2\sqrt{2600}}{105}|E_{35}(\mathcal{G}_2)| + \frac{2\sqrt{3960}}{139}|E_{36}(\mathcal{G}_2)| + \frac{2\sqrt{2058}}{91}|E_{37}(\mathcal{G}_2)| + \frac{2\sqrt{3066}}{115}|E_{38}(\mathcal{G}_2)| + \frac{\sqrt{3108}}{58}|E_{39}(\mathcal{G}_2)| + \frac{2\sqrt{3150}}{117}|E_{40}(\mathcal{G}_2)| + \frac{\sqrt{3192}}{59}|E_{41}(\mathcal{G}_2)| + \frac{2\sqrt{4242}}{143}|E_{42}(\mathcal{G}_2)| + \frac{\sqrt{2385}}{49}|E_{43}(\mathcal{G}_2)| + \frac{\sqrt{2835}}{54}|E_{44}(\mathcal{G}_2)| + \frac{2\sqrt{4140}}{137}|E_{45}(\mathcal{G}_2)| + \frac{\sqrt{2592}}{51}|E_{46}(\mathcal{G}_2)| + \frac{2\sqrt{3024}}{111}|E_{47}(\mathcal{G}_2)| + \frac{2\sqrt{3120}}{113}|E_{48}(\mathcal{G}_2)| + \frac{\sqrt{4416}}{70}|E_{49}(\mathcal{G}_2)| + \frac{2\sqrt{4752}}{147}|E_{50}(\mathcal{G}_2)| + \frac{2\sqrt{2646}}{103}|E_{51}(\mathcal{G}_2)| + \frac{\sqrt{3577}}{61}|E_{52}(\mathcal{G}_2)| + \frac{2\sqrt{3626}}{123}|E_{53}(\mathcal{G}_2)| + \frac{\sqrt{3675}}{62}|E_{54}(\mathcal{G}_2)| + \frac{2\sqrt{3724}}{125}|E_{55}(\mathcal{G}_2)| + \frac{\sqrt{4655}}{72}|E_{56}(\mathcal{G}_2)| + \frac{\sqrt{4949}}{75}|E_{57}(\mathcal{G}_2)| + \frac{2\sqrt{2862}}{107}|E_{58}(\mathcal{G}_2)| + \frac{\sqrt{3869}}{63}|E_{59}(\mathcal{G}_2)| + \frac{\sqrt{5035}}{74}|E_{60}(\mathcal{G}_2)| + 1|E_{61}(\mathcal{G}_2)| +$

$\frac{\sqrt{3996}}{64}|E_{62}(\mathcal{G}_2)| + \frac{\sqrt{4968}}{73}|E_{63}(\mathcal{G}_2)| + \frac{2\sqrt{5130}}{149}|E_{64}(\mathcal{G}_2)| + \frac{2\sqrt{5346}}{153}|E_{65}(\mathcal{G}_2)| + \frac{2\sqrt{5454}}{155}|E_{66}(\mathcal{G}_2)| + \frac{\sqrt{5832}}{81}|E_{67}(\mathcal{G}_2)| + \frac{\sqrt{4095}}{64}|E_{68}(\mathcal{G}_2)| + \frac{\sqrt{4599}}{68}|E_{69}(\mathcal{G}_2)| + \frac{2\sqrt{5796}}{155}|E_{70}(\mathcal{G}_2)| + \frac{\sqrt{6237}}{81}|E_{71}(\mathcal{G}_2)| + 1|E_{72}(\mathcal{G}_2)| + \frac{\sqrt{4875}}{70}|E_{73}(\mathcal{G}_2)| + \frac{\sqrt{6435}}{82}|E_{74}(\mathcal{G}_2)| + \frac{2\sqrt{5402}}{147}|E_{75}(\mathcal{G}_2)| + \frac{\sqrt{5475}}{74}|E_{76}(\mathcal{G}_2)| + \frac{\sqrt{6935}}{84}|E_{77}(\mathcal{G}_2)| + \frac{\sqrt{7373}}{87}|E_{78}(\mathcal{G}_2)| + 1|E_{79}(\mathcal{G}_2)| + \frac{\sqrt{5624}}{75}|E_{80}(\mathcal{G}_2)| + \frac{2\sqrt{7030}}{169}|E_{81}(\mathcal{G}_2)| + \frac{2\sqrt{7474}}{175}|E_{82}(\mathcal{G}_2)| + 1|E_{83}(\mathcal{G}_2)| + \frac{2\sqrt{5700}}{151}|E_{84}(\mathcal{G}_2)| + \frac{\sqrt{7575}}{88}|E_{85}(\mathcal{G}_2)| + 1|E_{86}(\mathcal{G}_2)| + \frac{2\sqrt{7676}}{177}|E_{87}(\mathcal{G}_2)| + \frac{2\sqrt{9108}}{191}|E_{88}(\mathcal{G}_2)| + \frac{\sqrt{9936}}{100}|E_{89}(\mathcal{G}_2)| + \frac{\sqrt{9595}}{98}|E_{90}(\mathcal{G}_2)| + \frac{2\sqrt{10260}}{203}|E_{91}(\mathcal{G}_2)| + 1|E_{92}(\mathcal{G}_2)| + \frac{2\sqrt{10692}}{207}|E_{93}(\mathcal{G}_2)| + 1|E_{94}(\mathcal{G}_2)| + \frac{2\sqrt{10908}}{209}|E_{95}(\mathcal{G}_2)| + 1|E_{96}(\mathcal{G}_2)|.$

After simplification, we get
$GA_5(\mathcal{G}_2) = 3n(3n^2-17n+24) + 4\sqrt{2}n(3n^2-15n+19) + \frac{12}{143}\sqrt{4242}(n^2-5n+6) + \frac{28}{25}\sqrt{101}(n^2-5n+6) + \frac{24}{13}\sqrt{42}(n^2-5n+6) + \frac{2424}{209}(n^2-5n+6) + \frac{36}{149}\sqrt{570}(3n-8) + 9n^3 - 21n^2 - 90n + \frac{252}{103}\sqrt{6}(n-2)^2 + \frac{6}{49}\sqrt{9595}(n-2) + \frac{12}{67}\sqrt{3705}(n-2) + \frac{72}{203}\sqrt{285}(n-2) + \frac{7}{6}\sqrt{95}(n-2) + \frac{21}{11}\sqrt{39}(n-2) + \frac{144}{17}\sqrt{2}(n-2) + \frac{144}{139}\sqrt{110}(n-3) + \frac{72}{17}\sqrt{66}(n-3) + \frac{9120\sqrt{33}(n-3)}{1127} + \frac{48}{11}\sqrt{30}(n-3) + \frac{16}{59}\sqrt{1919}(n-4)^2 + \frac{24}{59}\sqrt{798}(n-4)^2 + \frac{168}{125}\sqrt{19}(n-4)^2 + \frac{24}{175}\sqrt{7474}(n-4) + \frac{24}{169}\sqrt{7030}(n-4) + \frac{24}{113}\sqrt{2886}(n-4) + \frac{8}{25}\sqrt{1406}(n-4) + \frac{12}{29}\sqrt{777}(n-4) + \frac{36}{41}\sqrt{715}(n-4) + \frac{15}{11}\sqrt{303}(n-4) + \frac{1350}{791}\sqrt{195}(n-4) + \frac{9}{8}\sqrt{111}(n-4) + \frac{56}{41}\sqrt{74}(n-4) + \frac{240}{151}\sqrt{57}(n-4) + \frac{1496}{217}\sqrt{26}(n-4) + \frac{80}{13}\sqrt{14}(n-4) + \frac{210}{31}\sqrt{3}(n-4) + 12(n-4) + 36(n-5) + \frac{8\sqrt{7373}}{29} + \frac{2\sqrt{6935}}{7} + \frac{16\sqrt{5402}}{49} + \frac{6\sqrt{5035}}{37} + \frac{8\sqrt{3869}}{21} + \frac{48\sqrt{3066}}{115} + \frac{3\sqrt{2847}}{7} + \frac{12\sqrt{2067}}{23} + \frac{32\sqrt{851}}{43} + \frac{18\sqrt{511}}{17} + \frac{9\sqrt{455}}{8} + \frac{72\sqrt{318}}{107} + \frac{36\sqrt{265}}{49} + \frac{36\sqrt{259}}{25} + \frac{288\sqrt{253}}{191} + \frac{60\sqrt{219}}{37} + \frac{72\sqrt{185}}{41} + \frac{288\sqrt{161}}{155} + \frac{144\sqrt{138}}{73} + \frac{144\sqrt{115}}{137} + \frac{192\sqrt{111}}{85} + \frac{8\sqrt{77}}{3} + \frac{168\sqrt{73}}{61} + \frac{288\sqrt{70}}{103} + \frac{732\sqrt{69}}{175} + 4\sqrt{35} + \frac{192\sqrt{21}}{37} + 258.$ □

The Comparison graphs for ABC, GA, ABC_4 and GA_5 in case of a Hex Derived Cage networks $HDCN1(m,n)$ and $HDCN2(m,n)$ of dimension m and n are shown in Figures 3 and 4 respectively.

Table 4. Edge partition of Hex-Derived Cage network ($HDCN2$) based on sum of degrees of end vertices of each edge.

(S_u, S_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges	(S_u, S_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges
$E_{21} = (37,37)$	12	$E_{59} = (53,73)$	24
$E_{22} = (37,45)$	24	$E_{60} = (53,95)$	12
$E_{23} = (37,48)$	24	$E_{61} = (54,54)$	$9n^3 - 39n^2 + 48n - 30$
$E_{24} = (37,63)$	24	$E_{62} = (54,74)$	$2(6n-24)$
$E_{25} = (37,92)$	24	$E_{63} = (54,92)$	24
$E_{26} = (37,39)$	$6n-12$	$E_{64} = (54,95)$	$3(6n-16)$
$E_{27} = (39,49)$	$2(6n-12)$	$E_{65} = (54,99)$	$6(6n-18)$
$E_{28} = (39,53)$	24	$E_{66} = (54,101)$	$18n^2 - 90n + 108$
$E_{29} = (39,54)$	$2(6n-24)$	$E_{67} = (54,108)$	$18n^3 - 90n^2 + 114n$
$E_{30} = (39,73)$	24	$E_{68} = (63,65)$	24
$E_{31} = (39,74)$	$2(6n-24)$	$E_{69} = (63,73)$	24
$E_{32} = (39,95)$	$2(6n-12)$	$E_{70} = (63,92)$	24
$E_{33} = (40,48)$	$4(6n-18)$	$E_{71} = (63,99)$	24
$E_{34} = (40,63)$	24	$E_{72} = (65,65)$	$2(6n-30)$
$E_{35} = (40,65)$	$4(6n-24)$	$E_{73} = (65,75)$	$2(6n-24)$
$E_{36} = (40,99)$	$2(6n-18)$	$E_{74} = (65,99)$	$4(6n-24)$
$E_{37} = (42,49)$	$12n^2 - 60n + 72$	$E_{75} = (73,74)$	24
$E_{38} = (42,73)$	24	$E_{76} = (73,75)$	24
$E_{39} = (42,74)$	$2(6n-24)$	$E_{77} = (73,95)$	24
$E_{40} = (42,75)$	$4(6n-24)$	$E_{78} = (73,101)$	24
$E_{41} = (42,76)$	$12n^2 - 96n + 192$	$E_{79} = (74,74)$	$2(6n-30)$
$E_{42} = (42,101)$	$6n^2 - 30n + 36$	$E_{80} = (74,76)$	$2(6n-24)$
$E_{43} = (45,53)$	12	$E_{81} = (74,95)$	$2(6n-24)$
$E_{44} = (45,63)$	24	$E_{82} = (74,101)$	$2(6n-24)$
$E_{45} = (45,92)$	12	$E_{83} = (75,75)$	$2(6n-30)$

Table 4. *Cont.*

(S_u, S_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges	(S_u, S_v) where $ab \in E(\mathcal{G}_2)$	Number of Edges
$E_{46} = (48, 54)$	$2(6n - 12)$	$E_{84} = (75, 76)$	$2(6n - 24)$
$E_{47} = (48, 63)$	24	$E_{85} = (75, 101)$	$4(6n - 24)$
$E_{48} = (48, 65)$	$2(6n - 24)$	$E_{86} = (76, 76)$	$12n^2 - 108n + 240$
$E_{49} = (48, 92)$	24	$E_{87} = (76, 101)$	$12n^2 - 96n + 192$
$E_{50} = (48, 99)$	$4(6n - 18)$	$E_{88} = (92, 99)$	24
$E_{51} = (49, 54)$	$6n^2 - 24n + 24$	$E_{89} = (92, 108)$	12
$E_{52} = (49, 73)$	24	$E_{90} = (95, 101)$	$2(6n - 12)$
$E_{53} = (49, 74)$	$2(6n - 24)$	$E_{91} = (95, 108)$	$6n - 12$
$E_{54} = (49, 75)$	$2(6n - 24)$	$E_{92} = (99, 99)$	$2(6n - 24)$
$E_{55} = (49, 76)$	$6n^2 - 48n + 96$	$E_{93} = (99, 108)$	$4(6n - 18)$
$E_{56} = (49, 95)$	$2(6n - 12)$	$E_{94} = (101, 101)$	$6n^2 - 36n + 48$
$E_{57} = (49, 101)$	$12n^2 - 60n + 72$	$E_{95} = (101, 108)$	$12n^2 - 60n + 72$
$E_{58} = (53, 54)$	12	$E_{96} = (108, 108)$	$9n^3 - 51n^2 + 72n$

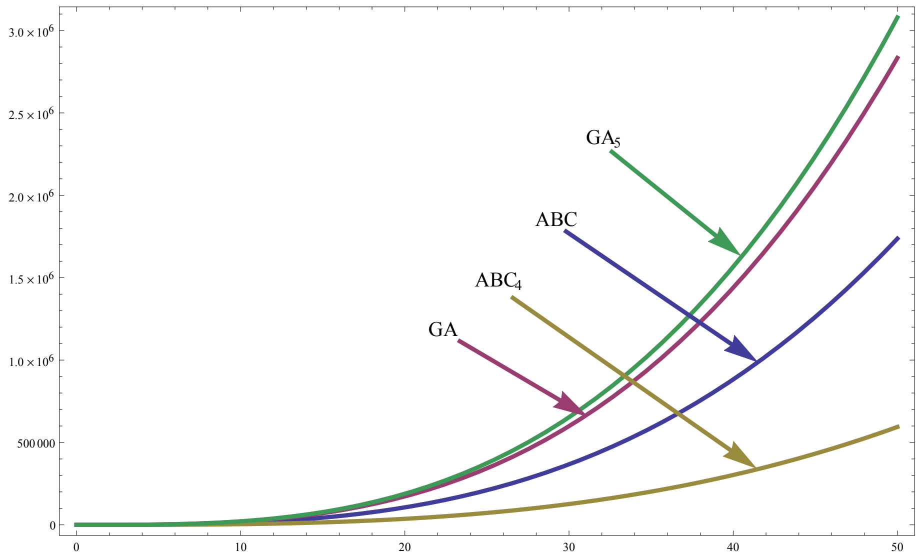

Figure 3. Comparison of ABC, GA, ABC_4 and GA_5 for $HDCN1(m, n)$.

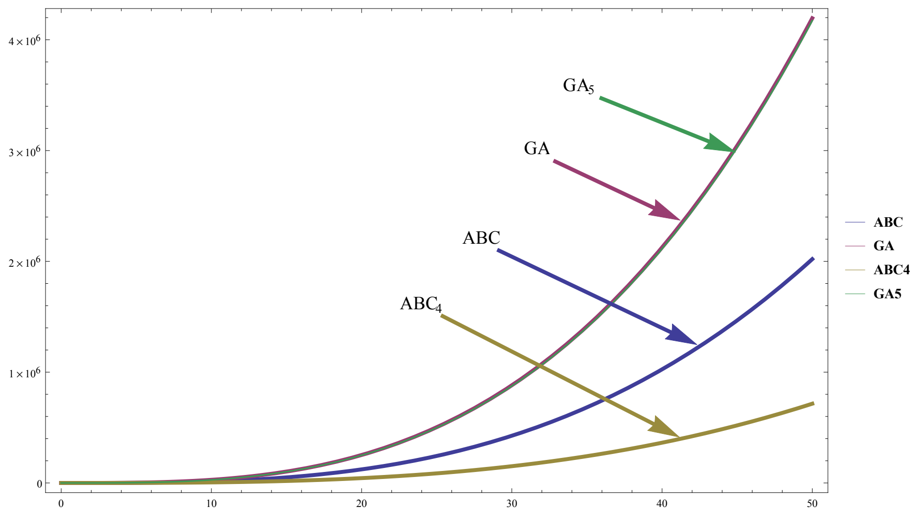

Figure 4. Comparison of ABC, GA, ABC_4 and GA_5 for $HDCN2(m, n)$.

3. Conclusions

In this paper, certain degree-based topological indices, namely the general Randić index, atomic-bond connectivity index (ABC), geometric-arithmetic index (GA) and first Zagreb index were studied for the first time and analytical closed formulas for $HDCN1(m,n)$ and $HDCN2(m,n)$ cage networks were determined which will help the people working in network science to understand and explore the underlying topologies of these networks.

For the future, we are interested in designing some new architectures/networks and then study their topological indices which will be quite helpful to understand their underlying topologies.

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