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On the Existence of the Solutions of a Fredholm Integral Equation with a Modified Argument in Hölder Spaces

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Abstract: This article concerns the entity of solutions of a quadratic integral equation of the Fredholm type with an altered argument, $x(t) = p(t) + x(t)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau$, where p, k are given functions, T is the given operator satisfying conditions specified later and x is an unknown function. Through the classical Schauder fixed point theorem and a new conclusion about the relative compactness in Hölder spaces, we obtain the existence of solutions under certain assumptions. Our work is more general than the previous works in the Conclusion section. At the end, we introduce several tangible examples where our entity result can be adopted.

Keywords: Fredholm integral equation; Schauder fixed point theorem; Hölder condition

1. Introduction

The work of differential equations, with an altered argument being latest, has continued for decades. For more data and consequences related to these equations, see [1–3]. These topics have linear modifications of their arguments and have been worked on by the authors in the papers [1–15]. Integral equations of course stem from several applications in specification numerous real-world problems (see [16,17] and the references therein). Quadratic integral equations arise naturally in applications to real-world problems. For example, problems in the kinetic theory of gases and in the theory of radiative transfer lead to the quadratic integral equation:

$$x(t) = 1 + tx(t)\int_0^1 \frac{\Phi(\tau)}{t+\tau}x(\tau)d\tau,$$

(see [18–21]). The integral equations of a similar form have been examined by several authors [22–28]. Furthermore, some studies using similar techniques have been dedicated to a micropolar porous body and vibrations in thermoelasticity [29,30].

Very recently, J. Banaś and R. Nalepa et al. [4] studied the following equation:

$$x(t) = p(t) + x(t)\int_a^b k(t,\tau)x(\tau)d\tau. \tag{1}$$

Further, J. Caballero, M. Darwish and K. Sadarangani et al. [5] studied the following equation:

$$x(t) = p(t) + x(t)\int_0^1 k(t,\tau)x(r(\tau))d\tau. \tag{2}$$

Furthermore, J. Cabelloro Mena, R. Nalepa and K. Sadarangani et al. [6] studied the following equation:

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau) \left\{ \max_{\eta \in [0,r(\tau)]} |x(\eta)| \right\} d\tau. \tag{3}$$

The purpose of this paper is to examine the existence of solutions of the following integral equation of the Fredholm type with a changed argument,

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau)(Tx)(\tau) d\tau, \quad t \in I = [0,1]. \tag{4}$$

Equation (4) is more general than many equations considered up to now and includes (1), (2) and (3) as special cases. Notice that Equation (1) in [4] for $a = 0$ and $b = 1$ is a particular case of (4) with $(Tx)(\tau) = x(\tau)$. Furthermore, it should be noted that Equation (4) is more general than Equation (2) considered in [5]. If we take $(Tx)(\tau) = x(r(\tau))$, then the equation:

$$x(t) = p(t) + x(t) \int_0^1 k(t,\tau) x(r(\tau)) d\tau$$

is obtained from Equation (4). Further, notice that Equation (3) in [6] is a particular case of (4), for $(Tx)(\tau) = \max_{\eta \in [0,r(\tau)]} |x(\eta)|$, where the function $r : [0,1] \to [0,1]$ is continuous and nondecreasing.

Compared to the previous works [4–6], we have further generalized the new assumptions in finding the solutions of (1), (2) and (3).

Our solutions substitute for the spaces of functions satisfactory the Hölder condition, and this is a source of the originality of the article. To do this, we will use a recent consequence about the classical Schauder fixed point theorem and the relative compactness in Hölder spaces.

2. Preliminaries and Notations

In this section, we present definitions, notations and theorems that are used along this paper. The following known definitions are available in [4,5,31,32].

Let $[a,b]$ be a closed interval in $\mathbb{R}$; by $C[a,b]$, we indicate the space of continuous functions defined on $[a,b]$ equipped with the supremum norm, i.e.,

$$\|x\|_\infty = \sup \{|x(t)| : t \in [a,b]\}$$

for $x \in C[a,b]$. For a fixed α with $0 < \alpha \le 1$, by $H_\alpha[a,b]$, we will indicate the spaces of the real functions x defined on $[a,b]$ and satisfying the Hölder condition, that is those functions x for which there exists a constant H_x^α such that:

$$|x(t) - x(s)| \le H_x^\alpha |t-s|^\alpha \tag{5}$$

for all $t,s \in [a,b]$. It is well proven that $H^\alpha[a,b]$ is a linear subspace of $C[a,b]$. Furthermore, for $x \in H^\alpha[a,b]$, by H_x^α, we will indicate the least possible stable value for which Inequality (5) is satisfied. Rather, we put:

$$H_x^\alpha = \sup \left\{ \frac{|x(t) - x(s)|}{|t-s|^\alpha} : t,s \in [a,b] \text{ and } t \neq s \right\}. \tag{6}$$

The space $H_\alpha[a,b]$ with $0 < \alpha \le 1$ may be equipped with the norm:

$$\|x\|_\alpha = |x(a)| + H_x^\alpha$$

for $x \in H_\alpha[a,b]$. Here, H_x^α is defined by (6). In [4], the authors demonstrated that $(H_\alpha[a,b], \|\cdot\|_\alpha)$ with $0 < \alpha \le 1$ is a Banach space.

Lemma 1. *For $0 < \alpha \leq 1$ and $x \in H_\alpha[a,b]$, we have:*

$$\|x\|_\infty \leq \max\left(1, (b-a)^\alpha\right) \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leq \|x\|_\alpha$ is satisfied for $a = 0$ and $b = 1$ [4].

Lemma 2. *For $0 < \alpha < \beta \leq 1$, we have:*

$$H_\beta[a,b] \subset H_\alpha[a,b] \subset C[a,b].$$

Furthermore, for $x \in H_\beta[a,b]$, we have:

$$\|x\|_\alpha \leq \max\left(1, (b-a)^{\beta-\alpha}\right) \|x\|_\beta.$$

Particularly, the inequality $\|x\|_\infty \leq \|x\|_\alpha \leq \|x\|_\beta$ is satisfied for $a = 0$ and $b = 1$, [4].

Lemma 3. *Let us assume that $0 < \alpha < \beta \leq 1$ and E is a bounded subset in $H_\beta[a,b]$, then E is a relatively compact subset in $H_\alpha[a,b]$ [5].*

Lemma 4. *Assume that $0 < \alpha < \beta \leq 1$, and by B_r^β, we indicate the ball centered at θ and radius r in the space $H_\beta[a,b]$, i.e., $B_r^\beta = \{x \in H_\beta[a,b] : \|x\|_\beta \leq r\}$. B_r^β is a closed subset of $H_\alpha[a,b]$ [5].*

Corollary 1. *Assume that $0 < \alpha < \beta \leq 1$ and B_r^β is a relatively compact subset in $H_\alpha[a,b]$ and a closed subset of $H_\alpha[a,b]$, then B_r^β is a compact subset in the space $H_\alpha[a,b]$, [5].*

Now let us give the following theorem, which is the base tool used in our study.

Theorem 1 (Schauder's fixed point theorem)**.** *Let E be a nonempty, compact subset of a Banach space $(X, \|\cdot\|)$, convex, and let $T : E \to E$ be a continuity mapping. Then, T has at least one fixed point in E [7].*

3. Main Result

Now, we are ready to give the main result of the paper. In this section, we introduce the following sufficient conditions for the main theorem in our study, and we will prove the solvability of Equation (4) in Hölder spaces.

Hereafter, we suppose unless stated otherwise that α and β are arbitrarily fixed numbers such that $0 < \alpha < \beta \leq 1$.

Theorem 2. *Assume that the following Conditions (i)–(iv) are satisfied:*

(i) $p \in H_\beta[0,1]$.

(ii) $k : [0,1] \times [0,1] \to \mathbb{R}$ *is a continuous function such that there exists a constant $k_\beta > 0$ such that:*

$$|k(t,\tau) - k(s,\tau)| \leq k_\beta |t-s|^\beta,$$

for any $t, s, \tau \in [0,1]$.

(iii) The operator $T : H_\beta[0,1] \to C[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, and there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, which is non-decreasing such that the inequality holds:

$$\|Tx\|_\infty \leq f\left(\|x\|_\beta\right),$$

for any $x \in H_\beta[0,1]$.

(iv) There exists a positive solution r_0 of the inequality:

$$\|p\|_\beta + (2K + k_\beta) r f(r) \leq r,$$

where the constant K is defined by:

$$\sup\left\{\int_0^1 |k(t,\tau)|d\tau : t \in [0,1]\right\} \leq K.$$

Then, Equation (4) has at least one solution $x = x(t)$ belonging to space $H_\alpha[0,1]$.

Proof. Now, let us consider $x \in H_\beta[0,1]$ and the operator F defined on the space $H_\beta[0,1]$ by the formula:

$$(Fx)(t) = p(t) + x(t)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau,$$

for $t \in [0,1]$. Then, for arbitrarily fixed $t, s \in [0,1], (t \neq s)$, in view of our assumptions, we get:

$$\begin{aligned}
& (Fx)(t) - (Fx)(s) \\
= \; & p(t) + x(t)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau - p(s) - x(s)\int_0^1 k(s,\tau)(Tx)(\tau)d\tau \\
= \; & p(t) - p(s) + x(t)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau - x(s)\int_0^1 k(s,\tau)(Tx)(\tau)d\tau \\
& + x(s)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau - x(s)\int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
= \; & p(t) - p(s) + (x(t) - x(s))\int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
& + x(s)\int_0^1 (k(t,\tau) - k(s,\tau))(Tx)(\tau)d\tau
\end{aligned}$$

and:

$$\begin{aligned}
& \frac{|(Fx)(t) - (Fx)(s)|}{|t-s|^\beta} \\
\leq \; & \frac{|p(t) - p(s)|}{|t-s|^\beta} + \frac{|x(t)-x(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)||(Tx)(\tau)|d\tau \\
& + \frac{|x(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau) - k(s,\tau)||(Tx)(\tau)|d\tau \\
\leq \; & H_p^\beta + \|x\|_\beta \|Tx\|_\infty \int_0^1 |k(t,\tau)|d\tau \\
& + |x(s)|\int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^\beta}|(Tx)(\tau)|d\tau \\
\leq \; & H_p^\beta + \|x\|_\beta \|Tx\|_\infty K + \|x\|_\infty \|Tx\|_\infty \int_0^1 k_\beta \frac{|t-s|^\beta}{|t-s|^\beta}d\tau \\
\leq \; & H_p^\beta + \|x\|_\beta \|Tx\|_\infty K + \|x\|_\beta \|Tx\|_\infty k_\beta \\
\leq \; & H_p^\beta + \|x\|_\beta f\left(\|x\|_\beta\right) K + \|x\|_\beta f\left(\|x\|_\beta\right) k_\beta \\
= \; & H_p^\beta + (K + k_\beta)\|x\|_\beta f\left(\|x\|_\beta\right).
\end{aligned} \tag{7}$$

This demonstrates that the operator F maps $H_\beta[0,1]$ into itself.

Besides, for any $x \in H_\beta[0,1]$, we get:

$$
\begin{aligned}
|(Fx)(0)| &\leq |p(0)| + |x(0)| \int_0^1 |k(0,\tau)||(Tx)(\tau)|d\tau \\
&\leq |p(0)| + \|x\|_\infty \|Tx\|_\infty K \\
&\leq |p(0)| + \|x\|_\beta \|Tx\|_\infty K \\
&\leq |p(0)| + \|x\|_\beta f\left(\|x\|_\beta\right) K.
\end{aligned} \tag{8}
$$

By the inequalities (7) and (8), we derive that:

$$
\|Fx\|_\beta \leq \|p\|_\beta + (2K + k_\beta)\|x\|_\beta f\left(\|x\|_\beta\right). \tag{9}
$$

Since positive number r_0 is the solution of the inequality given in Hypothesis (iv), from (9) and function $f : \mathbb{R}_+ \to \mathbb{R}_+$, which is non-decreasing, we conclude that the inequality:

$$
\|Fx\|_\beta \leq \|p\|_\beta + (2K + k_\beta) r_0 f(r_0) \leq r_0 \tag{10}
$$

holds. As a result, it follows that F transform the ball:

$$
B_{r_0}^\beta = \{x \in H_\beta[0,1] : \|x\|_\beta \leq r_0\}
$$

into itself. That is, $F : B_{r_0}^\beta \to B_{r_0}^\beta$. Thus, we have that the set $B_{r_0}^\beta$ is relatively compact in $H_\alpha[0,1]$ for any $0 < \alpha < \beta \leq 1$. Furthermore, $B_{r_0}^\beta$ is a compact subset in $H_\alpha[0,1]$.

In the sequel, we will demonstrate that the operator F is continuous on $B_{r_0}^\beta$ with respect to the norm $\|\cdot\|_\alpha$, where $0 < \alpha < \beta \leq 1$.

Let $y \in B_{r_0}^\beta$ be an arbitrary point in $B_{r_0}^\beta$. Then, we get:

$$
\begin{aligned}
& (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) \\
= \; & p(t) + x(t) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
& -p(t) - y(t) \int_0^1 k(t,\tau)(Ty)(\tau)d\tau \\
& -p(s) - x(s) \int_0^1 k(s,\tau)(Tx)(\tau)d\tau \\
& +p(s) + y(s) \int_0^1 k(s,\tau)(Ty)(\tau)d\tau
\end{aligned} \tag{11}
$$

for any $x \in B_{r_0}^\beta$ and $t, s \in [0,1]$. Equality (11) can be rewritten as:

$$
\begin{aligned}
& (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) \\
= \; & x(t) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau - y(t) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
& +y(t) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau - y(t) \int_0^1 k(t,\tau)(Ty)(\tau)d\tau \\
& -x(s) \int_0^1 k(s,\tau)(Tx)(\tau)d\tau + y(s) \int_0^1 k(s,\tau)(Tx)(\tau)d\tau \\
& -y(s) \int_0^1 k(s,\tau)(Tx)(\tau)d\tau + y(s) \int_0^1 k(s,\tau)(Ty)(\tau)d\tau.
\end{aligned} \tag{12}
$$

By (12), we have:

$$
\begin{aligned}
& (Fx)(t) - (Fy)(t) - ((Fx)(s) - (Fy)(s)) \\
= \; & (x(t) - y(t)) \int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
& +y(t) \int_0^1 k(t,\tau)((Tx)(\tau) - (Ty)(\tau))d\tau \\
& -(x(s) - y(s)) \int_0^1 k(s,\tau)(Tx)(\tau)d\tau \\
& -y(s) \int_0^1 k(s,\tau)((Tx)(\tau) - (Ty)(\tau))d\tau.
\end{aligned} \tag{13}
$$

(13) yields the following equality:

$$
\begin{array}{rl}
& ((Fx)(t)-(Fy)(t))-((Fx)(s)-(Fy)(s)) \\
= & [(x(t)-y(t))-(x(s)-y(s))]\int_0^1 k(t,\tau)(Tx)(\tau)d\tau \\
& +(x(s)-y(s))\int_0^1 (k(t,\tau)-k(s,\tau))(Tx)(\tau)d\tau \\
& +(y(t)-y(s))\int_0^1 k(t,\tau)((Tx)(\tau)-(Ty)(\tau))d\tau \\
& +y(s)\int_0^1 (k(t,\tau)-k(s,\tau))((Tx)(\tau)-(Ty)(\tau))d\tau.
\end{array} \tag{14}
$$

Hence, taking into account (14), we can write:

$$
\begin{array}{rl}
& \frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha} \\
\leq & \frac{|(x(t)-y(t))-(x(s)-y(s))|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(Tx)(\tau)|d\tau \\
& +\dfrac{|x(s)-y(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(Tx)(\tau)|d\tau \\
& +\dfrac{|y(t)-y(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)||(Tx)(\tau)-(Ty)(\tau)|d\tau \\
& +\dfrac{|y(s)|}{|t-s|^\alpha}\int_0^1 |k(t,\tau)-k(s,\tau)||(Tx)(\tau)-(Ty)(\tau)|d\tau \\
\leq & \|x-y\|_\alpha\|Tx\|_\infty K+\|x-y\|_\infty\|Tx\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau \\
& +\|y\|_\alpha\|Tx-Ty\|_\infty K+\|y\|_\infty\|Tx-Ty\|_\infty\int_0^1 k_\beta|t-s|^{\beta-\alpha}d\tau \\
\leq & K\|x-y\|_\alpha\|Tx\|_\infty+k_\beta\|x-y\|_\alpha\|Tx\|_\infty \\
& +K\|y\|_\alpha\|Tx-Ty\|_\infty+k_\beta\|y\|_\alpha\|Tx-Ty\|_\infty \\
\leq & Kf\left(\|x\|_\beta\right)\|x-y\|_\alpha+k_\beta f\left(\|x\|_\beta\right)\|x-y\|_\alpha \\
& +K\|y\|_\alpha\|Tx-Ty\|_\infty+k_\beta\|y\|_\alpha\|Tx-Ty\|_\infty \\
= & (K+k_\beta)f\left(\|x\|_\beta\right)\|x-y\|_\alpha+(K+k_\beta)\|y\|_\alpha\|Tx-Ty\|_\infty
\end{array} \tag{15}
$$

for all $t,s\in[0,1]$ with $t\neq s$. Besides, for $x,y\in B_{r_0}^\beta$, we obtain the following inequality:

$$
\begin{array}{rl}
& |(Fx)(0)-(Fy)(0)| \\
= & \Big|p(0)+x(0)\int_0^1 k(0,\tau)(Tx)(\tau)d\tau \\
& -p(0)-y(0)\int_0^1 k(0,\tau)(Ty)(\tau)d\tau\Big| \\
= & \Big|x(0)\int_0^1 k(0,\tau)(Tx)(\tau)d\tau-y(0)\int_0^1 k(0,\tau)(Tx)(\tau)d\tau \\
& +y(0)\int_0^1 k(0,\tau)(Tx)(\tau)d\tau-y(0)\int_0^1 k(0,\tau)(Ty)(\tau)d\tau\Big| \\
= & \Big|(x(0)-y(0))\int_0^1 k(0,\tau)(Tx)(\tau)d\tau \\
& +y(0)\int_0^1 k(0,\tau)((Tx)(\tau)-(Ty)(\tau))d\tau\Big| \\
\leq & \|x-y\|_\infty K\|Tx\|_\infty+\|y\|_\infty K\|Tx-Ty\|_\infty \\
\leq & K\|x-y\|_\alpha\|Tx\|_\infty+K\|y\|_\alpha\|Tx-Ty\|_\infty \\
\leq & Kf\left(\|x\|_\beta\right)\|x-y\|_\alpha+K\|y\|_\alpha\|Tx-Ty\|_\infty.
\end{array} \tag{16}
$$

From (15) and (16), we have that:

$$
\begin{array}{rl}
& \|Fx-Fy\|_\alpha \\
= & |(Fx-Fy)(0)|+H_{Fx-Fy}^\alpha \\
= & |(Fx)(0)-(Fy)(0)| \\
& +\sup\left\{\frac{|(Fx)(t)-(Fy)(t)-((Fx)(s)-(Fy)(s))|}{|t-s|^\alpha}:t,s\in[0,1]\text{ and }t\neq s\right\} \\
\leq & (2K+k_\beta)f\left(\|x\|_\beta\right)\|x-y\|_\alpha+(2K+k_\beta)\|y\|_\alpha\|Tx-Ty\|_\infty.
\end{array} \tag{17}
$$

Moreover, since $\|y\|_\alpha \le \|y\|_\beta \le r_0$ and $f\left(\|x\|_\beta\right) \le f(r_0)$, we derive from (17) that the following inequality holds:

$$\|Fx - Fy\|_\alpha \le (2K + k_\beta)f(r_0)\|x - y\|_\alpha + (2K + k_\beta)r_0\|Tx - Ty\|_\infty. \tag{18}$$

Since the operator $T : H_\beta[0,1] \to C[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, it is also continuous at the point $y \in B^\beta_{r_0}$. Let us take an arbitrary $\varepsilon > 0$. Since the operator T is continuous at the point $y \in B^\beta_{r_0}$, there exists $\delta > 0$ such that the inequality:

$$\|Tx - Ty\|_\infty < \frac{\varepsilon}{2(2K + k_\beta)r_0}$$

is satisfied for all $x \in B^\beta_{r_0}$, where $\|x - y\|_\alpha < \delta$ and:

$$0 < \delta < \frac{\varepsilon}{2(2K + k_\beta)f(r_0)}.$$

Then, taking into account (18), we derive the following inequality:

$$\begin{aligned}
\|Fx - Fy\|_\alpha &\le (2K + k_\beta)f(r_0)\|x - y\|_\alpha + (2K + k_\beta)r_0\|Tx - Ty\|_\infty \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

As a result, we infer that the operator F is continuous at the point $y \in B^\beta_{r_0}$. Because y was chosen arbitrarily, we deduce that F is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$. As $B^\beta_{r_0}$ is compact in $H_\alpha[0,1]$, from the classical Schauder fixed point theorem, we get the desired result. $\square$

4. Examples

In this part, we conclude the article by presenting two examples that illustrate the generality and efficiency of our results.

Example 1. *Let us consider the following quadratic integral equation:*

$$x(t) = \sqrt[6]{q\cos t + \hat{q}} + x(t)\int_0^1 \sqrt[5]{mt^2 + \tau}\,\sin x^2(\tau)d\tau, \quad t \in I = [0,1]. \tag{19}$$

Here, $q, \hat{q}$ and m are the suitable nonnegative constants to be determined such that Conditions (i)–(iv) of Theorem 2 hold.

Set $p(t) = \sqrt[6]{q\cos t + \hat{q}}$ and $k(t,\tau) = \sqrt[5]{mt^2 + \tau}$ for all $t, \tau \in [0,1]$.

It is easily seen that:

$$\begin{aligned}
|p(t) - p(s)| &= \left|\sqrt[6]{q\cos t + \hat{q}} - \sqrt[6]{q\cos s + \hat{q}}\right| \\
&\le \left|\sqrt[6]{q\cos t + \hat{q} - q\cos s - \hat{q}}\right| \\
&\le \sqrt[6]{q|\cos t - \cos s|} \\
&\le \sqrt[6]{2q\left|\sin\left(\frac{t+s}{2}\right)\right|\left|\sin\left(\frac{t-s}{2}\right)\right|} \\
&\le \sqrt[6]{q}\sqrt[6]{|t-s|} \\
&= \sqrt[6]{q}|t-s|^{\frac{1}{6}}
\end{aligned}$$

for all $t, s \in [0,1]$. *This says that* $p \in H_{\frac{1}{6}}[0,1]$ *and, moreover,* $H_p^{\frac{1}{6}} = \sqrt[6]{q}$. *Therefore, we can take the constants* α *and* β *as* $0 < \alpha < \frac{1}{6}$ *and* $\beta = \frac{1}{6}$. *Therefore, Assumption (i) of Theorem 2 holds. Note that:*

$$\begin{aligned} \|p\|_{\frac{1}{6}} &= |p(0)| + \sup\left\{ \frac{|p(t)-p(s)|}{|t-s|^{\frac{1}{6}}} : t,s \in [0,1] \text{ and } t \neq s \right\} \\ &= |p(0)| + H_p^{\frac{1}{6}} = \sqrt[6]{q+\hat{q}} + \sqrt[6]{q}. \end{aligned}$$

Further, we have:

$$\begin{aligned} |k(t,\tau) - k(s,\tau)| &= \left| \sqrt[5]{mt^2+\tau} - \sqrt[5]{ms^2+\tau} \right| \\ &\leq \left| \sqrt[5]{m(t^2-s^2)} \right| \\ &= \sqrt[5]{m}\sqrt[5]{|(t^2-s^2)|} \\ &= \sqrt[5]{m}\sqrt[5]{|t-s|}\sqrt[5]{|t+s|} \\ &\leq \sqrt[5]{2m}|t-s|^{\frac{1}{6}}|t-s|^{\frac{1}{30}} \\ &\leq \sqrt[5]{2m}|t-s|^{\frac{1}{6}} \end{aligned}$$

for all $t, s, \tau \in [0,1]$. *Assumption (ii) of Theorem 2 holds with* $k_\beta = k_{\frac{1}{6}} = \sqrt[5]{2m}$.

Since $(Tx)(\tau) = \sin x^2(\tau)$ *and:*

$$\left|\sin x^2(\tau)\right| \leq \left|x^2(\tau)\right| = |x(\tau)||x(\tau)| \leq \|x\|_\infty^2 \leq \|x\|_\beta^2$$

for all $x \in H_\beta[0,1]$ *and* $\tau \in [0,1]$, *the inequality:*

$$\|Tx\|_\infty = \sup_{\tau\in[0,1]} \left|\sin x^2(\tau)\right| \leq \|x\|_\beta^2$$

holds. Therefore, we can choose the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ *as* $f(x) = x^2$. *This function is non-decreasing and satisfies the inequality in Assumption (iii).*

We will show that the operator $T : H_\beta[0,1] \to C[0,1]$ *is continuous on* $H_\beta[0,1]$ *with respect to the norm* $\|\cdot\|_\alpha$. *Let us take* $x, y \in H_\beta[0,1]$ *and* $\tau \in [0,1]$.

It is clear that:

$$\begin{aligned} \left|\sin x^2(\tau) - \sin y^2(\tau)\right| &\leq \left|x^2(\tau) - y^2(\tau)\right| \\ &= |x(\tau)-y(\tau)|\,|x(\tau)+y(\tau)| \\ &= |x(\tau)-y(\tau)|\,|x(\tau)-y(\tau)+2y(\tau)| \\ &\leq |x(\tau)-y(\tau)|\,(|x(\tau)-y(\tau)| + 2\,|y(\tau)|) \\ &\leq \|x-y\|_\infty(\|x-y\|_\infty + 2\|y\|_\infty) \\ &\leq \|x-y\|_\alpha\,(\|x-y\|_\alpha + 2\|y\|_\alpha) \end{aligned}$$

and:

$$\|Tx - Ty\|_\infty \leq \sup_{\tau\in[0,1]} \left|\sin x^2(\tau) - \sin y^2(\tau)\right| \leq \|x-y\|_\alpha(\|x-y\|_\alpha + 2\|y\|_\alpha).$$

Now, we will show that T *is continuous at the point* $y \in H_\beta[0,1]$ *with respect to the norm* $\|\cdot\|_\alpha$*. Let us take an arbitrary* $\varepsilon > 0$*. Then, there exists the positive number* δ *such that* $\|x-y\|_\alpha < \delta$ *and the inequality:*

$$\|Tx - Ty\|_\infty < \delta\,(\delta + 2\|y\|_\alpha) < \varepsilon$$

is satisfied for all $x \in H_\beta[0,1]$*, where* $0 < \delta < \sqrt{\|y\|_\alpha^2 + \varepsilon} - \|y\|_\alpha$*. Therefore, we can choose the positive number* δ *as* $\delta = \frac{1}{2}\sqrt{\|y\|_\alpha^2 + \varepsilon} - \|y\|_\alpha$*. As a result, we infer that the operator* T *is continuous at the point* $y \in B_{r_0}^\beta$*. Since* y *was chosen arbitrarily, we deduce that* T *is continuous on* $H_\beta[0,1]$ *with respect to the norm* $\|\cdot\|_\alpha$*.*

Further, we can calculate that:

$$\begin{aligned}
\sup\left\{\int_0^1 |k(t,\tau)|d\tau : t\in[0,1]\right\} &= \sup\left\{\int_0^1 \left|\sqrt[5]{mt^2+\tau}\right| d\tau : t\in[0,1]\right\}\\
&= \sup\left\{\frac{5}{6}\left(\sqrt[5]{(mt^2+1)^6} - \sqrt[5]{(mt^2)^6}\right) : t\in[0,1]\right\}\\
&\le \sup\left\{\frac{5}{6}\sqrt[5]{(mt^2+1)^6} : t\in[0,1]\right\}\\
&= \frac{5}{6}\sqrt[5]{(m+1)^6}\\
&\le \sqrt[5]{(m+1)^6}\\
&= K.
\end{aligned}$$

In this case, the inequality appearing in assumption (vi) of Theorem 2 takes the following form:

$$\|p\|_{\frac{1}{6}} + (2K + k_\beta) r f(r) \le r$$

which is equivalent to:

$$\sqrt[6]{q} + \sqrt[6]{q+\hat{q}} + \left(2\sqrt[5]{(m+1)^6} + \sqrt[5]{2m}\right) r r^2 \le r. \tag{20}$$

There exists a positive number r_0 *satisfying (20) provided that the constants* $q, \hat{q}$ *and* m *are chosen as suitable.*

For example, if one chose $q = \frac{1}{10^{18}}, \hat{q} = 0$ *and* $m = \frac{1}{2^{16}}$*, then the inequality:*

$$\frac{2}{10^3} + \left(2\sqrt[5]{\left(\frac{1}{2^{16}}+1\right)^6} + 0.125\right) r^3 \le r$$

holds for $r = r_0 = \frac{1}{10}$*. Therefore, using Theorem 2, we infer that there is at least one solution* x *of Equation (19) in the space* $H_\alpha[0,1]$ *with* $0 < \alpha < \frac{1}{6}$*.*

Example 2. *Let us consider the following quadratic integral equation:*

$$x(t) = \frac{1}{10^6}\arctan\sqrt[3]{t+\ln q} + x(t)\int_0^1 \sqrt[3]{m\sin t + \tau}\sqrt{|x(\tau)|}d\tau,\;\; t\in I = [0,1], \tag{21}$$

where q *and* m *are the suitable positive constants to be selected for which Conditions (i)–(iv) of Theorem 2 hold.*

Set $p(t) = \frac{1}{10^6}\arctan\sqrt[3]{t+\ln q}$ *and* $k(t,\tau) = \sqrt[3]{m\sin t + \tau}$ *for all* $t,\tau \in [0,1]$*.*

It is obvious that the inequality:

$$\begin{aligned}
|p(t)-p(s)| &= \left|\frac{1}{10^6}\arctan\sqrt[3]{t+\ln q}-\frac{1}{10^6}\arctan\sqrt[3]{s+\ln q}\right| \\
&\le \left|\frac{1}{10^6}\arctan\left(\sqrt[3]{t+\ln q}-\sqrt[3]{s+\ln q}\right)\right| \\
&= \frac{1}{10^6}\left|\sqrt[3]{t+\ln q}-\sqrt[3]{s+\ln q}\right| \\
&\le \frac{1}{10^6}\left|\sqrt[3]{t+\ln q-s-\ln q}\right| \\
&\le \frac{1}{10^6}\sqrt[3]{|t-s|} \\
&= \frac{1}{10^6}|t-s|^{\frac{1}{3}}
\end{aligned}$$

holds for all $t,s\in[0,1]$. *Therefore,* $p\in H_{\frac{1}{3}}[0,1]$ *and* $H_p^{\frac{1}{3}}=\frac{1}{10^6}$. *Hence, the constants* α *and* β *can be taken as* $0<\alpha<\frac{1}{3}$ *and* $\beta=\frac{1}{3}$.

Therefore, Assumption (i) of Theorem 2 is satisfied. Note that:

$$\begin{aligned}
\|p\|_{\frac{1}{3}} &= |p(0)|+\sup\left\{\frac{|p(t)-p(s)|}{|t-s|^{\frac{1}{3}}}: t,s\in[0,1] \text{ and } t\neq s\right\} \\
&= |p(0)|+H_p^{\frac{1}{3}}=\frac{1}{10^6}\left|\arctan\sqrt[3]{\ln q}\right|+\frac{1}{10^6}.
\end{aligned}$$

Further, we have:

$$\begin{aligned}
|k(t,\tau)-k(s,\tau)| &= \left|\sqrt[3]{m\sin t+\tau}-\sqrt[3]{m\sin s+\tau}\right| \\
&\le \left|\sqrt[3]{m(\sin t-\sin s)}\right| \\
&\le \sqrt[3]{2m\left|\cos\left(\frac{t+s}{2}\right)\right|\left|\sin\left(\frac{t-s}{2}\right)\right|} \\
&\le \sqrt[3]{m}|t-s|^{\frac{1}{3}}
\end{aligned}$$

for all $t,s,\tau\in[0,1]$. *Assumption (ii) of Theorem 2 is satisfied with* $k_\beta=k_{\frac{1}{3}}=\sqrt[3]{m}$.

Since $(Tx)(\tau)=\sqrt{|x(\tau)|}$ *and:*

$$\sqrt{|x(\tau)|}\le\sqrt{\|x\|_\infty}\le\sqrt{\|x\|_\beta}$$

for all $x\in H_\beta[0,1]$ *and* $\tau\in[0,1]$, *the inequality:*

$$\|Tx\|_\infty=\sup_{\tau\in[0,1]}\left|\sqrt{|x(\tau)|}\right|\le\sqrt{\|x\|_\beta}$$

holds. Therefore, we can choose the function $f:\mathbb{R}_+\to\mathbb{R}_+$ *as* $f(x)=\sqrt{x}$. *This function is non-decreasing and satisfies the inequality in Assumption (iii).*

We will show that the operator $T: H_\beta[0,1]\to C[0,1]$ *is continuous on* $H_\beta[0,1]$ *with respect to the norm* $\|\cdot\|_\alpha$. *Let us take* $x,y\in H_\beta[0,1]$ *and* $\tau\in[0,1]$. *It is certain that:*

$$\left|\sqrt{|x(\tau)|}-\sqrt{|y(\tau)|}\right|\le\sqrt{|x(\tau)-y(\tau)|}\le\sqrt{\|x-y\|_\infty}\le\sqrt{\|x-y\|_\alpha}$$

and:

$$\|Tx - Ty\|_\infty \leq \sup_{\tau\in[0,1]} \left| \sqrt{|x(\tau)|} - \sqrt{|y(\tau)|} \right| \leq \sqrt{\|x-y\|_\alpha}.$$

Now, we will show that T is continuous at the point $y \in H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$. Let us take an arbitrary $\varepsilon > 0$. Then, there exists the positive number δ such that $\|x-y\|_\alpha < \delta$ and the inequality:

$$\|Tx - Ty\|_\infty \leq \sqrt{\|x-y\|_\alpha} < \varepsilon$$

is satisfied for all $x \in H_\beta[0,1]$. Here, we can choose the positive number δ as $\delta = \varepsilon^2$.

As a result, we infer that the operator T is continuous at the point $y \in B^\beta_{r_0}$. Since y was chosen arbitrarily, we deduce that T is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$.

Further, we can calculate that:

$$\begin{aligned}
&\sup\left\{\int_0^1 |k(t,\tau)|d\tau : t\in[0,1]\right\}\\
= &\sup\left\{\int_0^1 \left|\sqrt[3]{m\sin t+\tau}\right| d\tau : t\in[0,1]\right\}\\
= &\sup\left\{\frac{1}{3}\left(\sqrt[3]{(m\sin t+1)^4} - \sqrt[3]{(m\sin t)^4}\right) : t\in[0,1]\right\}\\
\leq &\sup\left\{\frac{1}{3}\sqrt[3]{(m\sin t+1)^4} : t\in[0,1]\right\}\\
\leq &\frac{1}{3}\sqrt[3]{(m+1)^4}\\
\leq &\sqrt[3]{(m+1)^4}\\
= &K.
\end{aligned}$$

In this case, the inequality appearing in Assumption (vi) of Theorem 2 takes the following form:

$$\|p\|_{\frac{1}{3}} + (2K + k_\beta) r f(r) \leq r$$

which is equivalent to:

$$\frac{1}{10^6}\left(\left|\arctan\sqrt[3]{\ln q}\right| + 1\right) + \left(2\sqrt[3]{(m+1)^4} + \sqrt[3]{m}\right) r\sqrt{r} \leq r. \tag{22}$$

There exists a positive number r_0 satisfying (22) for chosen suitable constants q and m.

For example, if one chooses $q = 1$ and $m = \frac{1}{5^{12}}$, then the inequality:

$$\frac{1}{10^6} + \left(2\sqrt[3]{\left(1+\frac{1}{5^{12}}\right)^4} + 0.0016\right) r^{\frac{3}{2}} \leq r$$

holds for $r = r_0 = \frac{1}{10^4}$. Therefore, using Theorem 2, we infer that there is at least one solution x of Equation (21) in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{3}$.

5. Conclusions

In this paper, we have investigated the existence of solutions of the integral Equation (4). It should be noted that Equation (4) is more general than many equations considered up to now. For example, it includes the equations examined in previous studies [4–6]. That is, if we take the operator T as

$(Tx)(\tau) = x(\tau)$, we obtain the integral Equation (1) in [4] with $a = 0$ and $b = 1$. On the other hand, if we take $(Tx)(\tau) = x(r(\tau))$, we have the integral Equation (2) in [5]. Further, if we take $(Tx)(\tau) = \max_{\eta \in [0, r(\tau)]} |x(\eta)|$, we have the integral Equation (3) in [6].

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