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On Coloring Catalan Number Distance Graphs and Interference Graphs

Venkataraman Yegnanarayanan [1,*,†], Gayathri Narayana Yegnanarayanan [2,†] and Marius M. Balas [3,†]

1 Department of Mathematics, SASTRA Deemed to be University, Thanjavur 613401, Tamilnadu, India
2 Department of Electronics and Communication Engineering, SSN College of Engineering, Chennai 603110, Tamilnadu, India; gayumithra19@gmail.com
3 Department of Automation, Industrial Engineering, Textiles and Transport, University 'Aurel Vlaicu', B-dul Revolutiei nr. 77, 310130 Arad, Romania; marius@drbalas.ro
* Correspondence: yegnanarayanan@maths.sastra.edu; Tel.: +91-44-9444036628
† These authors contributed equally to this work.

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Abstract: A vertex coloring of a graph G is a mapping that allots colors to the vertices of G. Such a coloring is said to be a proper vertex coloring if two vertices joined by an edge receive different colors. The chromatic number $\chi(G)$ is the least number of colors used in a proper vertex coloring. In this paper, we compute the χ of certain distance graphs whose distance set elements are (a) a finite set of Catalan numbers, (b) a finite set of generalized Catalan numbers, (c) a finite set of Hankel transform of a transformed sequence of Catalan numbers. Then while discussing the importance of minimizing interference in wireless networks, we probe how a vertex coloring problem is related to minimizing vertex collisions and signal clashes of the associated interference graph. Then when investigating the χ of certain $G(V, D)$ and graphs with interference, we also compute certain lower and upper bound for χ of any given simple graph in terms of the average degree and Laplacian operator. Besides obtaining some interesting results we also raised some open problems.

Keywords: chromatic number; distance graphs; interference graphs; Catalan numbers; Hankel transforms; chromatic threshold.

1. Introduction

A graph comprising vertices and edges is a discrete structure in which each edge joins only two different vertices. In this paper, we consider graphs with only finitely many vertices. Several graph classes model various problems wherein one can observe a lot of variance in their properties. Graph chromatic number is a pertinent parameter in graph theory. A coloring of a graph G can be deemed as a mapping that allots colors to elements in it. A usual type of coloring is a proper vertex coloring, where we color the vertices of G in such a way that two vertices joined by an edge receive different colors. The chromatic number χ of G is the least number of colors needed to properly color the vertices of G. Finding the χ of a given graph is a computationally hard task. Garey et al. [1], showed that the determination of χ is an NP-complete problem unless it is 1 or 2 where NP stands for Non-Deterministic Polynomial time. That is no polynomial-time algorithm could correctly find it. However, several practical algorithms that approximate the χ in polynomial time exist in literature [2,3].

An interesting instance where graph coloring is applied is the open-shop scheduling problem. It kindles us to find the least time to produce a set of products that has to pass through a sequence of processes on several machines. If all of them consume equal time where there is no compulsion of order, then this problem can be modeled using a bipartite graph. Here products and processes both act as vertices. We introduce an edge between them if and only if the product go through that process. The least time required to finish all processes is the least colors required in a proper edge-coloring of the respective graph. In the case of only two machines or only two products, the problem of finding the least time can be resolved in polynomial time. However, when we have three or more machines and three or more products to be produced then a best known algorithm is exponential as of now.

χ computation finds its use also in communication through mobile phones. Global System for mobile communication (GSM) network is the most commonly used cellular network. It means that the land area is partitioned into hexagonal cells with transceiver for each cell joining the mobile devices within the cell. To keep away from doing signal interference two neighboring cells should not share the same channel. The famous Four Color Theorem for maps provides four different channels independently of the shape of the cells. However, in real life instances the network is not that simple. This is because the users first should be able to shift from a given cell area to another without experiencing any signal loss. Next there are several phone carriers. Then, the same frequency using cells must be separated by a distance at least two or three to avoid interference. So such a conceivable model is highly complicated. To be precise, the respective graph will be non planar and hence not feasible for a four-coloring. This problem can be dealt with another method of graph coloring, named list coloring. Here each vertex has a list of available colors associated with it. On can assign to each vertex a color from its list without violating the concept of proper coloring. For the problem of the cellular network, every vertex is a transceiver and a list of its available colors are a list of frequencies available for assignment at that transceiver. To find the number of frequencies required one has to find the minimum cardinality of the lists that allows the vertices of the graph colored without violating proper coloring concept. This number is named as the list chromatic number and its computation is much more hard to find than the usual χ. For more, one can refer to [4–6].

Another interesting application of graph coloring related to interference is register allocation. Here a computer program has huge number of variables, but a processor has only a limited number say 32, of registers for basic operations. So a compiler has to decide how to provide the registers these variables. Several variables can be allotted to a given register, but variables that are in use at a given time cannot be allotted to the same register without spoiling their values. When we model this by means of a graph, vertices stand for temporary variables and two of them are joined by an edge if they are involved concurrently at some point in the program. The number of registers required to make the program run is then equal to the least number of colors in a proper vertex coloring of this interference graph. It may be the case that this number is more than the fixed number of registers. However, then it amounts to the fact that there exist variables which cannot be allotted to any register. Such extra variables can be shifted to Random Acces Memory (RAM) after each operation by a method named spilling. As RAM access speed is very low the aim is to optimize the number of spills. Chaitin's noteworthy algorithm [7] applies interference graph coloring in both register allocations and spilling. To begin, an interference graph is designed. Such a graph is sparse in reality, so in the place of adjacency matrix, an adjacency vector is placed for at each vertex. Then, get rid of unwanted register copy operations by combining vertices. To finish, all vertices of degree less than 32 are removed in succession. Also it leads to null graph at the end of the process, in the case of which it is possible to allot colors for vertices by reversing the adopted method and including the omitted vertices back in place. Now to probe the other possibility where each vertex in the modified graph has at least 32 neighbors, one also adds a spill code. To identify which node to spill, the procedure maintains a table of guessed costs of spills for vertices, and resolve to spill a vertex whose cost is least as dictated by its current degree. For more see [8].

A distance graph $G(V, D)$ is a simple graph with vertex set V and any two elements of V are said to form an edge if the absolute euclidean distance between them is a member of D. If D happens to be a singleton set with the element 1 and the set V happens to be the euclidean two dimensional space $R \times R$ then the computation of χ of such a distance graph $G(R \times R, \{1\})$ is the famous HNP

(Hadwiger-Nelson Problem). For this distance graph the best known lower bound is 4 and the best known upper bound is 7. These bound are due to Nelson and Isbell respectively.

If instead of $R \times R$ if we consider $R \times R \times R$ with the same distance set {1} then Raiskii [9] showed that $\chi(G(R \times R \times R, \{1\})) \geq 5$ and later it was modified to 6 in [10]. A lot of efforts were made to obtain an upper bound of 21 for this graph in [11] and it was modified to 18 in [12]. It was further modified into 15 in [13]. For higher dimensions of the form $R \times R \times \cdots \times R$ n-times the lower and upper bound for the $\chi(G(R \times R \times \cdots \times R, \{1\}))$ are $(1 + o(1)).1.2^d$ and $(3 + o(1))^d$ in [14] and [15] respectively.

2. Motivational Factor

Two closely and nicely interconnected research fields of current interest with similar structures and problems are graph theory and Network science. Some notable examples of networks are interference network, air network and scholarly networks. Lately the network science led to understanding of the real world networks from functional perspective. Real systems of contrasting nature can be visualized through functigraph structure that consists of two copies of the identical network. The authors' work in [16] similar to this paper work but on a different platform along with the articles in [17] have really served as a motivational factor.

3. Catalan Numbers

Catalan numbers are introduced by Eugene C. Catalan in 1838 [18]. They are defined as $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$, for $n \geq 0$. A C++ program to generate C_n for n =0 to 15 is given in Figure 1. The first few C_n are, 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420. It is interesting to note that C_{100} has 57 decimal digits. One more way of formulating C_n is $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!(n+1)!}$. This formulation asserts that $C_n \in Z^+, \forall n \in Z^+$. In general an approximate value for C_n can be got by using Stirling's approximation as $C_n \approx \frac{2^{2n}}{(n+1)\sqrt{n\pi}}$.

C_n can also be defined in the form of a recurrence relation as $C_0 = C_1 = 1, C_n = \frac{4n-2}{n+1}C_{n-1},\ n \geq 0$. It is easy to see that $C_n = \frac{4n-2}{n+1}C_{n-1} = \frac{(4n-2)(4n-6)}{(n+1)n}\ C_{n-2} = \frac{(4n-2)(4n-6)(4n-10)}{(n+1)n(n-1)}\ C_{n-3} = \ldots = \frac{1}{(n+1)}\binom{2n}{n}$. Now from $C_{n+1} = \frac{4n+2}{n+2}C_n$ it follows that $(n+2)\,C_{n+1} = (4n+2)\,C_n$. Suppose that C_n is prime for some n. If $n > 3$, then $\frac{C_n}{C_{n+1}} < 1$. So $C_{n+1} = \alpha C_n$ for some $\alpha \in Z^+$. That is $(4n+2) = \alpha(n+2)$ for $\alpha \in [1,3]$. Hence, $n \leq 4$. From this it follows that the only prime Catalan numbers are C_2 and C_3. In 1988, it was known to the world that the Catalan numbers C_n were used in China even during the year 1730 [19–22]. Originally C_n were made known to Goldbach in 1751 by Euler in his letter while he was in the task of determining in a convex polygon the number of ways of splitting the polygonal area into a set of traingles also called triangulations [23]. C_n are involved in the process of classification of objects that are either geometric or algebraic.

```
#include<iostream>
#define N 25
using namespace std;
unsigned long int comb(unsigned long
int n, unsigned long int k)
{
unsigned long int r = 1;
if (k > n-k)
k = n-k;
for (int i=0; i<k; ++i)
{
r*= (n - i);
r/= (i + 1);
}
return r;
}
unsigned long int cat(unsigned long int
n)
{
unsigned long int c = comb(2*n, n);
return c/(n+1);
}
unsigned long int catdisplay(unsigned
long int L)
{
for (unsigned long int i=0; i<L; i++)
{
cout <<"c("<<i+1<<")"<<<<" :
"<<cat(i) << endl;
}
cout<<endl;
}
unsigned long int ans(unsigned long
int L)
{
unsigned long int i,j,k,a[i];
a[0]=2;
for (i=1; i<L; i++)
{
a[i]=cat(i+1)+cat(i);
}
for (i=0; i<L; i++)
{
cout<<"h"<<i<<" = "<<a[i];
cout<<"\t";
cout<<endl;
}
for (k=0; k<(L+1)/2; k++)
{
cout<<"M"<<k+1<<"="<<endl;
for (i=0; i<k+1; i++)
{
for (j=0; j<k+1; j++)
{
cout<<a[i+j];
cout<<"\t";
}
cout<<endl;
}
}
cout<<endl;
}
int main()
{
unsigned long int L, mat[N][N];
cout<<"Enter L: "<<endl;
cin>>L;
catdisplay(L);
ans(L);
cout<<endl;
return 0;
}
```

Figure 1. C++ Program to generate C_n, h_n.

4. Variations of C_n, and Prime Numbers

We probe the hidden relationship between C_n, primes and twin primes. We know that $n \in N$ is a prime iff $(n-1)! \equiv -1\ (mod\ n)$, a Wilson's contribution. An easy consequence of this is $2^n \equiv 2\ (mod\ n)$ when n is a prime. However the reverse is false as $n = 341$ obeys this congruence and n is not a prime. We call such numbers as pseudoprimes.

It is interesting to note that if n is prime and odd, then $(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} \equiv 2\,(mod\ n)$. To see this first observe that $(n-1)$ can be thought of $\left[(\frac{n-1}{2})!\right]^2 (-1)^{\frac{(n-1)}{2}}$ as $(n-j) \equiv -j\ (mod\ n)$ for all j. So $C_{\frac{n-1}{2}} = \frac{1}{\frac{n+1}{2}} \binom{n-1}{\frac{n-1}{2}} = \frac{2}{n+1} \frac{(n-1)!}{[(\frac{n-1}{2})!]^2} \equiv 2(-1)^{\frac{n-1}{2}}$. Notice that a converse of this is false. That is if $n = 5907$ then $C_{\frac{n-1}{2}} \equiv -2\,(mod\ n)$ but n is a composite. One can call such $n's$ as Catalan pseudoprimes.

It is known that if n is non composite and $2^n \equiv 2(mod\ n^2)$ then n is called Weiferich prime. So far only 1093 and 3511 are known as Weiferich primes. It is not known whether they are finitely many or not. Also it is known that only $5907, 1194649 (= 1093^2)$ and $12327121 (= 3511^2)$ are the Catalan psuedoprimes. Also if $n \in N$ then n and $n+2$ are both non composite if and only if $4(n-1)! + n + 4 \equiv 0\ (mod\ n(n+2))$ by Clements result. From this one can observe easily that if n and $n+2$ are both non composite then $2^{n+2} \equiv 3n + 8(mod\ n(n+2))$.

Another interesting fact is that if $n \in N$ and $n, n+2$ are both non composite then $8(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} \equiv 7n + 16(mod\ n(n+2))$. This is because $(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} \equiv 2(mod\ \ n) \Rightarrow 8(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} \equiv 8.2 = 16 \equiv 7n + 16(mod\ n)$. Now, $\frac{C_{\frac{n-1}{2}}}{C_{\frac{n+1}{2}}} = \frac{n+3}{4n} \Rightarrow 8(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} = 8(-1)^{\frac{n-1}{2}} \frac{n+3}{4n} C_{\frac{n+1}{2}} \equiv -\frac{(n+3)}{n} 2(-1)^{\frac{n+1}{2}} \equiv -4(\frac{n+3}{n}) C_{\frac{n+1}{2}} \equiv 2 \equiv 7n + 16(mod\ n+2)$. As we are aware of only three Catalan pseudoprimes "So far, we also note that $x+2, x-2, y+2, y-2, z+2, z-2$ are not primes if $x = 5907$, $y = 1194649$, $z = 12327121$." That is Catalan pseudoprimes and their ± 2 successors do not form Catalan twin pseudoprimes.

5. Hankel Transform of Catalan Sequence

Suppose that $A = \{a_n\}_{n \geq 0}$. Let $H = (h_n)$, a matrix generated by A of order $n \times n$ with the property that (i, j)th entry of H is given by a_{i+j} for $0 \leq i, j \leq n-1$. We call H a Hankel matrix. By the Hankel transform $\mathscr{H}$ of $A = \{a_n\}$ we mean the $det((h_n))$ generated by A where $det(.)$ indicate the determinant of $(.)$.

Let $h_n = \frac{\binom{2n}{n}}{n+1} + \frac{\binom{2n+2}{n+1}}{n+2}$. Then h_n is the sum of nth and $(n+1)$th Catalan numbers. A first few (h_n) are: $h_0 = \frac{1}{1} + \frac{2}{2} = 2; h_1 = \frac{2}{2} + \frac{6}{3} = 1 + 2 = 3; h_2 = \frac{\binom{4}{2}}{3} + \frac{\binom{6}{3}}{4} = \frac{6}{3} + \frac{20}{4} = 2 + 5 = 7; h_4 = \frac{\binom{6}{3}}{4} + \frac{\binom{8}{4}}{5} = 5 + \frac{70}{5} = 17$. Now the $\mathscr{H}(h_n)$ are obtained as follows: $\mathscr{H}(h_0) = |(h_{0+0})| = |2| = 2$.

$$\mathscr{H}(h_1) = \left| \begin{pmatrix} h_{0+0} & h_{0+1} \\ h_{1+0} & h_{1+1} \end{pmatrix} \right| = \left| \begin{pmatrix} h_0 & h_1 \\ h_1 & h_2 \end{pmatrix} \right| = \left| \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \right| = 5.$$

A C++ program for computing h_n are given in Figure 1. We used MATLAB to compute $\mathscr{H}(h_n)$. A few $\mathscr{H}(h_i)$ are as follows: $\mathscr{H}(h_0)$= 2; $\mathscr{H}(h_1)$= 5; $\mathscr{H}(h_2)$= 13; $\mathscr{H}(h_3)$= 34; $\mathscr{H}(h_4)$= 89; $\mathscr{H}(h_5)$= 233; $\mathscr{H}(h_6)$= 610; $\mathscr{H}(h_7)$= 1597.

Interestingly Layman [24], see A001906) from his intuition raised the conjecture that $\mathscr{H}(h_n)$ includes every other Fibonacci number. However, Cvetkovic et al. in [25] positively settled this conjecture.

6. On k-Catalan Numbers

Jacob Bobroswski in [26] defined the nth Catalan number in terms of all realizable length 2n sequences endowed with the features: (a) Every term is equal to either 1 or –1; (b) The value of every partial sum is greater than or equal to 0; (c) Grand total of all terms of such a sequence is 0. Jacob [26] then called a sequence with length a multiple of k as k-Raney sequence where every term is equal to 1 or $1-k$, The value of every partial sum is greater than or equal to 0, and grand total of all terms of such a sequence is 0. Finally he defined the nth k-Catalan number, $C_k(n)$ as all realizable length k_n, k-Raney sequences. Following Lobb's idea [22], Jacob [26] considered only the features (a) and (c) for a k-Raney sequences anddenoted by $L^k_{n,m}$ the number of realizable sequences with the characteristic that the number of terms that are 1 is $(k-1)n + m$ and the number of terms that are $1-k$ is $n-m$. Jacob then by taking $m = 0$, defined the nth k-Catalan number as $C_k(n) = L^k_{n,0} = \binom{kn}{n} - (k-1)\binom{kn}{n-1}$. Table 1 shows the initial values of k-Catalan numbers for k = 2 to 7 through a C++ program shown in Figure 2.

Table 1. Computation of k-Catalan number.

j	$C_2(j)$	$C_3(j)$	$C_4(j)$	$C_5(j)$	$C_6(j)$	$C_7(j)$
1	1	1	1	1	1	1
2	2	3	4	5	6	7
3	5	12	22	35	51	70
4	14	55	140	285	506	819
5	42	273	969	2530	5481	10472
6	132	1428	7084	23751	62832	141778
7	429	7752	53820	231880	749398	1997688
8	1430	43263	420732	2330445	9203634	28989675
9	4862	246675	3362260	23950355	115607310	430321633
10	16796	1430715	27343888	250543370	1478314266	6503352856

```
#include<iostream>
using namespace std;
unsigned long int comb(unsigned long int
n, unsigned long int k)
{
unsigned long int r = 1;
if (k > n-k)
k = n-k;
for (int i=0; i<k; ++i)
{
r *= (n - i);
r /= (i + 1);
}
return r;
}
```

```
int main()
{
unsigned long int I,J,K,L,M,N;
cout<<"Enter K: "<<endl;
cin>>K;
cout<<endl;
for(N=1;N<=10;N++)
{
L=K*N;
I=combf(L,N);
J=comb(L,N-1);
M=I-(K-1)*J;
cout<<M<<endl;
}
return 0;
}
```

Figure 2. C++ program to generate k^{th} Catalan numbers.

7. Computation of $\chi(G)$ of Catalan Number Distance Set

Consider the distance graph $G(Z, D = \{d_1 < d_2 < ...\})$. Such objects were dealt in [27]. Suppose that $D = D_{Cat}(10) = \{1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796\}$ is a set of first 10 distinct Catalan numbers. Form $G(Z, D_{Cat})$ where D_{Cat} consists of the Catalan numbers. A minimal graph to realize each element of D_{Cat} as a distance between any two of its vertices exactly once is the simplest graph P_{11}. A coloring of the elements of $V(G)$ is deemed proper if any two adjacent elements in it are colored distinctly. The smallest number of colors used in such a process named as the chromatic number $\chi(G)$.

Let $V(P_{11}) = \{u_1, u_2, \ldots, u_{11}\}$. Set $l(u_1) = 1, l(u_2) = l(u_1) + D_{Cat}(1), l(u_i) = l(u_{i-1}) + D_{Cat}(i-1)$. As P_{11} is bipartite, $\chi(P_{11}) = 2$. Hence,

Proposition 1. *Let $G(Z, D_{Cat}(n))$ be any Catalan number distance graph. Let $|D_{Cat}(n)| < \infty$ and $n \in Z$. Then $\chi(G(Z, D_{Cat}(n))) = 2$ as $G(Z, D_{Cat}(n)) \cong P_{n+1}$, provided $V(P_{n+1}) = \{u_i : 1 \le i \le n+1\}$ with $l(u_1) = 1, l(u_i) = l(u_{i-1}) + D_{Cat}(i-1)$.*

Recall the following two results of Yegnanarayanan from [28].

Proposition 2. $\chi(G(Z, D)) = 2$ *if the elements of D are odd integers.*

Proposition 3. *If for a given positive integer n, D^n consists all those elements of D built by integers divisible by n, then $\chi(G(Z,D)) \leq min_{n\in N}\ n(|D^n|+1)$.*

Theorem 1. *Consider $G(Z, D_{Cat}(15))$ where $D_{Cat}(15)$ denotes the distance set of first 15 distinct Catalan numbers. Then $3 \leq \chi(G(Z, D_{Cat}(15))) \leq min_{n\in N}\ n(|D^n|+1) = 16$ where $D^n \subset D^n_{Cat}(15)$ is that subset of $D^n_{Cat}(15)$ built by integers divisible by n.*

Proof. The graph G_1 shown in Figure 3 namely $G_1(\{1,2,...,10\}, D_{Cat}(3))$ is a subgraph of $G(Z, D_{Cat}(15))$. We show in the next section that $\chi(G(\{1,2,...,10\}, D_{Cat}(3))) = 3$. Therefore, $3 = \chi(G(\{1,2,...,10\}, D_{Cat}(3))) \leq \chi(G(Z, D_{Cat}(15)))$ as χ is a monotone function. We now determine the upper bound as follows using Proposition 3. Clearly, $D_{Cat}(15) = \{1,2,5,14,42,132,429,1430,4862, 16796,58786,208012,742960,2674440,9694845\} \subseteq Z$. For $n = 1$, we get $n(|D^n_{cat}(15)|+1) = 1(|D^1_{cat}(15)|+1) = 1(15+1) = 16$; For $n = 2$, $2(|D^2_{cat}(15)|+1) = 2(|\{d_i \in D^2_{Cat}(15) : 2|d_i, i = 1,2,...,15\}|+1) = 2(|\{2,14,42,132,1430,4862,16796,58786,208012,742900,2674440\}|+1) = 2(11+1) = 24$; For $n=3$, $3(|D^3_{cat}(15)|+1) = 3(|\{42,132,429,2674440,9694845\}|+1) = 3(5+1) = 18$;. For $n=4$, $4(|D^4_{cat}(15)|+1) = 4(|\{132,16796,208012,742900,2674440\}|+1) = 4(5+1) = 24$; n =5,6,7,9 to 15 are not improving the above bounds. Also $n = 8$ yields a bound same as that for $n = 1$. Hence we conclude that $\chi(G(Z, D_{Cat}(15))) \leq 16$. □

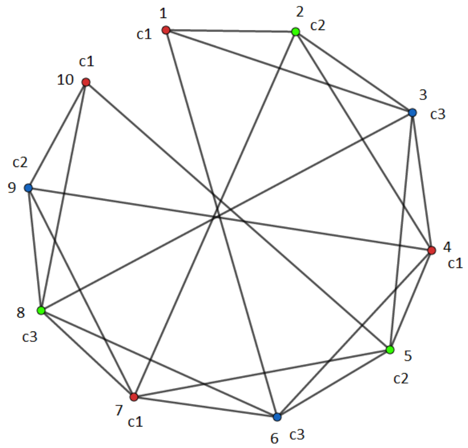

Figure 3. G_1: C_1 = Red, C_2 = Green, C_3 = Blue.

In general, in view of Theorem 1, we raise the following problem:

Problem 1. *What is $\chi(G(Z, D_{Cat}(n)))$ for any $n \in N$?*

Note 1. An easy upper bound for $\chi(G(Z, D_{Cat}(n)))$ is that $|D_{Cat}(n)|+1$.

Theorem 2. *Consider $G(Z,D)$ with $D = \{h_1,...,h_{15}\} = \{3,7,19,56,174,561,1859,6292,21658,75582, 266798,950912,3417340,12369285,11424336\}$ where h_i's are the sum of the ith and $(i+1)$th Catalan numbers. Then $3 \leq \chi(G(Z,D)) \leq min_{n\in N}\ n(|D^n|+1) = 16$.*

Proof. Voigt et al. in [29,30] have proved that if the elements of D includes two coprime elements of distinct parity then $\chi(G(Z,D)) = 3$. As $(3,6292) = 1$ and $3,6292 \in D$, we infer that $\chi(G(Z,D))$ is more than or equal to 3. Now the upper bound can be found by using the Proposition 3. For n = 1, $n(|D^1|+1) = 1(15+1) = 16$. For n=2, $2(|D^2|+1) = 2(9+1) = 20$; It is easy to check that $n = 3$ to 16 has not improved the upper bound 16 provided by $n = 1$. Therefore $3 \leq \chi(G(Z,D)) \leq 16$. □

Theorem 3. *Consider $G(Z,D)$ with $D = \{\mathscr{H}(h_1), \ldots, \mathscr{H}(h_8)\} = \{2, 5, 13, 34, 89, 233, 610, 1597\}$ where $\mathscr{H}(h_i)'s$ are the values of the Hankel transform of first 8 $h_i's$ of Theorem 2. Then $3 \leq \chi(G(Z,D)) \leq min_{n\in N}\ n(|D^n|+1) = 3$.*

Proof. The lower bound follows from the same reasoning of what is said in Theorem 2. We now find the upper bound by using the Proposition 3. For $n = 1$, we get $n(|D^n|+1) = 9$. For $n = 2$, we get $2(|D|^2+1) = 2(|\{2, 34, 610\}|+1) = 2(3+1) = 8$; For $n = 3$, we get $3(|D|^3+1) = 3(0+1) = 3$. Therefore $\chi(G(Z,D)) = 3$. □

Problem 2. *What is $\chi(G(Z,D))$ where D consists of any finite list of n $h_i's$ of Theorem 2.*

Problem 3. *What is $\chi(G(Z,D))$ where D consists of any finite list of n $\mathscr{H}(h_i)'s$ of Theorem 3.*

Note 2. In a similar manner as in Theorem 1 one can determine bounds (both lower and upper) for the chromatic number of generalized k-Catalan number distance graphs listed in Table 1. However finding the bounds (both lower and upper) of $\chi(G(Z,D))$ whose distance set elements are finite list of n such k-Catalan number is an open question.

Note 3. Presume that the vertex set of a Catalan number distance graph is labelled arbitrarily and an element from the Catalan distance set is allowed to repeat in the graph then we get several other non trivial graphs. For instance, Figure 3 shows a Catalan number distance graph whose χ is 3. This raises an interesting question: what is the χ of such an arbitrarily formed Catalan number distance graph?

8. Graph Coloring for Interference Networks

The problem of minimization of interference in wireless communication pose enormous challenges. If a vertex $x's$ interference range includes another vertex y then one can say x may interfere with y. This means the magnitude of interference on a vertex y can be measured by the magnitude of interference produced by all those vertices whose transmission area embraces y. It is known that in frequency division multiplexing of mobile networks the minimization of interference is directly related to the bringing down of number of channels which then leads to the increasing of bandwidth of the frequency channels. A small interference is useful to code overhead in systems where code division multiplexing is employed. However, in the case of networks employing battery driven devices it is better to minimize the interference to increase the longevity of the network. Interference can be kept to a minimum by keeping vertices with limited transmission power. In such a case, the area covered by such vertices result in low interference. However, then this action may lead to the disconnection of communication links. So, it is prudent to estimate the amount of reduction of transmission power of the vertices so that the connectivity characteristic of the network is preserved. Minimization of Interference in networks results in non-collisions and packet retransmissions and this factor saves power consumption and improves the longevity of the network.

Interference can be represented in the form of an Interference graph of a wireless network. Here the edges denote the interference occurring in their respective end points. The problem of minimizing the vertex collisions and signal clashes can be thought of as allotment of color to vertex task of the associated graph of interference. Distinctly colored vertices are assigned distinct channel of radio frequency. Robust graph vertex coloring methods suggest effective or correct channel picking ways that decreases wireless interference. The graph vertex coloring method is useful as it bars vertices from getting connected with other conflicting ones via radio frequency. These methods are pertinent as it conforms to mathematical rigidity. Prudent allotment of channels aligns allotment of color to vertex task and interference minimizing task in wireless network [31].

Suppose that η is the greatest number of vertices of equal degree j in a simple graph G with $j \geq \lfloor \frac{\Delta(G)+2}{2} \rfloor$. Let $r = \lceil \frac{\eta}{\eta+1}(\Delta(G)+2) \rceil$ and $d_1 \geq d_2 \geq \ldots \geq d_p$ be the degree sequence of G. If $d_r < \frac{\Delta(G)+2}{2}$

then clearly $d_r < r$ *as* $\eta \geq 1$. If $d_r \geq \frac{\Delta(G)+2}{2}$ then for $l = 1, 2, ..., r$. $d_l \leq \Delta(G) - \lceil \frac{l}{\eta} \rceil + 1 < \Delta(G) - \frac{l}{\eta} + 2$. That is $d_r < \Delta(G) - \frac{r}{\eta} + 2$. However, then this means that the G is r-degenerate or any subgraph of G includes in it a vertex of degree $< l$. Further it is known that at most r colors are needed to color any r-degenerate G properly. Hence,

Theorem 4. *If G is simple then $\chi(G)$ is at most $\lceil \frac{\eta}{\eta+1}(\Delta(G) + 2) \rceil$.*

Discussion. Consider the task of assignment of wi-fi channel in the language of graph theory. If G has k-colors as spectrum (also called k-channels). Let W be a $k \times k$ matrix of interferences associated with each vertex of G. The aim is to find a minimum threshold $t_0 \in Z, t > 0$ in such a way that (G, W) accepts a k-coloring f (allotment of channels) with reference to which $I_u(G, W, f) = \sum_{w \in N(u)} W(f(w), f(u)) \leq t_0$ for all $u \in V(G)$. t_0 is called least k-chromatic threshold of (G, W), denoted by $T_k^*(G, W)$.

Consider the graph G_1 in Figure 3 with $V(G_1) = \{1, 2, ..., 10\}$, $E(G_1) = \{(u, v) : d(u, v) \in D_{Cat}(3)\}$. Let $f : V(G_1) \to \{c_1, c_2, c_3\}$ be such that $f(3i + 1) = c_1$ for $0 \leq i \leq 3$; $f(3i + 2) = c_2$ for $0 \leq i \leq 2$; $f(3i) = c_3$ for $1 \leq i \leq 3$. Then f is a 3-chromatic coloring for G_1.

Let $W = (w_{ij}) = w(c_i, c_j) = \begin{pmatrix} (c_1, c_1) & (c_1, c_2) & (c_1, c_3) \\ (c_2, c_1) & (c_2, c_2) & (c_2, c_3) \\ (c_3, c_1) & (c_3, c_2) & (c_3, c_3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Then $W^T = W$. Now $I_1(G_1, W, f) = \sum_{u \in N(1)} W(f(u), f(1)) = W(f(2), f(1)) + W(f(3), f(1)) + W(f(6), f(1)) = W(c_2, c_1) + W(c_3, c_1) + W(c_3, c_1) = 1 + 1 + 1 = 3$; Similarly, $I_2(G_1, W, f) = 4$; $I_i(G_1, W, f) = 5$ for $3 \leq i \leq 8$; $I_9(G_1, W, f) = 4$; $I_{10}(G_1, W, f) = 3$. So here $t_o = 5$ and the least 3-chromatic threshold of (G, W), written as $T_3^*(G, W) = 5$.

In view of Theorem 4, the graph G, of Figure 3 has an upper bound for its χ as 5. Moreover the presence of a K_3 in G_1 indicates $3 \leq \chi(G_1) \leq 5$. This along with the fact that f is a 3-chromatic coloring shows $\chi(G_1) = 3$.

As the problem of determination of $T_k^*(G, W)$ fixes the parameter k and focus on minimizing t_0, it would be interesting to look at its complementary problem where the threshold $t_0 \in Z, t_0 > 0$ is fixed. The focus is on minimizing the number of colors k (channels) by allowing the size of the spectrum to be the cardinality of $V(G)$ in such a manner that (G, W) accepts a color function f with k-colors (allotmentt of channels) where at any vertex the interference is $\leq t_0$. t_0 is called as t_0-interference χ of (G, W), written as $\chi_{t_0}(G, W)$. A main feature of this determination lies in finding the least number of colors (or frequencies) to realize minimum throughput in the network for all users. Hence one can also think of deeming it as a task of finding to ensure quality of service the actual resources requirement.

Suppose that for a given (G, W, f) a color specific interference at a vertex u is defined as follows: $I_u^i(G, W, f) = \sum_{w \in N(u)} W(f(w), i)$. Then if $f(u) = i$ then $I_u^{f(u)}(G, W, f) = I_u(G, W, f)$. We call a k-coloring f of G as W-concrete if $I_u(G, W, f) \leq I_u^j(G, W, f) \forall j \in \{1, 2, ..., k\}$. Let $I^* = \sum_{(u,w) \in E(G)} W(f(u), f(w))$. Then one can clearly visualize that $I^* = \frac{1}{2} I_u(G, W, f)$. Also if $I_0 = min\{I^*\}$ for some specific coloring f_0 then it will be W-concrete. If for some $u \in V(G)$, $I_u^{f(u)} > I_u^j$ then one can attempt to recolor again and again to achieve I_0. Please note that each step in the process contributes $I_u^j(G, W, f)$ to I^* and decrements $I_u^{f(u)}(G, W, f)$. This procedure results in the reduction of I^* by a quantity $I_u^{f(u)} - I_u^j$ and terminates by achieving a W-concrete coloring of G. So

Theorem 5. *A W-concrete coloring for (G, W, f) with color spectrum cardinality atleast 2 exists. For the graph G_1 in Figure 3,*
$I_1^{c_1}(G, W, f) = 3$; $I_1^{c_2}(G, W, f) = 2$; $I_1^{c_3}(G, W, f) = 1$;
$I_2^{c_1}(G, W, f) = 2$; $I_2^{c_2}(G, W, f) = 4$; $I_2^{c_3}(G, W, f) = 3$;
$I_3^{c_1}(G, W, f) = 3$; $I_3^{c_2}(G, W, f) = 2$; $I_3^{c_3}(G, W, f) = 5$;
$I_4^{c_1}(G, W, f) = 5$; $I_4^{c_2}(G, W, f) = 3$; $I_4^{c_3}(G, W, f) = 2$;
$I_5^{c_1}(G, W, f) = 2$; $I_5^{c_2}(G, W, f) = 5$; $I_5^{c_3}(G, W, f) = 3$;
$I_6^{c_1}(G, W, f) = 2$; $I_6^{c_2}(G, W, f) = 3$; $I_6^{c_3}(G, W, f) = 5$;

$I_7^{c_1}(G,W,f) = 5;\quad I_7^{c_2}(G,W,f) = 2;\quad I_7^{c_3}(G,W,f) = 3;$
$I_8^{c_1}(G,W,f) = 3;\quad I_8^{c_2}(G,W,f) = 5;\quad I_8^{c_3}(G,W,f) = 2;$
$I_9^{c_1}(G,W,f) = 2;\quad I_9^{c_2}(G,W,f) = 3;\quad I_9^{c_3}(G,W,f) = 4;$
$I_{10}^{c_1}(G,W,f) = 3;\quad I_{10}^{c_2}(G,W,f) = 1;\quad I_{10}^{c_3}(G,W,f) = 2.$

It is easy to see that $I_l(G,W,f) > I_l^j(G,W,f)$ for $1 \le l \le 10$ and hence f is not a W-concrete coloring for G. Moreover, $I^*(G,W,f) = 22 = \frac{1}{2}I_u(G,W,f)$ for $u \in V(G_1)$.

Suppose that $t_0 = 4$ is fixed and $D_{Cat}(2) = \{1,2\}$ be a set of first two distinct Catalan numbers. Let $V(G_2) = \{1,2,3,4,5,6\}$ and $E(G_2) = \{(u,v) : d(u,v) \in D_{Cat}(2)\}$. Then the graph G_2 is shown in Figure 4.

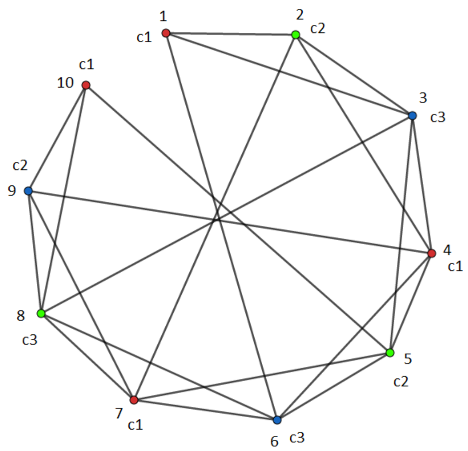

Figure 4. G_2: C_1 = Red,C_2 = Green, C_3 = Blue.

Clearly the matrix $W = (w_{ij})$ associated with G_2 is the distance between the elements of $V(G_2)$.

That is $W = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$ and $W = W^T$. Define $f : V(G_2) \to \{c_1, c_2, c_3\}$ as $f(3i+1) = c_1\ \ 0 \le i \le 1$; $f(3i+2) = c_2\ \ 0 \le i \le 1$; $f(3i) = c_3\ \ 0 \le i \le 2$; Then f is a 3-chromatic coloring for G_2. Now, $I_1(G_2,W,f) = 2$; $I_2(G_2,W,f) = 3$; $I_3(G_2,W,f) = 4$; $I_4(G_2,W,f) = 4$; $I_5(G_2,W,f) = 3$; $I_6(G_2,W,f) = 2$. Effortlessly $I_l(G_2,W,f) \le t_0 = 4$ for $1 \le l \le 6$. Hence, $\chi_{t_0=4}(G_2,W) = 4$ is the 4-interference chromatic number.

If we alter $t_0 = 4$ to $t_0 = 3$ then we can alter the size of $V(G_2)$ from 6 to 4 as $V(G_2) = \{1,2,3,4\}$. Then the resulting graph turns out to be K_4, the complete graph on 4 vertices. $G_3 = K_4$ is shown in Figure 5.

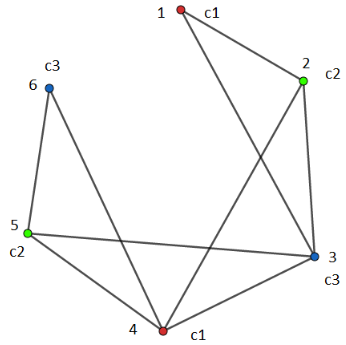

Figure 5. G_3: C_1=Red,C_2=Blue, C_3=Green, C_4=Purple.

$$W = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$ and $W = W^T$. Define $f : V(G_3) \to \{c_1, c_2, c_3, c_4\}$ as $f(i) = c_i$, $1 \leq i \leq 4$. Then f is a 4-chromatic coloring for G_3. Now $I_l(G_3, W, f) = 3$, $1 \leq l \leq 4$. So, $I_l(G_3, W, f) \leq t_0 = 3$. Hence, $\chi_{t_0=3}(G_3, W, f) = 3$ is the 3-interference chromatic number.

9. Some Lower and Upper Bounds for χ

For a simple graph $G = (V, E)$ with $|V(G)| = p, |E(G)| = q$, let $d(u_i)$ be the degree of $u_i \in V(G)$ and $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of G respectively. $d(G) = \sum_{i=1}^{p}(d(u_i)/p)$ is the average degree of G. Clearly $d(G) \in [0, p-1]$ and $d(G)$ need not always be an integer. In [19] the authors have called the ceiling of $d(G)$ ($\lceil d(G) \rceil$) as the top of G and denoted it as $\mu(G)$. It is easy to see that for all $i \in [1, p]$ and $d(u_i) \in [0, p-1]$ there is a $l \in [0, p-1]$ with $d(u_i) = l$. For any arbitrary $X \subseteq V(G)$ the authors called $w_X(l) = |\{x \in X : d(x) = l\}|$ as the frequency of degree u_i. If $X = V(G)$ then the frequency of degree u_i is denoted as $w(l)$. From the First Theorem of Graph Theory that $\sum_{u \in V(G)} deg(u) = 2|E(G)|$, it follows that $\sum_{l=1}^{p} lw(l) = 2q$. They further called p-times the difference, viz., $p(\mu(G) - d(G))$ as the gap $h(G)$. It is a trivial fact that if $d(G) \in Z^+$ then $h(G)$ is zero.

For instance consider the graph G_1 of Figure 3. Please note that $d(G_1) = 4.4$ and the $top(G_1) = \mu(G_1) = 5$. Furthermore, G_1 has 2 3-degree vertices, 2 4-degree vertices, and 6 5-degree vertices. In view of this it is easy to check that $\sum_{l=1}^{10} lw(l) = 2 \times 3 + 2 \times 4 + 6 \times 5 = 44 = 2|E(G_1)| = 2 \times 22$. Also the $gap(G_1) = h(G_1) = 10(5 - 4.4) = 6$ and this verifies the fact that if $d(G_1)$ is not an integer then the gap is non-zero.

Suppose that each vertex has a list of k colors endowed with G. A vertex proper coloring allots every vertex a color from its list. Obviously adjacent vertices do not share the same color in the said process. The smallest integer k, that results in a vertex proper coloring out of any given list of length k is the list chromatic number written as $\chi_l(G)$. It is clear that $\chi(G) \leq \chi_l(G)$. Moreover in [30] it is established that $\chi_l(G) \leq \alpha(d/logf)$ where the $N_G(u)$ for any $u \in V(G)$ has the largest of d^2/f edges for some $1 < f < d^2$ and α is some constant. By a combinatorial Laplacian operator L of $G = (V, E)$, we mean a matrix $L = D - A$. Here D is diagonal matrix and A is adjacency matrix of G. Clearly $D(u, u) = d(u)$ for every $u \in V(G)$. For any $V_1, V_2 \subseteq V(G), e(V_1, V_2) = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2, (v_1, v_2) \in E(G)\}$.

Theorem 6. *Presume $G = (V, E)$ as $(p(G), q(G), d(G), \sigma_i)$ graph with $\sigma_i, i \in [1, p]$ as the eigen values of L. If the absolute difference of d and σ_i is bounded above by β for some β and $i \neq 0$ then $\lceil \frac{d}{\beta} \rceil \leq \chi(G) \leq O(|\frac{d}{log(min\{\frac{p}{d}, \frac{d}{\beta}\})}|)$.*

Proof. The authors proved in [32] that $(\frac{p}{p-1})(d-\beta) \leq d(v) \leq (\frac{p}{p-1})(d+\beta) < d+\beta$ for each $v \in V(G)$. From this one can deduce that $\Delta(G) \leq d+\beta$. Moreover for any $V_1 \subseteq V(G)$, they proved that $|2e(V_1, V_1) - \frac{d|V_1|(|V_1|-1)}{p}| \leq \frac{2\beta}{p}|V_1|(p_1 - |V_1|/2)$. So for any $G_1 \subseteq G$ with $|p(G_1)| \leq d+\beta$ we have $|q(G_1)| \leq \frac{d(d+\beta)^2}{p} + 2\beta(d+\beta)$. Now by letting $\frac{1}{f} = \frac{d}{p} + \frac{2\beta}{d}$ we have by Vu's Theorem [30] that $\chi_l(G) = O(|\frac{d+\beta}{logf}|) = O(|\frac{d}{log(min\{\frac{p}{d},\frac{d}{\beta}\})}|)$. Further the authors in [32] have also showed along with the hypothesis that any independent subset V_0 of $V(G)$ satisfies $|V_0| \leq (\beta p)/d + 1$. So as $\chi(G) \leq \chi_l(G)$ we have $\lceil \frac{d}{\beta} \rceil \leq \chi(G) \leq O(|\frac{d}{log(min\{\frac{p}{d},\frac{d}{\beta}\})}|)$. □

Again consider the graph G_1 of Figure 3. The eigen values of the Laplacian matrix L of G_1 is calculated using MAtrix Laboratory called MATLAB. They are $\sigma_i, 1 \leq i \leq 10 :$ 0, 1.8279, 3.4384, 4, 4, 4.7002, 5, 5.7818, 7.5616, 7.6901. By letting $\beta = 5$ as the upper bound for the absolute difference, $|d - \sigma_i|$ for $1 \leq i \leq 10$ we have computed the lower bound, $\lceil \frac{d}{\beta} \rceil = \lceil 0.88 \rceil = 1$ and the upper bound $O(|\frac{d}{log(min\{\frac{10}{4.4},\frac{4.4}{5}\})}|) = O(79.2545319) = O(79)$.

10. Conclusions

We used C++ programming and Matrix Laboratory (MATLAB) to generate Catalan numbers, generalized Catalan numbers, Hankel transform of a transformed sequence of Catalan numbers, exhibited the output and the nature of their distribution by means of graphs. While computing the χ of certain distance graph class whose distance set elements are either of the above said numbers, we also computed the least k-chromatic threshold of an interference graph that arose out of modeling the problem of assignment of wi-fi channel. Then we proved for a given color specific interference graph the existence of a concrete coloring with color spectrum. Further we computed the lower and upper bound for $\chi(G)$ in relation to $d(G)$ and Laplacian operator. We also raised several open problems and hope to revert back on them elsewhere.

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References

1. Garey, M.R.; Johnson, D.S.; Stockmeyer, L. Some simplified NP-complete problems. In Proceedings of the Sixth Annual ACM Symposium on Theory of Computing, Seattle, WA, USA, 30 April–2 May 1974; pp. 47–63.
2. Leighton, F.T. A graph coloring algorithm for large scheduling problems. *J. Res. Natl. Bur. Stand.* **1979**, *84*, 489–506. [CrossRef]
3. Malkawi, M.; Hassan, M.A.; Hassan, O.A. A new exam scheduling algorithm using graph coloring. *Int. Arab J. Inf. Technol.* **2008**, *5*, 80–86.
4. Eisenblatter, A.; Grotschel, M.; Koster, A.M.C.A. Frequency planning and ramifications of coloring. *Discuss. Math. Graph Theory* **2002**, *22*, 51–88. [CrossRef]
5. Park, T.; Lee, C.Y. Application of the graph coloring algorithm to the frequency assignment problem. *J. Oper. Res. Soc. Jpn.* **1996**, *39*, 258–265. [CrossRef]
6. Tamura, H.; Nakano, K.; Sengoku, M.; Shinoda, S. On applications of graph/network theory to problems in communication systems. *ECTI Trans. Comput. Inf. Technol.* **2011**, *5*, 8–14.

7. Chaitin, G.J. Register allocation & spilling via graph coloring. In Proceedings of the 1982 SIGPLAN symposium on Compiler construction, Boston, MA, USA, 23–25 June 1982; pp. 98–105.
8. Anand, E.; Panneerselvam, R. Literature review of open shop scheduling problems. *Intell. Inf. Manag.* **2015**, *7*, 33–52. [CrossRef]
9. Raiskii, D.E. The realization of all distances in a decomposition of the space R^n into $(n+1)$ parts. *Math. Notes* **1970**, *7*, 194–196. [CrossRef]
10. Nechushtan, O. A note on the space chromatic number. *Discret. Math.* **2002**, *256*, 499–507. [CrossRef]
11. Szekely, L.A.; Wormald, N.C. Bounds on the measurable chromatic number of R^n. *Discret. Math.* **1989**, *75*, 343–372. [CrossRef]
12. Coulson, D. An interesting 18-coloring of 3-space omitting distance one. *Discret. Math.* **1997**, *170*, 241–247. [CrossRef]
13. Coulson, D. An interesting 15-coloring of 3-space omitting distance one. *Discret. Math.* **2002**, *256*, 83–90. [CrossRef]
14. Frankl, P.; Wilson, R.M. Intersection theorems with geometric consequences. *Combinatorica* **1981**, *1*, 357–368. [CrossRef]
15. Larman, D.G.; Rogers, C.A. The realization of distances within sets in Euclidean space. *Mathematika* **1972**, *19*, 1–24. [CrossRef]
16. Muhammad, F.; Muhammad, M.; Zafar, U.; Usman, A.; Imran, J. On the distinguishing number of functigraphs. *Symmetry* **2018**, *10*, 1–11, doi:10.3390/sym10080332. [CrossRef]
17. Rodriguez, J.M. Graph theory. *Symmetry* **2018**, *10*, 32. [CrossRef]
18. Alter, R. Some remarks and results on Catalan numbers. In Proceedings of the 2nd Louisiana Conference on Combinatorics, Graph theory and Computing, Baton Rouge, LA, USA, 8–11 March 1971.
19. Larcombe, P. On the history of the Catalan numbers: A first record in china. *Math. Today* **1999**, *35*, 89.
20. Liu, J.J. Ming Antu and Catalan numbers. *J. Math. Res. Expos.* **2002**, *22*, 589–594. (In Chinese)
21. Luo, J.J.; Ming Antu. The first discover of Catalan numbers. *Neimenggu Daxue Xuebao* **1988**, *19*, 239–245. (In Chinese)
22. Lobb, A. Deriving the nth Catalan number. *Math. Gaz.* **1999**, *83*, 109–110. [CrossRef]
23. Bezhanishvili, G.; Leung, H.; Lodder, J.; Pengelley, D.; Ranjan, D. Counting Triangulations of a Polygon. Teaching Discrete Mathematics via Primary Historical Sources. Available online: http://www.math.nmsu.edu/hist_projects/ (accessed on 15 May 2018).
24. Sloane, N.J.A. The On-Line Encyclopaedia of Integer Sequences. Available online: http://www.research.att.com (accessed on 15 May 2018).
25. Cvetkovic, A.; Rajkovic, P.; Inkovic, M. Catalan Numbers and Hankel Transform and Fibonacci Numbers. *J. Integer Seq.* **2002**, *5*, 1–8.
26. Bobroswski, J. Generalized Catalan Numbers and Some Divisibility Properties. Master's Thesis, The University of Nevada, Las Vegas, NV, USA, November 2015.
27. Eggleton, R.B.; Erdös, P.; Skilton, D.K. Coloring the real line. *J. Combin. Theory Ser. B* **1985**, *73*, 86–100. [CrossRef]
28. Yegnanarayanan, V. Chromatic number of graphs with special distance sets, I. *Algebra Discret. Math.* **2014**, *17*, 135–160. [CrossRef]
29. Damas, M.P.; de Abreu, L.M.N.M. New Concepts and Results on the Average Degree of a Graph. *Appl. Anal. Discret. Math.* **2007**, *1*, 284–292. [CrossRef]
30. Vu, V. A general upper bound on the list chromatic number of locally sparse graphs. *Comb. Probab. Comput.* **2002**, *11*, 103–111. [CrossRef]
31. Bertossi, A.A.; Pinotti, C.M.; Tan, R.B. Efficient use of radio spectrum in wireless networks with channel separation between close stations. In Proceedings of the 4th International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications, Boston, MA, USA, 11 August 2000; pp. 18–27.
32. Chung, F. Discrete isoperimetric inequalities. *Surv. Differ. Geom. IX* **2004**, *9*, 53–82. [CrossRef]