



Article Linking Time-Domain Vibration Behaviors to Spatial-Domain Propagating Waves in a Leaf-like Gradient-Index Phononic Crystal Lens

Kuo-Chih Chuang¹, Dan-Feng Wang¹, Jun-Jie Liu² and Chan-Yi Liao^{3,*}

- Key Laboratory of Soft Machines and Smart Devices of Zhejiang Province, Institute of Applied Mechanics, School of Aeronautics and Astronautics, Zhejiang University, Hangzhou 310027, China; chuangkc@zju.edu.cn (K.-C.C.); 21824001@zju.edu.cn (D.-F.W.)
- ² Applied Mechanics and Structure Safety Key Laboratory of Sichuan Province, School of Mechanics and Engineering, Southwest Jiaotong University, Chengdu 610031, China; junjie_liu@swjtu.edu.cn
- Department of Mechanical Engineering, National Central University, Taoyuan 320317, Taiwan
- * Correspondence: chanyi@ncu.edu.tw

Abstract: It is known that a propagating wave at a certain spatial point can be decomposed into plane waves propagating at different angles. In this work, by designing a gradient index phononic crystal lens (GRIN PCL) with transverse-continuous leaf-like unit cells, we theoretically and experimentally show that the spatial-domain propagating waves in finite periodic structures can be linked to their time-domain vibration behaviors. The full-field instantaneous focusing behaviors of Lamb waves in the proposed leaf-like GRIN PCL give an example of the wave-vibration linkage in finite periodic structures while allowing a certain complexity. The conclusion in this paper can help one skip iterative time-consuming finite element analysis (e.g., time-stepping solutions) to avoid possible numerical instabilities occurred in calculating transient wave field on practical finite metamaterials or phononic crystals having unit cells with complicated configurations.

Keywords: phononic crystals; lamb waves; gradient-index lens; vibration

1. Introduction

Elastic metamaterials and phononic crystals are artificial engineered materials with band gaps capable of suppressing wave propagation [1–7]. They have attracted significant interest due to their capabilities of tailoring, focusing, or guiding elastic waves. Generally, research regarding propagating waves on periodic structures such as metamaterials or phononic crystals uses finite element time-dependent solver or finite-difference timedomain (FDTD) method to observe the manipulated wave field. Take phononic crystals as examples, borrowing from the design concept of optical lens, Lin et al. designed a GRIN phononic crystal lens to control the propagation of elastic waves, where the focusing results were numerically investigated by the finite-difference time-domain method [8]. Recently, based on destructive interference mechanism, Yan et al. designed a phononic crystal with coupled lanes to enhance elastic wave attenuation in the low-frequency regime and the finite element method (FEM) analysis as well as theoretical modeling are presented [9]. Danawe et al. designed a conformal gradient-index phononic crystal lens that are integrated within a pipe and applied the amplified guided wave modes to detect the damage of the pipelines, with the validation using the FEM [10]. Zareei et al. proposed a continuous profile GRIN lens to focus flexural waves in a thin plate and used the FEM to validate the theoretical design [11]. However, when using a time-dependent solver in a commercial finite element software for calculating transient wave field, large and time-consuming computations are inevitably involved to obtain time-stepping solutions that based on iteration from the newest calculated time-step. For long-time wave propagations, numerical instabilities might grow sufficiently large and Nyquist-like error (i.e., related to reasonable



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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). mesh size) might propagate to dominate the solution and distort the physical behavior [12]. If metamaterials or phononic crystals have unit cells with a continuous or complicated profile, a theoretical-based modeling approach that can skip time-stepping solutions would be desired.

The full-field traveling waves in a metamaterial or phononic crystal is essentially a transient process, which should be a superposition of the instantaneous vibrational normal modes of the whole structure. Thus, for practical finite metamaterials or phononic crystals with complicated configurations, a transient analytical method based on the vibrational normal modes would be promising in skipping time-stepping solutions and can provide another route to appreciate the behaviors of the modulated waves by linking the long-time transient process to information of vibrational characteristics [13,14]. Linking time-domain vibration behaviors to spatial-domain propagating waves can also help us understand instantaneous wave responses at any time or at any interval with identical high calculation efficiency, if compared to a full finite element analysis.

To give an example to show the linkage between the time-domain vibration behaviors and spatial-domain propagating waves while allowing a certain complexity, we theoretically and experimentally investigate the full-field instantaneous focusing behaviors of the Lamb waves on a gradient-index phononic crystal lens (GRIN PCL) with complex unitcell profiles. Originally inspired from lens in optics, GRIN PCLs are artificial engineered periodic structures constructed by arranging unit cells in plates that obeys a prescribed gradient refractive index distribution capable of focusing or imaging elastic or sound waves [15–21]. Currently, most published phononic crystal lenses arrange discrete unit cells that are either permanently perforated, engraved, or decorated with added masses on the host plates in achieving wave focusing [10,22,23]. In this paper, a GRIN PCL having unit cells with continuous leaf-like profiles is designed and investigated as an example to show the transient vibration/wave linkage.

This paper is organized as follows: In Section 2, we briefly describe the design approach of the GRIN PCL and then provide the transient wave solution, based on the vibrational characteristics, to investigate the full-field instantaneous focusing phenomenon. In Section 3, we study the relation between the vibration modes and instantaneous wave responses, with a specific attention on the required number of modes and study the transmission paths of Lamb waves on the leaf-like GRIN PCLs by connecting the vibration-based transient model to a power flow analysis technique. Finally, in Section 4, experimental and theoretical validations are given to support the linkage between the vibration behaviors and wave behaviors in periodic structures.

2. Design of the GRIN PCLs and the Linkage between the Time-Domain Vibration Behaviors and Spatial-Domain Wave Behaviors

In this paper, a GRIN PCL arranged on a plate is designed and considered as an example to show the transient vibration/wave linkage. Before expressing the vibration-wave linkage, we first briefly describe the design principle of the proposed GRIN PCL with blind-grooved unit cells having a leaf-like profile as shown in Figure 1a. Although the unit cells along the wave propagation direction are still discretely arranged, the generally-discrete unit cells are continuous merged along the transversion direction. The leaf-like profile allows the central groove of each unit cell to be wider than those at the two ends, which reduces the wave velocities of the Lamb waves.



Figure 1. (a) The configuration and meshing of the unit cell of the continuous leaf-like GRIN PCL. (b) Band structure of the A_0 mode Lamb wave for various groove distances over a range of frequencies including the design frequency (15 kHz). (c) The profile and layout of the continuous leaf-like GRIN PCL. In our design, the lattice constant of each sub-unit cell is a = 10 mm and the meshing distance 1 mm. The thickness of the host plate on which waves are to be focused is 3 mm and the depth of the symmetric blind groove is 1 mm. To obtain the band diagram using the FEM, the following parameters are used: the elastic constants of the plate are E = 72.6 GPa and $\rho = 2700 \text{ kg/m}^3$ and the Poisson's ratio is 0.33.

According to the design principles of the GRIN PCLs, waves can be focused if the lens parameters transverse to the propagation direction of the Lamb wave has a hyperbolic secant gradient distribution as [8]:

$$n(y) = n_0 \operatorname{sech}(\alpha y), \tag{1}$$

where n_0 is the refractive index along the center axis and α is a gradient coefficient. The refractive index is defined as the ratio between the phase velocity of the A_0 mode Lamb wave in the host homogenous plate without introducing gradient unit cells to that within PC lens along the wave propagation direction. The gradient refractive index variation gradually bends the propagating Lamb waves toward the center axis of the plate, leading to a convergence of the lowest antisymmetric A_0 mode Lamb waves at a focal spot.

As shown in Figure 1a, the continuous profile of a symmetric blind groove can be meshed as discrete I-shape slices (i.e., sub-unit cells) before being integrated. The band structures of the A_0 mode Lamb waves along the x direction for the sub-unit cells with different groove distances d = [8, 7, 6, 5, 4, 3, 2, 1] mm are plotted in Figure 1b, where d = 1 associates with the groove distance of the I-shape slice at the two ends and d = 8 associates with that at the center. The band diagram in Figure 1b enables us to calculate the refractive index and shows that the phase velocity decreases as the groove distance decreases.

From Equation (1), we can calculate the corresponding locations of the unit cells, with d = 8 arranged for the central sub-unit cell and the remaining seven sub-unit cells have gradient groove distances symmetrically arranged with respect to the central sub-unit cell. The corresponding locations for these sub-unit cells are $[0, \pm 0.733, \pm 1.332 \pm 1.83 \pm 2.334 \pm 2.929 \pm 3.792, \pm 5] \times a$, respectively. Then, by performing a curve fitting operation, an integrated unit cell with a continuous leaf-like profile can be designed as shown in the first profile in Figure 1c. As shown in Figure 1c, the whole leaf-like GRIN PCL consists of 15 leaf-like unit cells.

Now we link the instantaneous full-field transient waves to vibration behaviors of the periodic structures, which practically has a finite dimension. Since the given example is a PCL arranged on a thin plate, we consider a homogeneous PCL on a thin flat plate with an external loading $p(x_1,x_2,t)$ applied on the upper surface of the plate. The governing equation for the out-of-plane deformation is:

$$D\left(\frac{\partial^4 u_3}{\partial x_1^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u_3}{\partial x_2^4}\right) + \rho_s h \frac{\partial^2 u_3}{\partial t^2} = p(x_1, x_2, t)$$
(2)

where $D = \frac{Eh^3}{12(1-\nu^2)}$, *E* is Young's modulus, ρ_s is the density, ν is Poisson's ratio, and $h(x_1, x_2)$ is the thickness at position (x_1, x_2) of a plate. According to the normal mode method, the transient displacement solution based on the normal mode method is [14]:

$$u_{3}(x,y,t) = \sum_{i=1}^{\infty} \left[\frac{W_{i} \cdot \int_{0}^{b} \int_{0}^{a} R \cdot W_{i} dx_{1} dx_{2}}{\rho_{s} \omega_{i} \cdot \int_{0}^{b} \int_{0}^{a} (h \cdot W_{i}^{2}) dx_{1} dx_{2}} \right] \cdot \int_{0}^{t} F(\tau) \sin \omega_{i} (t-\tau) d\tau = \sum_{i=1}^{\infty} u_{3,i}$$
(3)

where $W_i(x_1,x_2)$ is the *i*-th mode shape at a resonant frequency ω_i of the PCL, and the external loading $p(x_1,x_2,t)$ applied within a specific range $R(x_1,x_2)$ with a loading history F(t), and $u_{3,i}$ is the contribution of the *i*-th mode to the transient displacement. By observing Equation (3) we can see that only the second term is dependent of time. The other terms, such as resonant frequencies and mode shapes, are all vibration characteristics of the PCLs plate which are independent of the observation time, magnitude of the external loadings, and excitation locations. Thus, if one can obtain resonant frequencies and mode shapes of the finite periodic structures (here the PCLs) in advance, the transient solution in Equation (3) could be used to estimate the full-field transient wave propagation corresponding to various forms of loadings and at any time. Although the unit cell configuration of the PCLs is complex, finite element analysis is an effective method to extract the vibration characteristics of PCLs plate with various boundary conditions before being substituted into transient solution to obtain the full-field transient wave field.

3. Study the Vibration/Wave Linkage of the Leaf-like GRIN PCLs

Before performing experimental validations, we study the focusing performance of the proposed leaf-like GRIN PCLs based on the proposed vibration/wave linkage concept, specifically associated with a power-flow analysis. We first study the relation between the vibration modes and instantaneous wave responses with a specific attention on the number of modes required to analyze the focusing behaviors and the contribution of modes with respect to the resonant frequencies. After realizing the mode contributions, we study the transmission paths of Lamb waves on the leaf-like GRIN PCLs by connecting the vibration-based transient model to a power flow analysis technique.

3.1. Relation between the Vibration Modes and Instantaneous Wave Responses

Here, we study the relation between the vibration modes and instantaneous wave responses by considering a line source-excited leaf-like GRIN PCL (as shown in Figure 1c, see Supplementary Information for detailed parameters regarding this section). The considered GRIN PCL has a boundary condition that fixed at the upper and lower edges and free at the left and right edges. From Equation (3) in Section 3, we know that each vibration

$$(u_3)_{rms} = \sqrt{\frac{\sum\limits_{i=1}^{N} [u_3^2(t_1) + u_3^2(t_2) + u_3^2(t_3) + \dots + u_3^2(t_N)]}{N}}$$
(4)

The number of the composed vibration modes with respect to the RMS displacement field at different excitation frequencies are considered as shown in Figure 2a. We can see that only a finite number of vibrational modes are required for the solution to converge. However, the convergent speed of the RMS displacement field is related to the excitation frequency. For higher frequency excitations (i.e., shorter wavelength), more modes are required to converge the RMS displacement field. In addition, when the excitation frequency is lower, the convergent speed would be higher.



Figure 2. (a) The relation between the root-mean-square (RMS) transient displacement and the required number of the vibrational modes. (b) The relation between the RMS transient displacement due to the *i*-th mode $(u_{3,i})_{rms}$ and the resonant frequency.

After investigating the relation between the RMS transient displacement and the required vibrational mode numbers, we consider the contribution of each mode to the overall transient responses. The relation between RMS transient displacement due to the *i*-th mode (i.e., $(u_{3,i})_{rms}$) and resonant frequencies is plotted in Figure 2b. We can see that only a specific range of resonant frequencies contributes to the RMS transient displacement, indicating the dominating modes of the transient responses are centered and around the excitation frequency (i.e., marked as red dashed lines in Figure 2b). From the above discussions, we know that only vibration modes with resonant frequency nearby the excitation frequency will dominate the overall transient response, which might further enhance the calculation efficiency. In addition, for convergence of the transient response, a highest mode with resonant frequency higher than the excitation frequency needs to

be considered in the vibration-based transient model. As an empirical rule based on the Nyquist sampling theory in signal analysis, for excitation frequency at *N* Hz, the highest mode considered in modal contribution should have resonant frequency higher than 2*N* Hz for the convergence of the transient solution [12].

3.2. The Transmission Paths of the Power Flow

In this subsection, based on the concept of vibration/wave linkage vibration-based transient model, we further incorporate the concept of Power-Flow Analysis (PFA) to study the instantaneous trend of the propagating waves. The power flow analysis is a method capable of describing and tracing the energy transmission paths from one location to another within the plate [24]. In this paper, unlike the average power flow or modal power flow considered in most studies [25,26], an instantaneous power flow is considered. The instantaneous power flow density vector is defined by the dot product of the velocity vector and stress tensor in the form of [24]:

$$q_j = -\sigma_{ji}v_i , \, i, \, j = 1, 2, 3 \tag{5}$$

Since in this work only the out-of-plane deflections due to lateral loading is considered, the component of the power flow vector on the plate based on the classical plate theory can be rewritten as [25]:

$$q_{1} = \frac{-1}{2} \left(Q_{1}v_{3} - M_{12}\dot{\theta}_{1} - M_{11}\dot{\theta}_{2} \right) q_{2} = \frac{-1}{2} \left(Q_{2}v_{3} - M_{12}\dot{\theta}_{2} - M_{22}\dot{\theta}_{1} \right),$$
(6)

where Q_1 and Q_2 are the transverse shear forces per unit length, expressed as:

$$Q_{\alpha} = -D(u_{3,11} + u_{3,22})_{\alpha}, \, \alpha = 1,2 \tag{7}$$

and M_{11} are M_{22} are the bending moments per unit length and M_{12} is the torsional moment per unit length, expressed as:

$$M_{\alpha\beta} = -D \left| (1-\nu)u_{3,\alpha\beta} + \nu (u_{3,11} + u_{3,22})\delta_{\alpha\beta} \right|. \ \alpha, \beta = 1,2 \tag{8}$$

The full-field transient wave field is given in Figure 3, where green color refers to the region of zero displacement and the deep-red and deep-blue regions refer to the location of the maximum positive and negative amplitude of the transient displacement. By substituting the transient displacement u_3 into Equations (7) and (8) and using the relation of power flow in Equation (6), we can obtain instantaneous power flow vector in Figure 3, where the arrows refer to the intensity of the vector and the arrows clearly demonstrate the instantaneous transmission paths of the focusing process. We can see that, as time increases, the proposed leaf-like GRIN PCL can alter the propagating path of the input line plane waves and gradually focus the Lamb waves to a focal spot at the center of the PCL. From the given power transmission paths we can see that the theoretical transient solution can easily be transformed to other physical quantities interested in wave studies. Finally, apart from the general power flow analysis that mainly considers individual vibrational modes, the proposed instantaneous power flow transmission paths give the energy transport at every location and at every specific time, which is suitable for PCL designs.



Figure 3. The full-field displacement response within 0 to 0.8 ms and the power transmission paths.

4. Experimental Results and Discussion

In this section, as a reasonable example to link the time-domain vibration behaviors to spatial-domain propagating waves, we perform experiments to validate the focusing performance of the proposed leaf-like gradient-index PCLs. Here, a cantilever gradient-index PCL is considered. The photo of the fabricated leaf-like GRIN PCL on an aluminum plate is shown in Figure S1.2 in the Supplementary Information, where 15 leaf-like unit cells are grooved one-dimensionally in a row along the direction of wave propagation using a computerized numerical control (CNC) milling technology on both the top and bottom surfaces. An array of the piezoelectric patches (Haiying Group, Wuxi, China), glued on the left side of the plate, is used to excite the A_0 mode Lamb waves. A 15 kHz five-cycle tone burst is excited by the piezoelectric patches using a function generator (DG4102, Rigol, Suzhou, Jiangsu, China) and then amplified by a voltage amplifier (PZD700A, Trek, Lockport, NY, USA). In order to save space for piezoelectric patches that excite the Lamb waves and for a clamp that are used to fix the plate, the fabricated plate is 180 mm longer (140 mm to the left and 40 mm to the right) than the theoretical and FEM model considered in the Section 3.1.

The out-of-plane propagating velocity wave field is measured by a non-contact scanning laser Doppler vibrometer (LDV) system (PSV-F-500, Polytec, Waldbronn, Baden-Württemberg, Germany) over a grid of points inside the GRIN-PCL domain of the plate. The LDV system has a velocity resolution of 10 mm/s and a built-in bandpass filter (from 10 kHz to 20 kHz). To minimize the environmental disturbances on the measurement results, we average five measurements of instantaneous waves. It should be noted that, the averaging is done in one experiment, where the instantaneous wave at each location is measured five times before scanning the next location. Since the excitation is triggered with a duration longer than that of the signal, the scanning at each location collects the wave responses with the same excitation signals and the overall responses are the fullfield instantaneous waves. The measured velocity field at 15 kHz frequencies is given in Figure 4, where the focusing behavior agrees well with the concept of the linkage between the transient vibration behaviors and the instantaneous propagating waves. The results shown in Figure 4 indicate that, an easy-to-fabricate GRIN PCL with a continuous profile can focus incident low-frequency planar Lamb waves on a plate. In addition, the linkage to the transient vibrational modes can accurately and efficiently predict the focusing behavior the GRIN PCLs having unit cells with complex configurations. It should be noted that the LDV system measures the waves on the surface of the designed leaf-like GRIN PCL, which has grooves on the plate and this might inevitably cause measurement errors.



Figure 4. Comparisons of the velocity field between the theoretical and experimental results.

The vibration/wave linkage for linear finite periodic structures, validated in Figure 4 for the proposed GRIN PCL, can be generalized by a schematic illustration as shown in Figure 5. Full-field transient wave propagations (here the displacement field) in practical finite periodic structures at successive time are essentially constructed by the vibration mode shapes at the corresponding time, where the contribution of each modes is determined by the corresponding weighting coefficient (i.e., $u_3(x_1, x_2, t) = \sum_{i=1}^{\infty} q_i(t)W_i(x_1, x_2)$ where $q_i(t)$ is the time function representing the contribution of the *i*-th mode shapes at a specific time to the transient wave field). The concept of linking the time-domain vibration behaviors to the spatial-domain propagating waves can be applied to other kind of host structures such as shells or even three-dimensional continua. We note that since in this work a thin plate with

a thickness much smaller than its length and width is considered, the transient responses of the in-plane vibration modes can be neglected. Thus, we only have to consider the linear superposition of the out-of-plane vibrational modes to obtain rather accurate instantaneous

wave propagation responses. In other words, when the finite periodic structures are thick plates, shells, or even three-dimensional continua, the vibrational modes of different types might couple and the coupling should be considered in the proposed vibration-based transient model.



Figure 5. A schematic illustration for the time-domain vibration behaviors/spatial-domain propagating waves linkage for finite periodic structures.

5. Conclusions

The main focus of this work is twofold. First, we show that the full-field transient wave propagations in practical finite periodic structures at successive time can be linked to successive vibration modes at the corresponding time. In addition, the proposed concept can be used as a method to simulate the full-field transient wave propagations while avoiding possible numerical instabilities occurred in calculating transient wave responses on practical finite phononic crystals or metamaterials in the iterative time-consuming finite element analysis. As an example of periodic structures which manipulate wave propagations, we link the transient vibration behaviors of a gradient index phononic crystal lens (GRIN PCL) to the instantaneous wave focusing behavior and experimentally and theoretically validate the linkage. Despite being "periodic", periodic structures in real applications are finite structures which have boundary conditions. We note that the analytical solution in Equation (3), where we have mentioned that the vibration characteristics can be obtained either numerically or experimentally, can be applied to PCLs with any boundary conditions or complex geometrical configurations. On the other hand, when the excitation locations and loading histories are changed, one can still calculate the transient wave field through Equation (3) with identical vibration characteristics that have been obtained beforehand in our database. In addition, by taking advantage of Equation (3), the time-consuming iterative time-stepping process is not required in the calculation of timedependent integrand, which allows us to obtain the full-field wave propagation results at a specific time directly and straightly. Finally, the analytical transient solutions can easily be associated with other physical quantities without performing any additional numerical post-processing operations, which can minimize the temporal or spatial resolutions and reduce the involved noises. While taking wave focusing on phononic crystal lens as an example, we note that the proposed linkage concept between the time-domain vibration behaviors to spatial-domain propagating waves in finite periodic structures can be applied to acoustic black holes or elastic topological metamaterials on thin plates.

Supplementary Materials: The following are available online at https://www.mdpi.com/article/ 10.3390/cryst11121490/s1, Supplementary information of leaf-like GRIN PCL model and detailed experimental setup. Figure S1.1. The CFCF plate with leaf-like GRIN PCL model, composed of 15 leaf-like unit cells. Figure S1.2. The experimental setup of a cantilever plate with the proposed leaf-like GRIN PCL.

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