

Article

Bartering: Price-Setting Newsvendor Problem with Barter Exchange

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Abstract: Barter exchange is a system of swapping goods or services for other goods or services in a moneyless and direct manner. Barter has become an effective model of a circular economy because it reduces the consumption impact. Bartering maximizes the utility of assets and existing resources, and can unleash the unspent social, economic, and environmental value of underutilized assets. The present article analyzes the price-setting newsvendor problem with a barter exchange option. The retailer facing a stochastic price-dependent demand sells a product on the market and, additionally, needs another product for its own purposes. Therefore, first, the retailer trades the unsold product for the product it needs by means of barter, and next disposes of the unsold product at a discounted price at the end of the selling season. The retailer's optimal order quantity and optimal price are derived assuming additive uncertainty in demand. This type of demand function has special characteristics, for example, the actual demand may attain negative values in times of economic uncertainty. The possibility of negative demand realizations is taken into consideration in the study. It proves that, in certain cases, the optimal solution belongs to the set of high barter prices which implies that the actual demand may be negative.



Citation: Bieniek, M. Bartering: Price-Setting Newsvendor Problem with Barter Exchange. *Sustainability* **2021**, *13*, 6684. <https://doi.org/10.3390/su13126684>

Academic Editor: Sungyong Choi

Received: 20 April 2021

Accepted: 9 June 2021

Published: 12 June 2021

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Keywords: inventory management; price-setting newsvendor; additive demand; bartering

1. Introduction

Bartering is described as a system of a direct exchange of goods or services for other goods or services without using a medium of exchange, such as money [1]. In other words, bartering denotes an exchange of something one might no longer need for something one does. Barter economy constitutes a circular economy because a circular sharing society is driven by necessity, implied from the scarcity of food, materials, and objects [2]. Circular economy is defined as an economic model whose objective is to produce goods and services in a sustainable way by limiting the consumption and waste of resources, such as raw materials, water, and energy, as well as the production of waste [3–5]. Today's barter economy specifies a solution to improving the “use” part of the cycle and to reducing the consumption impact. The barter system may become an effective model of a circular economy. At present, the traditional barter system has been redefined and should not promote commercialism, but positive values, such as generosity, honesty, integrity, and ecological awareness. In bartering, swapping goods and services is based on the mutual and voluntary decision of two persons or enterprises. It is not treated as selling, because goods or services are exchanged based on value, rather than price. Barter swap is not synonymous with donating. The price of the item or service is never equal, but the satisfaction should be so [6].

Bartering is a new way of understanding property because the process belongs to the field of sharing economy, which has affected circular economy. The sharing economy, also termed “collaborative consumption”, takes place in organized systems or networks in which participants engage in sharing activities, for example, in the form of bartering, and swapping goods, services, transportation solutions, or space [7]. Recently, conceptual links

between the sharing economy and circular economy have been examined systematically in [8]. It was found that notable links in the fields of sustainability, business models, sustainable consumption, and governance exist. The sharing economy was conceptualized as a subset of circular economy [8]. Rather than simple consumption, the sharing economy is founded on the principle of maximizing the utility of assets, for example, via bartering. Barter provides the ability to unlock the untapped social, economic, and environmental value of assets which may not have been fully exploited. It also helps maximize existing resources, that is, vehicles not being parked and left unused, and food not going to waste [9–11]. The increase of collaborative consumption barter platforms offers a new sense of ownership and allows to share resources when they are needed. Passive consumers become collaborators, which leads to a more sustainable and altruistic way of shopping. Barter intends to bring this concept forward by building trust among peers, saving resources, and reducing spending. Circular economy mainly works on an enterprise or multinational level, but barter can bring the circular economy into households. Exchange has had certain disadvantages in the past, namely, finding the right product and the risk of moving stock. This problem was solved with the emergence of new technologies and easy access to relevant data [12].

Bartering is employed in the modern design as well. Mass production causes environmental problems because it generates a high volume of waste. In bartering, the relationship between objects and people is closer and more complicated. Through bartering, people prevent the lives of objects from ending prematurely due to people's preferences for new objects. The process also allows the objects' value to be continuously revealed and the purpose of sustainability to be achieved. However, in the concept of barter, with the aid of networking technology, the objects are allowed to circulate among people longer in order to reduce the generation of waste and achieve sustainability more effectively [13]. For instance, the peer-to-peer bartering economy is especially popular among fashion consumers. They can directly trade clothing on websites and in fashion swap groups. Very frequently, one item is traded for another item, and their shelf prices are irrelevant. These clothing swaps create a sense of community and extend the life of clothes. A circulated object is reused and bartered [14].

A telling example of introducing bartering is an idea brought into life in Mexico. In 2012, the organization steering the capital's environmental policy launched an original initiative entitled the Mercado de Trueque—, a "barter market". In the barter market, people were able to exchange recyclable waste for local farm produce [15]. Bartering was also popular in Argentina during the crisis in 2001. At that time, approximately one million Argentinians turned to an alternative, moneyless economy to overcome their financial problems. The barter system rescued a major part of the Argentinian lower and middle classes from the money crisis. However, the benefits of barter exchange were noticeable in many other areas [16–18].

In a slow economy, a barter system can help small businesses to find new customers and move their inventory, which supports the circular economy. Old-fashioned barter exchange has an exceptional power to attract new customers and sell excess inventory in a reality of checks, electronic funds transfers, and credit cards and cash on deliveries. Bartering provides member businesses with a cashless alternative that reduces bad debts, converts surplus stock into profits, and mitigates cash expenditures for common business and personal expenses. Business owners can use the barter system to pay for a wide range of common expenses, such as business trips, auto repair, dental visits, or printing materials [19]. Barter relationships between businesses, non-profits, and other organizations can be set up individually on a case-by-case basis, as well as with the use of personal barter. Internet resources are designed to help firms make the connections they need. Businesses can find bartering sites online. This helps to reach specific demographics, such as busy mothers, particular geographic areas, or specific types of barterers [20]. Online social networks help spread the practice of product exchange, which is a trend of sustainable

consumption [21]. It can be stated that the barter system may support the circular economy and economic, social, and environmental sustainability [22].

Wrong inventory management can also definitely affect sustainability and resource efficiency. Holding excess stock ties up working capital and can bring extra problems, such as the spoilage of product, damaged goods, and an increased risk of theft. All of these have adverse effects for businesses and promote inefficiency within the supply chain. Maintaining minimum inventory levels offers several environmental benefits. Lower excess production helps to eliminate carbon emissions, reduces water required to manufacture products, lowers transportation needs, and decreases the cost of storing extra stock. There is a growing focus on the environmental footprint of raw materials. Reduction of environmental impacts is vital for sustainability goals to be achieved. By improving inventory control, idle stock can be kept at a minimum, meaning a faster cash turnaround for the business. This excess stock bears strong negative impacts for both company profit and influence on the environment. By focusing on inventory control, companies can plan their inventory requirements to minimize waste and increase overall sustainability efforts [23].

This article focuses on retail—commercial—barter in the newsvendor problem, which is a core concept in the supply chain and inventory management [24]. The commercial barter may operate on traditional barter platforms, and also on digital ones. In the commercial barter, the sequence of events is as follows: first, Firm A registers on a barter platform and provides fundamental information regarding the product to be swapped. With the help of the broker, or autonomously, it finds Firm B that seeks the offered product. Those two firms engage in a moneyless transaction and pay a commission of 5–15% of the traded value. A noteworthy fact is that barter has specific requirements, such as fairness, reputation, product indivisibility, and the fact that the barter price is the same as market price. It distinguishes such an approach from principles followed on the secondary market on which the price is discounted. The firm can obtain the product it needs on the barter platform without money. Conversely, to achieve the same end on the secondary market, the firm needs funds. In addition, economists distinguish gift economies from barter in many ways, that is, barter features an immediate reciprocal exchange which is not delayed in time; barter transactions are purely economic, involving no mutual obligation between partners; barter is strictly taxable, and gifts in general are not [25]. A modern barter is a multilateral exchange conducted through a cycle or chains. In the multilateral barter, if the broker cannot find the firm for reciprocal transaction, that is, Firm A needs a product from Firm B, but Firm B does not need a product from Firm A, then the broker finds Firm C, which supports the multilateral barter. In modern barter, transactions may also involve barter currency or barter credit, whose use is usually strictly limited to the specific barter platform. Bartering has many advantages compared to traditional retail-like moving of a distressed inventory, and also compared to the disposal of excess inventory at a very low cost. Additionally, barter allows firms to obtain products without using money, which helps to maintain liquidity. By means of barter, the firm can increase sales volumes, find new customers, develop new markets, and face uncertain demand. Bartering retains cash for other needs, delays, and obsolescence of unwanted goods. It also preserves resources [20].

The above considerations justify the use of bartering in the proposed newsvendor model. The aim of this study is to introduce and investigate the overall mathematical model, which works well for any kind of commercial barter. In the present article, the newsvendor problem with barter exchange considered by [26] is supplemented with the non-negative price-dependent demand, and the optimization is conducted from the beginning. Commercial barter with the following sequence of events is considered. The retailer sets the optimal order quantity and price, which maximizes its profit to satisfy customer demand on Product X at the beginning of the selling season. Subsequently, the retailer sells Product X on the market according to the stochastic demand, and moves excess inventory to the barter platform. On the barter platform, it exchanges unsold Product X for Product Y, which the firm needs for its own purposes. Finally, the retailer salvages unsold

inventory or buys the remaining needed portion of Product Y on the market. Two cases are studied:

- There is a co-movement of the prices of Products X and Y; namely, those prices are highly positively correlated. It is possible, for example, in a well-functioning market in which the prices of processed products are strongly positively correlated with the purchase price of the basic raw material. There is also the co-movement of food prices and oil price [27,28], of crude oil and petroleum-product prices [29,30], and of energy prices and agricultural commodity prices [31];
- The price of Product Y is exogenous and uncorrelated with the price of Product X.

In the study, the additive demand is considered because of its special feature, that is, the possibility that the actual additive demand is negative. This characteristic is unique and does not concern multiplicative demand. The modified non-negative form of demand function was taken into account, and the studied newsvendor problem with barter exchange is resolved in relation to correlated and uncorrelated barter prices. Finally, a numerical example is presented which vividly illustrates the obtained results. The main conclusion of this study is that, in certain cases, the optimal solution of the price-setting newsvendor problem with barter exchange belongs to the set of high retail prices for which the realized demand may be negative. The numerical calculations were made using Mathematica software.

2. Related Literature

The present study is related to the price-setting newsvendor problem, barter exchange, and Operations Research (OR) problems regarding the non-negative demand. The review of recent literature concerning such subjects is presented below.

2.1. Price-Setting Newsvendor Problem

DeYong [32] states that the newsvendor problem is a staple of OR practice and pedagogy. In the basic formulation, the newsvendor problem aims at finding an optimal replenishment policy for a perishable product in the face of uncertain demand. From its roots as a single-period problem for a price-taking newspaper seller, the newsvendor problem and its solutions have inspired generations of researchers while contributing to inventory management [33,34]. The newsvendor model has been investigated for almost 60 years. The model provides a coherent description of the problem, and it is worth mentioning that the newsvendor problem has a number of applications far beyond the original ones. The model is employed to manage distribution channels, operating room time, tax-sheltered income, service level capacity, air cargo, airlines, and hotels [32]. Many modifications of this basic model have been introduced, adding to the complexity of the problem [24,35]. One of the major generalizations of the classic newsvendor problem is the use of price as a decision variable. This modification has been applied for decades. However, over the recent years, the issue has become an increasingly popular subject matter studied by scholars and practitioners [32].

Recently, the price-setting newsvendor problem with multiple criteria was considered in [36–44]. The price-setting newsvendor problem with the mean-variance analysis used in the objective function was carried out in [36,37]. These works focused on the impact of stockout costs upon optimal decisions when comparing classic models with mean-variance models. The newsvendor problem with additive and isoelastic demand under the aforementioned criteria was studied in [38,39]. Ye and Sun [40] investigated the newsvendor problem with pricing and strategic customers under two demand cases: additive and multiplicative. For each case, they demonstrated that neglecting the price-sensitivity of demand leads to sub-optimal decisions. Bai et al. [43] studied the joint optimization of pricing and ordering decisions in different scenarios. They proved that the effects of reference dependence and loss aversion were different depending on a particular scenario. Yu et al. [41] explored the price-setting newsvendor model, which involves a manufacturer with random yield and a retailer with uncertain demand using stochastic comparisons. Kirshner and Shao [42] studied optimism and overconfidence of a

newsvendor modelled as weights on demand and profit. They used the prospect theory to show that optimism increases inventory. Schulte and Sachs [44] put forward a solution to the price-setting newsvendor with Poisson demand. They analyzed differences between continuous and discrete versions of the model and presented the decision rule of using heuristics to solve the optimization problem.

2.2. Barter Exchange

Barter is a process constructed to maintain trade volumes and balance them, while maximizing the utility of participants. Several studies aim to derive the optimal allocation and efficient exchange algorithms for barter in many industries, for example, the medical industry, education, energy industry, tourism, material processing industries, and knowledge exchange. This article studies commercial barter, which occurs in business alongside personal barter. A few studies examined the social, economic, marketing, and logistics-related issues of commercial barter. The economic effect of barter concentrates on the liquidity problem [45], capitalist economic crisis reduction [46], lightening the tax burden [47], market segmentation and price discrimination [47,48], and negotiations [49]. Modern technologies can overcome the limitations of barter. Digital barter platforms, for example, iBarter, IMS Barter, b2b-barter, and barterxyz, can be useful during a crisis such as the one caused by COVID-19 [49]. Barter can represent a strategic answer to small and medium enterprises' growth and dexterity, allowing to increase sales and to pave the way to new markets [50]. Several studies explore barter's influence on logistics-related issues, such as inventory, purchasing, and supply chain management [51–53]. However, there are few modeling research studies focusing on the presence of barter in inventory decisions [26,54,55]. Zhang et al. [55] considered barter exchange in a two-stage wholesale-price pull contract in which the retailer sets the wholesale price and the manufacturer determines the stocking quantity. In their model, the demand does not depend on the retail price, and is defined as a random variable with a given distribution function. In [26], the authors applied a commercial barter exchange to the newsvendor problem. They considered the exogenous barter price and gave the optimal solution to the problem for the non-random or random barter supply. The results of [26] are extended by considering the endogenous barter price instead of the exogenous one, and the stochastic price-dependent demand instead of the stochastic demand being a random variable.

2.3. Operations Research Models with Non-Negative Demand

The presence of the negative realized demand in OR models was examined in [56–58]. Krishnan [56] pointed out that the non-negativity assumption should be imposed on the demand to ensure the generality of the study. If the non-negativity constraint is not introduced, the solution may be sub-optimal, and the expected profit can be underestimated. However, the non-negativity assumption can create problems with the sufficiency of the first-order conditions in a monopoly and the existence of equilibrium in an oligopoly. These challenges have been frequently ignored in OR studies. Kyparisis and Koulamas [57] examined the price-setting newsvendor problem with the non-negative demand, and proved that this problem always has an optimal solution, even in adverse market conditions. Bieniek [58] investigated a two-stage Vendor-Managed Consignment Inventory (VMCI) contract with a similar constraint imposed on demand. It was proved that in this case, there is no guarantee that the retailer's expected profit is quasi-concave with respect to price. However, it was shown that VMCI always has at least one optimal solution, which is possibly non-unique.

In this study, the stochastic demand is additive and that is why, in general, it may attain actual negative values in times of major economic uncertainty. The overall solution to the price-setting newsvendor problem with barter exchange is presented, taking into account a special feature of the additive demand. It is proved that, in certain cases, the optimal solution belongs to the region of high prices, and possibly negative realizations

of the demand. Therefore, limiting the considerations exclusively to the positive actual demand implies the loss of generality of the results.

3. Preliminary Facts

Most OR papers start with a specific function for demand. The price-dependent demand function with uncertainty in a common form is denoted by $D(p, \varepsilon)$, where p is the price and ε is the uncertainty parameter, which is mainly a random variable with a particular known distribution. Usually, the uncertainty is incorporated into deterministic demand functions by introducing the additive or multiplicative uncertainty parameter. The general additive demand model is presented as $D(p, \varepsilon) = d(p) + \varepsilon$, and the multiplicative one is defined by $D(p, \varepsilon) = d(p)\varepsilon$, where $d(p)$ is the deterministic demand. Without the assumption of non-negativity, negative demand realizations are possible in additive models. They may occur for high values of price p and significant negative realizations of uncertainty parameter ε . Restricting the set of possible parameters leads to an incomplete characterization of the optimal price. In the multiplicative case, the non-negativity of demand is satisfied if both $d(p)$ and ε are non-negative, and then the expected revenue $pE[D(p, \varepsilon)] = pd(p)E[\varepsilon] < \infty$ is bounded. The additive case is quite different. If both $d(p)$ and ε are non-negative, the company can increase its expected revenue $pE[D(p, \varepsilon)] = p(d(p) + E[\varepsilon])$ without any upper bound by setting price p arbitrarily high. Therefore, for additive uncertainty, non-negativity cannot be imposed on $d(p)$ and ε separately. Instead, it has to be imposed simultaneously, for instance, by defining the random demand as $D^+(p, \varepsilon) = \max\{d(p) + \varepsilon, 0\} = (d(p) + \varepsilon)^+$ [56]. For the deterministic demand, which constitutes a linear function $d(p) = a - bp$, $a, b > 0$, the uncertain additive demand is given by

$$D(p, \varepsilon) = a - bp + \varepsilon. \quad (1)$$

In the present study, ε is a random variable with expectation $\mu = 0$ and variance $Var(\varepsilon)$, cumulative distribution function F , and continuously differentiable probability density function f with support $[A, B]$, where $A < 0$ and $B > 0$; hazard rate $\bar{F}(z) = 1 - F(z)$ and increasing failure rate (IFR) $h(z) = f(z)/\bar{F}(z)$. Without the loss of generality, it is assumed that $\mu = 0$, since if $\mu \neq 0$, its value can be added to a . This can be done provided $a + \mu > 0$. If ε is defined on an open interval, an efficient truncation capturing as much information as possible is taken into account. Most of the distributions used in OR problems are IFR and have a twice-differentiable F with a continuous second derivative. The common IFR distributions used in inventory problems include the uniform, normal, Gamma, Power, χ^2 , Logistic, and Weibull distributions with specific parameters. Under the given assumptions, it can be stated that demand realizations $D(p^*, \varepsilon) = a - bp^* + \varepsilon$, where p^* is the optimal price, can be negative. This situation is allowed for $p^* > \frac{A+a}{b}$ and large negative ε . Therefore, in this case, it is assumed that there is no demand. In view of the foregoing, instead of (1), the demand function expressed by

$$D^+(p, \varepsilon) = (a - bp + \varepsilon)^+ \quad (2)$$

will be used [57]. Note that if $p > \frac{B+a}{b}$, then $D(p, \varepsilon)$ is always negative; therefore, the considerations are limited to $p \leq \frac{B+a}{b}$.

4. Model with Correlated Barter Prices and Non-Random Barter Supply

This article thoroughly analyzes the single-period price-setting newsvendor problem with barter exchange from the retailer's perspective. In this inventory model, the customer stochastic demand is additive and depends linearly on the retail price, which equals barter price. The retailer sets the optimal stocking and pricing policy through profit maximization to satisfy customer demand on Product X at the beginning of the selling season. The retailer disposes of the unsold Product X at a low price at the end of the selling season. In the present model, the retailer also has to buy certain Product Y it needs from the market. Therefore, the retailer can use a barter platform to barter the unsold Product X at almost

the full selling price for the needed Product Y. It is assumed that the selling prices of Products X and Y are highly positively correlated; that is, in this case, the price of Product Y is proportional to the price of Product X. In this section, the optimal solution to the newsvendor problem with non-random barter supply is presented.

The notation used in the paper is as follows.

Assumptions:

1. c —supplier's wholesale price;
2. v —salvage value per unit;
3. s —shortage penalty cost per unit;
4. p —retail (selling) price of Product X;
5. r —per-unit commission for the product the retailer pays to the barter platform given in a share of retail price, in practice $r \in [0.05, 0.15]$;
6. q_0 —needed quantity of Product Y in units at price $p_0 = kp$, $k \in R^+$; equivalently it can be said that the retailer needs $Q_0 = kq_0$ units at price p ;
7. $(1 - r)p \geq c > v$;
8. $\mu(z) = \int_z^B (z - u)f(u)du$, $\frac{d\mu(z)}{dz} = \bar{F}(z)$, $z \in [A, B]$;
- 9.

$$A + a - bc \frac{1+r}{1-r} - r(A + Q_0 - \mu(A + Q_0)) > 0. \quad (3)$$

Decision variables:

- p —barter price per unit of Product X equal to the retail price per unit of Product X;
- Q —order quantity of Product X.

In the price-setting newsvendor problem with barter exchange, the retailer's profit is described in the following three cases. Note that after imposing the non-negativity constraint on demand, the modified demand defined by (2) is considered.

Case 1. If $0 \leq Q \leq D^+(p, \varepsilon)$, the retailer pays the shortage penalty cost for the unsatisfied demand and buys Product Y on the market. The retailer's profit is given by

$$\tilde{\Pi}(p, Q) = (p - c + s)Q - sD^+(p, \varepsilon) - pQ_0.$$

Case 2. If $D^+(p, \varepsilon) < Q \leq D^+(p, \varepsilon) + Q_0$, then $Q - D^+(p, \varepsilon)$ units remain unsold. The retailer trades $Q - D^+(p, \varepsilon)$ units of Product X for Product Y on the barter platform, pays the commission $rp(Q - D^+(p, \varepsilon))$, and buys the remaining needed Product Y of value $p(Q_0 - (Q - D^+(p, \varepsilon)))$ on the market. Consequently, the retailer's profit is equal to

$$\tilde{\Pi}(p, Q) = ((1 - r)p - c)Q + rpD^+(p, \varepsilon) - pQ_0.$$

Case 3. If $Q - Q_0 > D^+(p, \varepsilon)$, the retailer barter its Product X for Product Y at the cost of the commission rpQ_0 and disposes of the remaining Product X at v . As a consequence, the retailer's profit is given by

$$\tilde{\Pi}(p, Q) = (v - c)Q + (p - v)D^+(p, \varepsilon) - (rp + v)Q_0.$$

Transforming $D^+(p, \varepsilon)$ to $D(p, \varepsilon)$ in the above formulas, it is obtained for $Q \geq Q_0$,

$$\tilde{\Pi}(p, Q) = \begin{cases} (p - c + s)Q - sD(p, \varepsilon) - pQ_0, & D(p, \varepsilon) \geq Q; \\ ((1 - r)p - c)Q + rpD(p, \varepsilon) - pQ_0, & Q - Q_0 \leq D(p, \varepsilon) < Q; \\ (v - c)Q + (p - v)D(p, \varepsilon) - (rp + v)Q_0, & 0 \leq D(p, \varepsilon) < Q - Q_0; \\ (v - c)Q - (rp + v)Q_0, & D(p, \varepsilon) < 0, \end{cases} \quad (4)$$

and for $Q < Q_0$,

$$\tilde{\Pi}(p, Q) = \begin{cases} p(Q - Q_0) - (c - s)Q - sD(p, \varepsilon), & D(p, \varepsilon) \geq Q; \\ p(Q - Q_0) + rp(D(p, \varepsilon) - Q), & 0 \leq D(p, \varepsilon) < Q; \\ (-rp - c)Q + p(Q - Q_0), & D(p, \varepsilon) < 0. \end{cases} \quad (5)$$

Since (5) attains negative values for $Q < Q_0$, the considerations are limited to the case $Q \geq Q_0$. If $Q = a - bp + z \geq Q_0$ and $D(p, \varepsilon) = a - bp + \varepsilon$, the profit can be written as

$$\tilde{\Pi}(p, z) = \begin{cases} (p - c)(a - bp + z) + s(z - \varepsilon) - pQ_0, & z \leq \varepsilon \leq B; \\ (p - c)(a - bp + z) - rp(z - \varepsilon) - pQ_0, & z - Q_0 \leq \varepsilon < z; \\ (p - c)(a - bp + z) - (p - v)(z - \varepsilon) + (p(1 - r) - v)Q_0 - pQ_0, & bp - a \leq \varepsilon < z - Q_0; \\ (v - c)(a - bp + z) - (rp + v)Q_0, & A \leq \varepsilon < bp - a. \end{cases} \quad (6)$$

The following lemma will be useful in calculating the retailer's expected profit. All proofs are relegated to the Appendix A.

Lemma 1. The following identities hold for any $A + Q_0 \leq z \leq B$

$$\begin{aligned} \int_A^{z-Q_0} (z - \varepsilon)f(\varepsilon)d\varepsilon &= z - \mu(z - Q_0) - Q_0\bar{F}(z - Q_0), \\ \int_{z-Q_0}^z (z - \varepsilon)f(\varepsilon)d\varepsilon &= \mu(z - Q_0) + Q_0\bar{F}(z - Q_0) - \mu(z), \\ \int_{bp-a}^{z-Q_0} (z - \varepsilon)f(\varepsilon)d\varepsilon &= \mu(bp - a) + z\bar{F}(bp - a) \\ &\quad - \mu(z - Q_0) - Q_0\bar{F}(z - Q_0) - (bp - a)\bar{F}(bp - a). \end{aligned}$$

4.1. Non-Negative Realized Demand

Let us consider the optimization problem $\max E\tilde{\Pi}(p, z)$, where $\tilde{\Pi}(p, z)$ is defined by (6) restricted to feasible set $\{(p, z) : p \in [\frac{c}{1-r}, \frac{A+a}{b}], z \in [A + Q_0, B]\}$. Then, $\tilde{\Pi}(p, z) = \Pi(p, z)$ and profit $\Pi(p, z)$ can be written as

$$\Pi(p, z) = \begin{cases} (p - c)(a - bp + z) + s(z - \varepsilon) - pQ_0, & z \leq \varepsilon \leq B; \\ (p - c)(a - bp + z) - rp(z - \varepsilon) - pQ_0, & z - Q_0 \leq \varepsilon < z; \\ (p - c)(a - bp + z) - (p - v)(z - \varepsilon) + (p(1 - r) - v)Q_0 - pQ_0, & A \leq \varepsilon < z - Q_0. \end{cases} \quad (7)$$

Therefore, the retailer's expected profit is given by

$$\begin{aligned} E\Pi(p, z) &= (p - c)(a - bp + z) - pQ_0 + s\mu(z) - rp \int_{z-Q_0}^z (z - \varepsilon)f(\varepsilon)d\varepsilon \\ &\quad - (p - v) \int_A^{z-Q_0} (z - \varepsilon)f(\varepsilon)d\varepsilon + (p(1 - r) - v)Q_0 \int_A^{z-Q_0} f(\varepsilon)d\varepsilon, \end{aligned}$$

which using Lemma 1, gives

$$\begin{aligned} E\Pi(p, z) &= (p - c)(a - bp + z) - (rp + v)Q_0 + (s + rp)\mu(z) \\ &\quad + (p(1 - r) - v)\mu(z - Q_0) - (p - v)z. \end{aligned} \quad (8)$$

Therefore, the following optimization task will be considered

$$\max_{\substack{p \in [\frac{c}{1-r}, \frac{A+a}{b}], \\ z \in [A+Q_0, B]}} \text{EII}(p, z) \quad (9)$$

with $\text{EII}(p, z)$ given by (8).

Now, the sequential optimization method proposed in [59] is employed. The method finds the optimal price that maximizes the performance measure for a given z . The existence of the optimal price requires the concavity of the performance measure with respect to p . Then, the objective function is expressed in terms of only one variable z , and is optimized. In view of such an approach, the first step in the optimization is to find the price that maximizes (8) for any given $z \in [A + Q_0, B]$. By solving the first-order condition $d\text{EII}(p, z)/dp = -2bp + a + bc - rQ_0 + r\mu(z) + (1-r)\mu(z - Q_0) = 0$, the optimal price is obtained equal to

$$p^*(z) = \frac{a + bc - rQ_0 + r\mu(z) + (1-r)\mu(z - Q_0)}{2b}. \quad (10)$$

Moreover, $d^2\text{EII}(p, z)/dp^2 = -2b < 0$, $d\text{EII}(p, z)/dp|_{p(1-r)=c} > A + a - bc\frac{1+r}{1-r} - r(A + Q_0 - \mu(A + Q_0)) > 0$ under assumption (3) and $\lim_{p \rightarrow \infty} \text{EII}(p, z) = -\infty$, which assures that $p^*(z)$ is a unique maximum. The first and the second derivatives of the optimal price are given by $p^{*'}(z) = \frac{1}{2b}(r\bar{F}(z) + (1-r)\bar{F}(z - Q_0)) > 0$ and $p^{*''}(z) = -\frac{1}{2b}(rf(z) + (1-r)f(z - Q_0)) < 0$, respectively. These formulas lead to the following lemma describing the shape of $p^*(z)$.

Lemma 2. *Optimal price $p^*(z)$ uniquely determined by (10) is increasing and concave for $z \in [A + Q_0, B]$.*

At this point, it should be established whether function $p^*(z)$ is hedged in interval $[\frac{c}{1-r}, \frac{A+a}{b}]$. Let us define the hedged optimal price function in the form of a piecewise function, that is,

$$\pi^*(z) = \begin{cases} p^*(z), & z \in [A + Q_0, z_p]; \\ \frac{A+a}{b}, & z \in [z_p, B], \end{cases}$$

where

$$z_p = \min \left\{ \left\{ z : p^*(z) = \frac{A+a}{b} \right\}, B \right\}. \quad (11)$$

Consequently and additionally, the following useful functions of $z \in [A + Q_0, B]$: $P_1(z) = \text{EII}(p^*(z), z)$ and $P_2(z) = \text{EII}(\frac{A+a}{b}, z)$ ought to be defined. Then, (9) can be transformed into

$$\max_{z \in [A+Q_0, B]} P^*(z),$$

where

$$P^*(z) = \text{EII}(\pi^*(z), z) = \begin{cases} P_1(z), & z \in [A + Q_0, z_p]; \\ P_2(z), & z \in [z_p, B]. \end{cases}$$

Function $P^*(.)$ is continuous, and its first derivative is equal to

$$P^{*'}(z) = \begin{cases} v - c + (rp^*(z) + s)\bar{F}(z) + (p^*(z)(1-r) - v)\bar{F}(z - Q_0), & z \in [A + Q_0, z_p]; \\ v - c + \left(r\frac{A+a}{b} + s\right)\bar{F}(z) + \left(\frac{A+a}{b}(1-r) - v\right)\bar{F}(z - Q_0), & z \in [z_p, B]. \end{cases} \quad (12)$$

Now, the second derivative of $P^*(.)$ is needed. It is given by

$$P^{*''}(z) = \begin{cases} r\bar{F}(z)(p^{*'}(z) - (p^*(z) + \frac{s}{r})h(z)) \\ + (1-r)\bar{F}(z - Q_0)(p^{*'}(z) - (p^*(z) - \frac{v}{1-r})h(z - Q_0)), z \in [A + Q_0, z_p]; \\ - (r\frac{A+a}{b} + s)f(z) - (\frac{A+a}{b}(1-r) - v)f(z - Q_0), z \in [z_p, B]. \end{cases} \quad (13)$$

In the following lemma, the concavity of P_1 and P_2 under certain conditions is proved.

Lemma 3.

1. $P_1(.)$ is first increasing and concave in $[A + Q_0, B]$ if

$$\lim_{z \rightarrow A^+} h(z) > \frac{1}{2b(p^*(A + Q_0) - \frac{v}{1-r})}. \quad (14)$$

2. $P_2(.)$ is increasing-decreasing and concave in $[A + Q_0, B]$ if

$$\left(\frac{A+a}{b}(1-r) - v\right)\bar{F}(B - Q_0) - c + v < 0. \quad (15)$$

The above lemma leads us to the theorem which presents the solution to the considered optimization problem.

Theorem 1. The equilibrium decisions of the decision-maker of the problem (9) are as follows.

1. $p^* = p^*(z^*)$ and z^* is the unique solution to $P_1'(z^*) = 0$ if $P_1'(z_p) < 0$ and (14) holds, or
2. $p^* = \frac{A+a}{b}$ and z^* is the unique solution to $P_2'(z^*) = 0$ if $P_1'(z_p) > 0$ and (15) holds, where z_p is defined by (11).

4.2. Possibly Negative Realized Demand

Let us consider the optimization problem $\max E\tilde{\Pi}(p, z)$ with $E\tilde{\Pi}(p, z)$ defined by (6) restricted to feasible set $\{(p, z) : p \in [\frac{A+a}{b}, \frac{B+a}{b}], z \in [bp - a + Q_0, B]\}$. By Lemma 1, the restricted version of the problem can be presented as

$$\max_{\substack{p \in [\frac{A+a}{b}, \frac{B+a}{b}], \\ z \in [bp - a + Q_0, B]}} E\tilde{\Pi}(p, z), \quad (16)$$

where

$$E\tilde{\Pi}(p, z) = (p - v)(\mu(z) - \mu(bp - a)) - (c - v)(a - bp + z) - (p(1 - r) - v)(\mu(z) - \mu(z - Q_0)) - (rp + v)Q_0 + s\mu(z). \quad (17)$$

The following results are obtained.

Lemma 4. For a given p , the expected profit function defined by (17) is concave with respect to z .

Theorem 2. The problem defined by (16) always has an optimal—, possibly non-unique—solution. There exists a unique solution if the objective function is quasi-concave with respect to p for a given z .

4.3. Complete Solution

Considering the previous deliberations, the unified and complex mathematical solution to the maximization problem is presented. The solution generalizes the results of [26] in two directions. First of all, in the present model, the barter price is set by the retailer. Moreover, the common restriction on the optimal price which exclusively creates

non-negative realizations of the additive demand is removed. On the basis of the previous results, it can be concluded that the problem given by $\max_{p,z} E\tilde{\Pi}(p,z)$ can be written as

$$\max \left(\begin{array}{cc} \max_{\substack{p \in [\frac{c}{1-r}, \frac{A+a}{b}], \\ z \in [A+Q_0, B]}} E\Pi(p,z), & \max_{\substack{p \in [\frac{A+a}{b}, \frac{B+a}{b}], \\ z \in [bp-a+Q_0, B]}} E\tilde{\Pi}(p,z) \end{array} \right). \quad (18)$$

5. Model with Correlated Barter Prices and Uncertain Barter Supply

In this section, the single-period newsvendor problem with the uncertain barter supply and the modified demand defined by (2) is considered. The notation and assumptions 1–7 given in Section 4 are used. Moreover, the following assumptions are made:

- w —random variable with pdf g , mean u_w and variance $\sigma_w^2 < \infty$;
- wQ_0 —uncertain barter supply;

$$A + a - bc \frac{1+r}{1-r} - r(A + Q_0 - \mu(A + Q_0)) - (1-r)Q_0 \int_0^1 (1-w)g(w)dw > 0. \quad (19)$$

Decision variables:

- p —barter price per unit of Product X in the case of the uncertain barter supply;
- Q —order quantity of Product X in the case of uncertain barter supply.

The following two cases are considered: I. barter supply is higher than barter demand if $w > 1$, or II. barter supply is lower or equal to barter demand if $0 \leq w \leq 1$. These cases are presented below.

- I. If $w > 1$, the model is described in the same manner as the model in the previous section;
- II. For $0 \leq w \leq 1$,

Case 1. If $Q \leq D^+(p, \varepsilon)$, the retailer pays the shortage penalty cost for the unsatisfied demand and buys Product Y of value pQ_0 on the market. The retailer's profit is given by

$$\tilde{\Pi}_u(p, Q) = (p - c + s)Q - sD^+(p, \varepsilon) - pQ_0.$$

Case 2. If $D^+(p, \varepsilon) < Q \leq D^+(p, \varepsilon) + wQ_0$, then $Q - D^+(p, \varepsilon)$ units of Product X remain unsold. The retailer trades $Q - D^+(p, \varepsilon)$ units of Product X for Product Y on the barter platform, pays the commission $rp(Q - D^+(p, \varepsilon))$, and buys Product Y of value $p(Q_0 - (Q - D^+(p, \varepsilon)))$ on the market. Then, the retailer's profit is equal to

$$\tilde{\Pi}_u(p, Q) = ((1-r)p - c)Q + rpD^+(p, \varepsilon) - pQ_0.$$

Case 3. If $Q - wQ_0 > D^+(p, \varepsilon)$, the retailer barter its Product X for Product Y at the cost of commission $rpwQ_0$ and disposes of the remaining Product X at v . Consequently, the retailer's profit is given by

$$\tilde{\Pi}_u(p, Q) = (v - c)Q + (p - v)D^+(p, \varepsilon) - (rp + v)wQ_0 - pQ_0(1 - w).$$

Substituting (2) and $Q = a - bp + z$, in the above formulas

$$\tilde{\Pi}_u(p, z) = \begin{cases} \tilde{\Pi}(p, z), & w > 1; \\ (p - c)(a - bp + z) + s(z - \varepsilon) - pQ_0, & z \leq \varepsilon \leq B, \quad 0 \leq w \leq 1; \\ (p - c)(a - bp + z) - rp(z - \varepsilon) - pQ_0, & z - Q_0 \leq \varepsilon < z, \\ & 0 \leq w \leq 1; \\ (p - c)(a - bp + z) - (p - v)(z - \varepsilon) - (p(1 - r) - v)Q_0 - pQ_0, & bp - a \leq \varepsilon < z - Q_0, \quad 0 \leq w \leq 1; \\ (v - c)(a - bp + z) - (rp + v)Q_0, & A \leq \varepsilon < bp - a, \quad 0 \leq w \leq 1. \end{cases} \quad (20)$$

Now, the considerations are restricted to $Q \geq Q_0$ because, otherwise, the profit attains negative values.

5.1. Non-Negative Realized Demand

Let us consider the optimization problem $\max_{p,z} E\tilde{\Pi}_u(p,z)$, where $\tilde{\Pi}_u(p,z)$ is defined by (20) restricted to feasible set $\{(p,z) : p \in [\frac{c}{1-r}, \frac{A+a}{b}], z \in [A+Q_0, B]\}$. Then, $\tilde{\Pi}_u(p,z) = \Pi_u(p,z)$ and $\Pi_u(p,z)$ is given by

$$\Pi_u(p,z) = \begin{cases} \Pi(p,z), & w > 1; \\ (p-c)(a-bp+z) + s(z-\varepsilon) - pQ_0, & z \leq \varepsilon \leq B, \quad 0 \leq w \leq 1; \\ (p-c)(a-bp+z) - rp(z-\varepsilon) - pQ_0, & z - wQ_0 \leq \varepsilon < z, \\ & 0 \leq w \leq 1; \\ (p-c)(a-bp+z) - (p-v)(z-\varepsilon) + (p(1-r)-v)wQ_0 - pQ_0, & A \leq \varepsilon < z - wQ_0, \quad 0 \leq w \leq 1. \end{cases} \quad (21)$$

Using Lemma 1, the expected profit under the uncertain barter supply is established, which is equal to

$$\begin{aligned} E\Pi_u(p,z) &= (p-c)(a-bp+z) - (rp+v)Q_0 + (s+rp)\mu(z) \\ &+ (p(1-r)-v)\mu(z-Q_0) - (p-v)z \\ &+ (p(1-r)-v) \int_0^1 (\mu(z-wQ_0) + wQ_0 - \mu(z-Q_0) - Q_0)g(w)dw \end{aligned} \quad (22)$$

for $Q \geq Q_0$. At this point, it should be noted that for $Q < Q_0$, the profit is non-positive in all cases. Therefore, below, the optimization task for $Q \geq Q_0$ is considered, which implies $z \geq A+Q_0$. The following optimization problem is studied:

$$\max_{\substack{p \in [\frac{c}{1-r}, \frac{A+a}{b}], \\ z \in [A+Q_0, B]}} E\Pi_u(p,z). \quad (23)$$

Using the method proposed by [59], the first step in the optimization is to find the price that maximizes the objective function for any given $z \in [A+Q_0, B]$. By solving the first-order condition

$$\begin{aligned} \frac{dE\Pi_u(p,z)}{dp} &= -2bp + a + bc - rQ_0 + r\mu(z) + (1-r)\mu(z-Q_0) \\ &+ (1-r) \int_0^1 (\mu(z-wQ_0) + wQ_0 - \mu(z-Q_0) - Q_0)g(w)dw = 0, \end{aligned}$$

the following is obtained:

$$\begin{aligned} p_u^*(z) &= \frac{a + bc - rQ_0 + r\mu(z) + (1-r)\mu(z-Q_0)}{2b} \\ &+ \frac{(1-r) \int_0^1 (\mu(z-wQ_0) + wQ_0 - \mu(z-Q_0) - Q_0)g(w)dw}{2b}. \end{aligned} \quad (24)$$

Moreover, $d^2E\Pi_u(p,z)/dp^2 = -2b < 0$, and by assumption (19)

$$\begin{aligned} \frac{dE\Pi_u(p,z)}{dp} \Big|_{p(1-r)=c} &> A + a - bc \frac{1+r}{1-r} - r(A+Q_0 - \mu(A+Q_0)) \\ &- (1-r) \int_0^1 (1-w)Q_0g(w)dw > 0, \end{aligned}$$

and $\lim_{p \rightarrow \infty} E\Pi_u(p, z) = -\infty$, which assures that $p_u^*(z)$ is a unique maximum. The first and the second derivatives of the optimal price are given by $p_u^{*'}(z) = \frac{1}{2b}(r\bar{F}(z) + (1-r) \int_1^\infty \bar{F}(z - Q_0)g(w)dw + (1-r) \int_0^1 \bar{F}(z - wQ_0)g(w)dw) > 0$ and $p_u^{*''}(z) = -\frac{1}{2b}(rf(z) + (1-r) \int_1^\infty f(z - Q_0)g(w)dw + (1-r) \int_0^1 f(z - wQ_0)g(w)dw) < 0$. Based on these formulas, the lemma describing the shape of $p_u^*(.)$ has been proved.

Lemma 5. The optimal price $p_u^*(z)$ uniquely determined by (24) is increasing and concave for $z \in [A + Q_0, B]$.

It should be examined whether $p_u^*(z)$ is hedged in interval $\left[\frac{c}{1-r}, \frac{A+a}{b}\right]$ or not. Let us define the piecewise hedged optimal price function as

$$\pi_u^*(z) = \begin{cases} p_u^*(z), & z \in [A + Q_0, z_{pu}); \\ \frac{A+a}{b}, & z \in [z_{pu}, B], \end{cases}$$

where

$$z_{pu} = \min\left\{\left\{z : p_u^*(z) = \frac{A+a}{b}\right\}, B\right\}. \quad (25)$$

Furthermore, the following functions of $z \in [A + Q_0, B]$ will also be used: $P_{1u}(z) = E\Pi(p_u^*(z), z)$ and $P_{2u}(z) = E\Pi\left(\frac{A+a}{b}, z\right)$ and transform (23) into

$$\max_{z \in [A+Q_0, B]} P_u^*(z), \quad (26)$$

where

$$P_u^*(z) = E\Pi(\pi_u^*(z), z) = \begin{cases} P_{1u}(z), & z \in [A + Q_0, z_{pu}); \\ P_{2u}(z), & z \in [z_{pu}, B]. \end{cases}$$

Function $P_u^*(.)$ is continuous, and its first derivative is equal to

$$P_u^{*'}(z) = \begin{cases} v - c + (rp_u^*(z) + s)\bar{F}(z) + (p_u^*(z)(1-r) - v) \left(\int_0^1 \bar{F}(z - wQ_0)g(w)dw + \int_1^\infty \bar{F}(z - Q_0)g(w)dw \right), & z \in [A + Q_0, z_{pu}); \\ v - c + \left(r\frac{A+a}{b} + s\right)\bar{F}(z) + \left(\frac{A+a}{b}(1-r) - v\right) \left(\int_0^1 \bar{F}(z - wQ_0)g(w)dw + \int_1^\infty \bar{F}(z - Q_0)g(w)dw \right), & z \in [z_{pu}, B]. \end{cases} \quad (27)$$

Now, the second derivative of $P_u^*(.)$ is needed, which is given by

$$P_u^{*''}(z) = \begin{cases} \left[rp_u^{*'}(z)\bar{F}(z) - (rp_u^*(z) + s)f(z) \right] + \left[(1-r)p_u^{*'}\bar{F}(z - Q_0) - (p_u^*(z)(1-r) - v)f(z - Q_0) \right] \int_1^\infty g(w)dw \\ + p_u^{*'}(z)(1-r) \int_0^1 \bar{F}(z - wQ_0)g(w)dw \\ - (p_u^*(z)(1-r) - v) \int_0^1 f(z - wQ_0)g(w)dw, & z \in [A + Q_0, z_{pu}); \\ - \left(r\frac{A+a}{b} + s \right) f(z) - \left(\frac{A+a}{b}(1-r) - v \right) f(z - Q_0) \int_1^\infty g(w)dw \\ - \left(\frac{A+a}{b}(1-r) - v \right) \int_0^1 f(z - wQ_0)g(w)dw, & z \in [z_{pu}, B]. \end{cases} \quad (28)$$

In the lemma, the concavity of P_{1u} and P_{2u} under certain given conditions is proved.

Lemma 6.

1. $P_{1u}(\cdot)$ is first increasing and concave in $[A + Q_0, B]$ if

$$p_u^*(A + Q_0)(1 - r) - c + \bar{F}(A + Q_0)(rp^*(A + Q_0) + s) - (p_u^*(A + Q_0)(1 - r) - v) \int_0^1 (1 - \bar{F}(A + (1 - w)Q_0))g(w)dw > 0 \quad (29)$$

and

$$\lim_{z \rightarrow A^+} h(z) > \frac{1}{2b(p_u^*(A + Q_0) - \frac{v}{1-r})} \quad (30)$$

and for $z \in [A + Q_0, B]$

$$\frac{\int_0^1 f(z - wQ_0)g(w)dw}{\int_0^1 \bar{F}(z - wQ_0)g(w)dw} > \frac{1}{2b(p_u^*(A + Q_0) - \frac{v}{1-r})}. \quad (31)$$

2. $P_{2u}(\cdot)$ is increasing-decreasing and concave in $[A + Q_0, B]$ if

$$\left(\frac{A+a}{b}(1-r) - v \right) \bar{F}(B - Q_0) - c + v + \left(\frac{A+a}{b}(1-r) - v \right) \int_0^1 (\bar{F}(B - wQ_0) - \bar{F}(B - Q_0))g(w)dw < 0. \quad (32)$$

Remark 1. If $\varepsilon \sim U[A, -A]$ then the constraint (31), which should generally hold for any $z \in [A + Q_0, -A]$, reduces to $\frac{\int_0^1 g(w)dw}{\int_0^1 (wQ_0 - Q_0 - 2A)g(w)dw} > \frac{1}{2b(p_u^*(A + Q_0) - \frac{v}{1-r})}$.

Pursuant to the foregoing, the mathematical solution to the problem (23) which generalizes the results presented in [26] is proved.

Theorem 3. The equilibrium decisions of the decision-maker of the problem (23) are as follows:

1. $p_u^* = p_u^*(z_u^*)$ and z_u^* is the unique solution to $P'_{1u}(z_u^*) = 0$ if $P'_{1u}(z_{pu}) < 0$ and (29)–(31) are satisfied, or
2. $p_u^* = \frac{A+a}{b}$ and z_u^* is the unique solution to $P'_{2u}(z_u^*) = 0$ if $P'_1(z_{pu}) > 0$ and (32) are satisfied, where z_{pu} is defined by (25).

5.2. Possibly Negative Realized Demand

Let us consider the optimization problem $\max_{p,z} E\tilde{\Pi}_u(p, z)$ with $\tilde{\Pi}_u(p, z)$ given by (20) restricted to feasible set $\left\{ (p, z) : p \in \left[\frac{A+a}{b}, \frac{B+a}{b} \right], z \in [bp - a + Q_0, B] \right\}$. Based on Lemma 1, the restricted version of the problem can be written as

$$\max_{\substack{p \in \left[\frac{A+a}{b}, \frac{B+a}{b} \right], \\ z \in [bp - a + Q_0, B]}} E\tilde{\Pi}_u(p, z), \quad (33)$$

where

$$E\tilde{\Pi}_u(p, z) = (p - v)(\mu(z) - \mu(bp - a)) - (c - v)(a - bp + z) - (p(1 - r) - v)(\mu(z) - \mu(z - Q_0)) - (rp + v)Q_0 + s\mu(z) + (p(1 - r) - v) \int_0^1 (\mu(z - wQ_0) + wQ_0 - Q_0 - \mu(z - Q_0))g(w)dw. \quad (34)$$

The following results are presented.

Lemma 7. For a given p , the expected profit function defined by (34) is concave with respect to z .

Theorem 4. The problem defined by (33) always has an optimal—, possibly non-unique—solution. There exists the unique solution if the objective function is quasi-concave with respect to p for a given z .

5.3. Complete Solution

On the basis of the results obtained in previous sections, it can be concluded that the problem given by $\max_{p,z} E\tilde{\Pi}_u(p, z)$ can be written as

$$\max \left(\max_{\substack{p \in [\frac{c}{1-r}, \frac{A+a}{b}], \\ z \in [A+Q_0, B]}} E\Pi_u(p, z), \max_{\substack{p \in [\frac{A+a}{b}, \frac{B+a}{b}], \\ z \in [bp-a+Q_0, B]}} E\tilde{\Pi}_u(p, z) \right). \quad (35)$$

6. Numerical Example—Model with Correlated Barter Prices

This section outlines the numerical examples which illustrate and verify the preceding theoretical analysis. It is assumed that the random part of demand follows the uniform distribution, and w follows the normal distribution. The newsvendor problem with the non-random and random barter supply is examined separately.

In the first example, let us consider the linear demand function $D(p, \varepsilon) = 65 - p + \varepsilon$, where $\varepsilon \sim U[-10, 10]$ and $w \sim N[2, 1]$. It is assumed that the cost of commodity $c = 10$, the salvage value $v = 3$, the shortage cost $s = 2$, the quantity $Q_0 = 3$, and the commission $r = 0.1$. All required conditions are satisfied for models with the non-random and uncertain barter supply (Tables 1 and 2). Optimal solutions to both models belong to the set of low barter prices, for which demand has non-negative realizations. In the case of the non-random barter supply, the optimal expected profit equals $E\Pi(z^*) = 660.775$ for the optimal retail price $p^* = 37.135$ and the optimal service level $z^* = 8.651$. In the case of the random barter supply, the optimal expected profit equals $E\Pi(z_u^*) = 656.872$ for $p_u^* = 37.063$ and $z_u^* = 8.512$ (Tables 3 and 4).

In the second example, let us investigate the model with linear demand function $D(p, \varepsilon) = 37 - p + \varepsilon$, where $\varepsilon \sim U[-15, 15]$ and $w \sim N[2, 1]$. It is assumed that the cost of commodity $c = 10$, $v = 1$, $s = 1$, $Q_0 = 1$ and $r = 0.1$. All required conditions are satisfied for models with the non-random and uncertain barter supply (Tables 1 and 2). It appears that optimal solutions to both models belong to the set of high barter prices for which actual demand is possibly negative. For the non-random barter supply, the optimal expected profit $E\tilde{\Pi}(\hat{z}^*) = 89.355$ for $\hat{p}^* = 22.349$ and $\hat{z}^* = 3.774$ is greater than the expected profit $E\Pi(z^*) = 89.291$. In the case of the uncertain barter supply, the optimal expected profit attains value $E\tilde{\Pi}_u(\hat{z}_u^*) = 88.758$ for the optimal retail price $\hat{p}_u^* = 22.305$ and the optimal service level $\hat{z}_u^* = 3.706$, which is greater than the expected profit $E\Pi_u(z^*) = 88.708$. Solutions belonging to the set of low retail prices—where the demand always attains non-negative values—are sub-optimal, and expected profits are underestimated (Tables 3 and 4). In this example, limiting considerations only to the set of low barter prices would imply underestimated profit and a suboptimal order quantity and barter price.

Table 1. Conditions of the solution to the model with the non-random barter supply, $\varepsilon \sim U[A, -A]$.

$(A, a, b, c, v, s, Q_0, r)$	z_p	(3)	(14)	(15)
$(-10, 65, 1, 10, 3, 2, 3, 0.1)$	10	Yes	Yes	Yes
$(-15, 37, 1, 10, 1, 1, 1, 0.1)$	2.71	Yes	Yes	Yes

Table 2. Conditions of the solution to the model with the uncertain barter supply, $\varepsilon \sim U[A, -A]$, $w \sim N(2, 1)$.

$(A, a, b, c, v, s, Q_0, r)$	z_{pu}	(19)	(29)	(30)	(31)	(32)
$(-10, 65, 1, 10, 3, 2, 3, 0.1)$	10	Yes	Yes	Yes	Yes	Yes
$(-15, 37, 1, 10, 1, 1, 1, 0.1)$	2.773	Yes	Yes	Yes	Yes	Yes

Table 3. Solutions to the model with the non-random barter supply, $\varepsilon \sim U[A, -A]$.

$(A, a, b, c, v, s, Q_0, r)$	$\frac{c}{1-r} \leq p \leq \frac{A+a}{b}$	$\frac{A+a}{b} \leq p \leq \frac{-A+a}{b}$
$(-10, 65, 1, 10, 3, 2, 3, 0.1)$	$p^* = 37.135$ $z^* = 8.65$ $E\Pi(z^*) = 660.775$	$\hat{p}^* = 55$ $\hat{z}^* = 9.99$ $E\tilde{\Pi}(\hat{z}^*) = 344.038$
$(-15, 37, 1, 10, 1, 1, 1, 0.1)$	$p^* = 22$ $z^* = 3.582$ $E\Pi(z^*) = 89.291$	$\hat{p}^* = 22.349$ $\hat{z}^* = 3.774$ $E\tilde{\Pi}(\hat{z}^*) = 89.355$

Table 4. Solutions to the model with the uncertain barter supply, $\varepsilon \sim U[A, -A]$, $w \sim N(2, 1)$.

$(A, a, b, c, v, s, Q_0, r)$	$\frac{c}{1-r} \leq p \leq \frac{A+a}{b}$	$\frac{A+a}{b} \leq p \leq \frac{-A+a}{b}$
$(-10, 65, 1, 10, 3, 2, 3, 0.1)$	$p_u^* = 37.063$ $z_u^* = 8.512$ $E\Pi(z_u^*) = 656.872$	$\hat{p}_u^* = 55$ $\hat{z}_u^* = 9.856$ $E\tilde{\Pi}(\hat{z}_u^*) = 337.578$
$(-15, 37, 1, 10, 1, 1, 1, 0.1)$	$p_u^* = 22$ $z_u^* = 3.537$ $E\Pi(z_u^*) = 88.708$	$\hat{p}_u^* = 22.305$ $\hat{z}_u^* = 3.706$ $E\tilde{\Pi}(\hat{z}_u^*) = 88.758$

7. Model with Uncorrelated Barter Prices

In this section, the general model described in Section 4 with uncorrelated barter prices is investigated. Now, the barter price of Product Y is exogenous and is not correlated with the price of Product X, which is endogenous. It is assumed that the value of Product Y needed by the retailer equals $w_0 = p_0 Q_0$, where Q_0 and p_0 are the pre-specified quantity and selling price, respectively. The current considerations are limited to the non-random barter supply because, even in this case, the problem is mathematically complicated, and giving precise solutions is not possible. Other assumptions of the model in this section are the same as the assumptions 1–7 presented in Section 4.

Similarly to (4), it is obtained that for $Q \geq \frac{w_0}{p}$, the profit can be written as

$$\tilde{\Pi}(p, Q) = \begin{cases} (p - c + s)Q - sD(p, \varepsilon) - w_0, & D(p, \varepsilon) \geq Q; \\ ((1 - r)p - c)Q + rpD(p, \varepsilon) - w_0, & Q - \frac{w_0}{p} \leq D(p, \varepsilon) < Q; \\ (v - c)Q + (p - v)D(p, \varepsilon) - (rp + v)\frac{w_0}{p}, & 0 \leq D(p, \varepsilon) < Q - \frac{w_0}{p}; \\ (v - c)Q - (rp + v)\frac{w_0}{p}, & D(p, \varepsilon) < 0, \end{cases}$$

which can be transformed into

$$\tilde{\Pi}(p, z) = \begin{cases} (p - c + s)(a - bp + z) - s(a - bp + \varepsilon) - w_0, & B \geq \varepsilon \geq z; \\ ((1 - r)p - c)(a - bp + z) + rp(a - bp + \varepsilon) - w_0, & z - \frac{w_0}{p} \leq \varepsilon < z; \\ (v - c)(a - bp + z) + (p - v)(a - bp + \varepsilon) - (rp + v)\frac{w_0}{p}, & \\ bp - a \leq \varepsilon < z - \frac{w_0}{p}; \\ (v - c)(a - bp + z) - (rp + v)\frac{w_0}{p}, & A \leq \varepsilon < bp - a. \end{cases} \quad (36)$$

Otherwise, for $Q < \frac{w_0}{p}$ the profit is negative. Therefore, by Lemma 1,

$$\begin{aligned} E\tilde{\Pi}(p, z) = & (p - v)(\mu(z) - \mu(bp - a)) - (c - v)(a - bp + z) \\ & - (p(1 - r) - v)\left(\mu(z) - \mu\left(z - \frac{w_0}{p}\right)\right) + s\mu(z) - (rp + v)\frac{w_0}{p}. \end{aligned} \quad (37)$$

Let us consider the optimization problem $\max_{p,z} E\tilde{\Pi}(p,z)$ with $E\tilde{\Pi}(p,z)$ given by (37) restricted to feasible set $\{(p,z) : p \in [\frac{c}{1-r}, \frac{B+a}{b}], z \in [\max\{A, bp-a\} + \frac{w_0}{p}, B]\}$. Solving this problem, the following lemma is needed.

Lemma 8. For a given p , function $E\tilde{\Pi}(p,z)$ defined by (37) is concave with respect to z .

Using the above lemma, the following theorem is proved.

Theorem 5. Problem $\max_{\{p \in [\frac{c}{1-r}, \frac{B+a}{b}], z \in [\max\{A, bp-a\} + \frac{w_0}{p}, B]\}} E\tilde{\Pi}(p,z)$ with $E\tilde{\Pi}(p,z)$ defined by (37) always has an optimal—possibly non-unique—solution. There exists the unique solution to the problem if the objective function is quasi-concave with respect to p for a given z .

Now, numerical examples are presented which illustrate the above result.

In the first example, let us consider linear demand function $D(p, \varepsilon) = 65 - p + \varepsilon$, where the random variable ε is uniformly distributed $\varepsilon \sim U[-10, 10]$ and $w_0 = 30$. It is assumed that the cost of commodity $c = 10$, the salvage value $v = 3$, the shortage cost $s = 2$, and the commission $r = 0.1$. The optimal solution to this model belongs to the set of low barter prices, for which realizations of demand are non-negative. The optimal expected profit equals $E\Pi(z^*) = 689.488$ for the optimal retail price $p^* = 37.407$, and the optimal service level $z^* = 8.651$ (Table 5).

In the second example, let us investigate the model with linear demand function $D(p, \varepsilon) = 37 - p + \varepsilon$, where $\varepsilon \sim U[-15, 15]$ and $w_0 = 10$. It is assumed that the cost of commodity $c = 10$, $v = 1$, $s = 1$ and $r = 0.1$. It appears that the optimal solution to the model belongs to the set of high barter prices for which actual demand is possibly negative. The optimal expected profit equal to $E\tilde{\Pi}(\hat{z}^*) = 95.472$ for the optimal retail price $\hat{p}^* = 22.621$ and the optimal service level $\hat{z}^* = 3.443$ is greater than the expected profit $E\Pi(z^*) = 95.267$. In this example, the optimal solution restricted to the set of low retail prices—where the actual demand always attains non-negative values—is sub-optimal, and the optimal expected profit is underestimated (Table 5).

Table 5. Solution to the model with uncorrelated barter prices, $\varepsilon \sim U[A, -A]$.

$(A, a, b, c, v, s, w_0, r)$	$\frac{c}{1-r} \leq p \leq \frac{A+a}{b}$	$\frac{A+a}{b} \leq p \leq \frac{-A+a}{b}$
$(-10, 65, 1, 10, 3, 2, 30, 0.1)$	$p^* = 37.407$ $z^* = 8.630$ $E\Pi(p^*, z^*) = 689.488$	$\hat{p}^* = 55$ $\hat{z}^* = 7.877$ $E\tilde{\Pi}(p^*, z^*) = 381.102$
$(-15, 37, 1, 10, 1, 1, 10, 0.1)$	$p^* = 22$ $z^* = 3.116$ $E\Pi(p^*, z^*) = 95.267$	$\hat{p}^* = 22.621$ $\hat{z}^* = 3.443$ $E\tilde{\Pi}(p^*, \hat{z}^*) = 95.472$

8. Discussion

In this article, the issue of the negative demand realizations in the price-setting newsvendor problem with barter exchange is investigated. This feature is characteristic of the additive stochastic demand. Numerous studies concerning the supply chain management literature ignore the problems associated with the additive demand [38,60–62]. Only a few studies take into account the non-negativity issue in OR models. According to our best knowledge, this subject was surveyed in [56–58] only. In [57,58], it was shown that applying the non-negativity assumption on demand could change the optimal order quantity and retail price. Similarly, in this paper, it is shown that the negative actual demand may influence optimal solutions to the newsvendor problem with barter exchange, and not considering this issue may cause sub-optimal solutions to the problem. These sub-optimal results often translate into underestimated profits. That is why the need for

obtaining complete optimal solutions requires the non-negativity constraint to be imposed on the demand in OR problems with the additive stochastic demand.

The mathematical model of this article extends the model of [26]. The retail price as a decision variable and price-sensitive demand are introduced. The analytical results on the optimal order quantity are similar to those given in the aforementioned work, but additionally, the formulas for the optimal selling price are presented.

9. Conclusions

Recently, especially at a time of crisis, such as during the COVID-19 pandemic, firms have increasingly moved their excess or obsolete inventory to barter platforms, including traditional and digital barter platforms [63,64]. This approach supports sustainability goals, that is, reduces the risk of wasteful stock disposal, because excess inventory wastes carbon and energy. Digital barter platforms, in particular, allow organizations to free up their cash flow by trading what they have to obtain what they need. By means of digital barter platforms, one can partly exchange excess products for other budgeted products or services from other network members. For example, restaurants can fill empty tables and use such earnings to complete necessary renovations; hotel owners can find creative ways to fill vacancies and use such income to pay for advertising [65].

The objective of this article was to determine the general mathematical model adapted to any kind of commercial barter. The problem of the negative actual demand in the price-setting newsvendor problem with barter exchange is considered. The examined problem was solved assuming that the barter price is endogenous and demand is price-dependent and additive. The considerations are built on the results of [26], where the barter price was exogenous and demand was simply a random variable. The basic model has to be transformed due to the fact that the barter price is not fixed and becomes a decision variable. Two cases have to be considered: products on the barter platform have correlated or uncorrelated barter prices. Moreover, the additive form of the demand is taken into account because of its special feature—, that is, the possibility of negative realizations—described above.

The newsvendor model with barter exchange is constructed as follows. A retailer sells Product X and needs a specific quantity of Product Y. The retailer can barter unsold Product X for Product Y on a barter platform. Next, if needed, it can buy the remaining portion of Product Y on the market. The retailer decides the order quantity and barter price which maximizes its expected profit. Two cases are considered—with correlated and uncorrelated barter prices of Products X and Y. The author indicates that the optimal order quantity and optimal barter price sometimes belong to the set of high barter prices for which demand realizations may be negative. This fact is illustrated in numerical examples. It turns out that restricting the investigation exclusively to positive demand realizations considerably reduces the generality of insights, and in addition, it may produce sub-optimal solutions.

This article definitely extends the newsvendor model with barter exchange studied in [26] by adapting the basic model to reality in view of pricing and uncertainty. In the new model, the retail price is a decision variable along with the order quantity. The important issue of possible negative demand realizations is also addressed.

The newsvendor problem has numerous applications for decision-making in manufacturing and services, as well as decision-making by individuals. The newsvendor model is used, for example, to address supply chain contracts which constitute agreements between buyers and suppliers. Future research can examine the implications of barter exchange for firms using newsvendor-type settings. It can be especially useful in times of economic uncertainty when barter may be a viable option for overcoming the crisis. Finally, one can also investigate the newsvendor problem with barter exchange in terms of other forms of demand function—, that is, multiplicative form—characteristics for specific products.

Funding: This work was supported by grant no. 2019/35/D/HS4/00801, funded by the National Science Centre, Poland.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

Proof of Lemma 1. Using standard algebraic derivations, the statement of the lemma is obtained. $\square$

Proof of Lemma 3. Using (12) it can be seen that $P'_1(A + Q_0) > 0$ under assumption (3) which implies that $P_1(\cdot)$ increases in $A + Q_0$. Moreover, if

$$h(z - Q_0) > \frac{p^{*'}(z)}{p^*(z) - \frac{v}{1-r}} \quad (\text{A1})$$

and

$$h(z) > \frac{p^{*'}(z)}{p^*(z) + \frac{s}{r}}, \quad (\text{A2})$$

then the second derivative given by (13) is negative and therefore $P_1(z)$ is concave for $z \in [A + Q_0, B]$. Let us note that (A1) implies (A2). From (A1) and the fact that $p^*(z)$ is increasing by Lemma 2, $p^{*'}(z) < \frac{1}{2b}$ and F is IFR, (14) is obtained.

Next, by (13) it can be stated that the second derivative $P''_2(z) < 0$ for any $z \in [A + Q_0, B]$, which implies that $P_2(\cdot)$ is concave. Moreover, $P'_2(A + Q_0) > 0$ and $P'_2(B) < 0$ by (15), which proves that $P_2(\cdot)$ is increasing-decreasing on $[A + Q_0, B]$. $\square$

Proof of Theorem 1. Using Lemma 3, it can be seen that there are two possible cases. They were described in this theorem in Points 1. and 2. It ought to be noted that function $P^*(\cdot)$ is smooth. Therefore, under the assumptions, in the first case, the maximum belongs to interval $[A + Q_0, z_p]$ since $P_1(\cdot)$ and $P_2(\cdot)$ decrease in z_p , and in the latter case the maximum belongs to interval $[z_p, B]$, which ends the proof. $\square$

Proof of Lemma 4. Using (17) for a given p implies

$$\frac{dE\tilde{\Pi}(p, z)}{dz} = v - c + (rp + s)\bar{F}(z) + (p(1 - r) - v)\bar{F}(z - Q_0),$$

and consequently

$$\frac{d^2E\tilde{\Pi}(p, z)}{dz^2} = -(rp + s)f(z) - (p(1 - r) - v)f(z - Q_0) < 0.$$

Therefore, $E\tilde{\Pi}(p, z)$ is concave with respect to z . $\square$

Proof of Theorem 2. By the Extreme Value Theorem, continuous function $E\tilde{\Pi}(p, z)$ defined by (16) attains at least one maximum value on convex set $\{(p, z) : p \in [\frac{A+a}{b}, \frac{B+a}{b}], z \in [bp - a + Q_0, B]\}$. By Lemma 4, the solution is unique if $E\tilde{\Pi}(p, z)$ is quasi-concave with respect to p for a given z . The proof is complete. $\square$

Proof of Lemma 6. The proof is similar to the proof of Lemma 3. It can be seen that $P'_{1u}(A + Q_0) > 0$ if (29) is satisfied. P_{1u} is concave if (30) and (31) hold. Moreover, $P'_{2u}(A + Q_0) > 0$ from the definition and $P'_{2u}(B) < 0$ if (32) is satisfied, which ends the proof. $\square$

Proof of Theorem 3. The proof is similar to the proof of Theorem 1. $\square$

Proof of Lemma 7. The proof is similar to the proof of Lemma 4. $\square$

Proof of Theorem 4. The proof is similar to the proof of Theorem 2. $\square$

Proof of Lemma 8. Using (37) for a given p implies

$$\frac{dE\tilde{\Pi}(p, z)}{dz} = v - c + (rp + s)\bar{F}(z) + (p(1 - r) - v)\bar{F}\left(z - \frac{w_0}{p}\right),$$

and consequently

$$\frac{d^2 E\tilde{I}(p, z)}{dz^2} = -(rp + s)f(z) - (p(1 - r) - v)f\left(z - \frac{w_0}{p}\right) < 0,$$

which implies that $E\tilde{I}(p, z)$ is concave with respect to z . $\square$

Proof of Theorem 5. By the Extreme Value Theorem, continuous function $E\tilde{I}(p, z)$ given by (37) attains at least one maximum value on convex set $\{(p, z) : p \in \left[\frac{c}{1-r}, \frac{B+a}{b}\right], z \in \left[\max\{A, bp - a\} + \frac{w_0}{p}, B\right]\}$. This function is concave with respect to z by Lemma 8 which implies that the solution in a specific convex set is unique if $E\tilde{I}(p, z)$ is quasi-concave with respect to p for a given z . The proof is complete. $\square$

References

- O'Sullivan, A.; Sheffrin, S.M. (Eds.) *Economics: Principles in Action*; Pearson Prentice Hall: Upper Saddle River, NJ, USA, 2003; p. 243.
- Stahel, W.R.; MacArthur, E. *The Circular Economy—A User's Guide*; Routledge: London, UK, 2019. [CrossRef]
- Merli, R.; Preziosi, M.; Acampora, A. How do scholars approach the circular economy? A systematic literature review. *J. Clean. Prod.* **2018**, *178*, 703–722. [CrossRef]
- Andersen, M. An introductory note on the environmental economics of the circular economy. *Sustain. Sci.* **2007**, *2*, 133–140. [CrossRef]
- Kirchherr, J.; Reike, D.; Hekkert, M. Conceptualizing the circular economy: An analysis of 114 definitions. *Resour. Conserv. Recycl.* **2017**, *127*, 221–232. [CrossRef]
- Batapa-Sigue, J. Bartering to promote circular economy. *SunStar BACOLOD* **2021**. Available online: <https://www.sunstar.com.ph/article/1887094/Bacolod/Opinion/Batapa-Sigue-Bartering-to-promote-circular-economy> (accessed on 15 March 2021).
- Mohlmann, M. Collaborative consumption: Determinants of satisfaction and the likelihood of using a sharing economy option again. *J. Consum. Behav.* **2015**, *14*, 193–207. [CrossRef]
- Henry, M.; Schraven, D.; Bocken, N.; Frenken, K.; Hekkert, M.; Kirchherr, J. The battle of the buzzwords: A comparative review of the circular economy and the sharing economy concepts. *Environ. Innov. Soc. Transit.* **2021**, *38*, 1–21. [CrossRef]
- Plewnia, F.; Guenther, E. Mapping the sharing economy for sustainability research. *Manag. Decis.* **2018**, *56*, 570–583. [CrossRef]
- Innovation and New Business Models Dialogue. Technical Report, World Economic Forum—Young Global Leaders. 2013. Available online: www3.weforum.org/docs/WEF_YGL_CircularEconomyInnovation_PositionPaper_2013.pdf (accessed on 25 March 2021).
- Curtis, S.K.; Mont, O. Sharing economy business models for sustainability. *J. Clean. Prod.* **2020**, *266*, 121519. [CrossRef]
- Bao, J. HULA Combines Exchange and Technology in Order to Contribute to a More Sustainable World. Technical Report. 2019. Available online: <https://ideas.unite.un.org/circulareconomy/Page/ViewIdea?ideaId=1641> (accessed on 15 March 2021).
- Chun-wei, L.; Wong, J.J. Barter, a dimension of sustainable design. In *The Value of Design Research, Proceedings of the 11th International Conference of the European Academy of Design, Boulogne Billancourt, France, 22–24 April 2015*; The European Academy of Design: Boulogne Billancourt, France, 2015.
- Linn, T.M. The Future Of Fashion: A Circular Economy The evolution of how we buy, sell, and make clothing. *Age Aware.* **2020**. Available online: <https://medium.com/age-of-awareness/the-future-of-fashion-a-circular-economy-4684d7ed98a4> (accessed on 15 March 2021).
- Veolia. Mexico City's Roving Barter Market and Short Circuit Food. Technical Report. 2021. Available online: <https://www.livingcircular.veolia.com/en/city/mexico-citys-roving-barter-market-and-short-circuit-food> (accessed on 15 March 2021).
- Pearson, R. Argentina's Barter Network: New Currency for New Times? *Bull. Lat. Am. Res.* **2003**, *22*, 214–230. [CrossRef]
- Seyfang, G. Bartering for a Better Future? Community Currencies and Sustainable Consumption. Local Pathways to Global Wellbeing. In *Proceedings of the 2nd International Conference on Gross National Happiness Rethinking Development, Antigonish, NS, Canada, 24–26 June 2005*.
- Escobar, A.L.; Lopez, R.R.; Solano-Sanchez, M.A.; de los Banos Garcia-Moreno Garcia, M. The Role of Complementary Monetary System as an Instrument to Innovate the Local Financial System. *J. Open Innov. Technol. Mark. Complex.* **2020**, *6*, 141. [CrossRef]
- The Barter Company. How can a Barter System help my company attract new customers in a slow economy? *Barter. Pandemic* **2021**. Available online: <https://barterco.com/barter-in-the-pandemic/> (accessed on 2 March 2021).
- Green America. Get What You Need without Money. *Green Am. Mag.* **2017**. Available online: <https://www.greenamerica.org/green-living/get-what-you-need-without-money> (accessed on 2 March 2021).

21. Lima, L.; Silva, J.; Franca, B.; Souto, V.; Ferraz, F. Online barter. Trade of goods in digital communities. In Proceedings of the Fourth International Conference on Smart Systems, Devices and Technologies (SMART 2015), Brussels, Belgium, 21–26 June 2015; Delsing, J., de Bonis, R., Eds.; International Academy, Research, and Industry Association (IARIA): Wilmington, DE, USA, 2015.
22. United Nations General Assembly. 2005 World Summit Outcome A/60/L.1, 48. Sustainable Development: Managing and Protecting Our Common Environment. 2005. Available online: <https://www.who.int/hiv/universalaccess2010/worldsummit.pdf> (accessed on 2 March 2021).
23. Chan, M. What Do You Do with Excess Inventory? 2018. Available online: <https://www.unleashedsoftware.com/blog/what-do-with-excess-inventory> (accessed on 2 March 2021).
24. Khouja, M. The single-period (news-vendor) problem: Literature review and suggestions for future research. *Omega* **1999**, *27*, 537–553. [\[CrossRef\]](#)
25. Chapman, A. Barter as a Universal Mode of Exchange. *L'Homme* **1980**, *20*, 33–83. [\[CrossRef\]](#)
26. Hua, G.; Zhang, Y.; Cheng, T.; Wang, S.; Zhang, J. The newsvendor problem with barter exchange. *Omega* **2020**, *92*, 102149. [\[CrossRef\]](#)
27. Pal, D.; Mitra, S.K. Interdependence between crude oil and world food prices: A detrended cross correlation analysis. *Phys. A Stat. Mech. Appl.* **2018**, *492*, 1032–1044. [\[CrossRef\]](#)
28. Esmaeili, A.; Shokoohi, Z. Assessing the effect of oil price on world food prices: Application of principal component analysis. *Energy Policy* **2011**, *39*, 1022–1025. [\[CrossRef\]](#)
29. Polanco Martinez, J.M.; Abadie, L.M.; Fernandez-Macho, J. A multi-resolution and multivariate analysis of the dynamic relationships between crude oil and petroleum-product prices. *Appl. Energy* **2018**, *228*, 1550–1560. [\[CrossRef\]](#)
30. Liu, L.; Ma, G. Cross-correlation between crude oil and refined product prices. *Phys. A Stat. Mech. Appl.* **2014**, *413*, 284–293. [\[CrossRef\]](#)
31. Koirala, K.H.; Mishra, A.K.; D'Antoni, J.M.; Mehlhorn, J.E. Energy prices and agricultural commodity prices: Testing correlation using copulas method. *Energy* **2015**, *81*, 430–436. [\[CrossRef\]](#)
32. DeYong, G.D. The price-setting newsvendor: Review and extensions. *Int. J. Prod. Res.* **2020**, *58*, 1776–1804. [\[CrossRef\]](#)
33. Whitin, T.M. Inventory Control and Price Theory. *Manag. Sci.* **1955**, *2*, 61–68. [\[CrossRef\]](#)
34. Porteus, E.L. Chapter 12 Stochastic inventory theory. In *Stochastic Models. Handbooks in Operations Research and Management Science*; Heyman, D.P., Sobel, M.J., Eds.; Elsevier: Amsterdam, The Netherlands, 1990; Volume 2, pp. 605–652. [\[CrossRef\]](#)
35. Qin, Y.; Wang, R.; Vakharia, A.J.; Chen, Y.; Seref, M.M. The newsvendor problem: Review and directions for future research. *Eur. J. Oper. Res.* **2011**, *213*, 361–374. [\[CrossRef\]](#)
36. Choi, T.M.; Li, D.; Yan, H. Mean-Variance Analysis for the Newsvendor Problem. *IEEE Trans. Syst. Man Cybern. Part A Syst. Humans* **2008**, *38*, 1169–1180. [\[CrossRef\]](#)
37. Wu, J.; Li, J.; Wang, S.; Cheng, T. Mean variance analysis of the newsvendor model with stockout cost. *Omega* **2009**, *37*, 724–730. [\[CrossRef\]](#)
38. Rubio-Herrero, J.; Baykal-Gursoy, M. On the unimodality of the price-setting newsvendor problem with additive demand under risk considerations. *Eur. J. Oper. Res.* **2018**, *265*, 962–974. [\[CrossRef\]](#)
39. Rubio-Herrero, J.; Baykal-Gursoy, M. Mean-variance analysis of the newsvendor problem with price-dependent, isoelastic demand. *Eur. J. Oper. Res.* **2019**, *283*, 942–953. [\[CrossRef\]](#)
40. Ye, T.; Sun, H. Price-setting newsvendor with strategic consumers. *Omega* **2016**, *63*, 103–110. [\[CrossRef\]](#)
41. Yu, L.; Pei, J.; Liu, X.; Fan, W.; Pardalos, P.M. The effect of yield rate in a general price-setting newsvendor model with a yield-dependent secondary market. *Int. Trans. Oper. Res.* **2019**, *26*, 2337–2361. [\[CrossRef\]](#)
42. Kirshner, S.N.; Shao, L. The overconfident and optimistic price-setting newsvendor. *Eur. J. Oper. Res.* **2019**, *277*, 166–173. [\[CrossRef\]](#)
43. Bai, T.; Wu, M.; Zhu, S.X. Pricing and ordering by a loss averse newsvendor with reference dependence. *Transp. Res. Part E Logist. Transp. Rev.* **2019**, *131*, 343–365. [\[CrossRef\]](#)
44. Schulte, B.; Sachs, A.L. The price-setting newsvendor with Poisson demand. *Eur. J. Oper. Res.* **2020**, *283*, 125–137. [\[CrossRef\]](#)
45. Williams, C.C. The New Barter Economy: An Appraisal of Local Exchange and Trading Systems (LETS). *J. Public Policy* **1996**, *16*, 85–101. [\[CrossRef\]](#)
46. Melis, C.; Guidici, E. Is Barter a Strategic Response to the Global Capitalist Crisis? In *Capitalism and the Social Relationship*; Kazeroony, H., Stachowicz-Stanusch, A., Eds.; Palgrave Macmillan: London, UK, 2014. [\[CrossRef\]](#)
47. Seabright, P. (Ed.) *The Vanishing Rouble: Barter Networks and Non-monetary Transactions in Post-Soviet Societies*; Cambridge University Press: Cambridge, UK, 2000.
48. Guriev, S.; Kvassov, D. Barter for price discrimination. *Int. J. Ind. Organ.* **2004**, *22*, 329–350. [\[CrossRef\]](#)
49. Gunia, B.C.; Lewicki, R.J. Bartering as a Blind Spot: A Call to Action from COVID-19. *Negot. J.* **2020**, *36*, 561–572. [\[CrossRef\]](#)
50. Battistella, L.M.; Hynes, N. Something New Under the Sun? A Case Study of Modern Italian Barter Systems. *J. Appl. Bus. Econ.* **2019**, *21*. [\[CrossRef\]](#)
51. Erridge, A.; Zhabykenov, D. The role of purchasing in countertrade. *Eur. J. Purch. Supply Manag.* **1998**, *4*, 97–107. [\[CrossRef\]](#)
52. Plank, R.E.; Reid, D.A.; Bates, F. Barter: An Alternative to Traditional Methods of Purchasing. *Int. J. Purch. Mater. Manag.* **1994**, *30*, 51–57. [\[CrossRef\]](#)

-
53. Efremova, O.I.; Gabdullin, L.V. Logistic barter features as a new concept of supply chain management. *Acad. Strateg. Manag. J.* **2016**, *15*, 19–25.
 54. Anderson, R.; Ashlagi, I.; Gamarnik, D.; Kanoria, Y. Efficient Dynamic Barter Exchange. *Oper. Res.* **2017**, *65*, 1446–1459. [[CrossRef](#)]
 55. Zhang, B.; Wei, L.; Chen, M. Implications of Barter Exchange and Decision Biases in a Pull Contract. *Econom. Econom. Stat. Methods Spec. Top. eJournal* **2020**. [[CrossRef](#)]
 56. Krishnan, H. A note on demand functions with uncertainty. *Oper. Res. Lett.* **2010**, *38*, 436–440. [[CrossRef](#)]
 57. Kyparisis, G.; Koulamas, C. The price-setting newsvendor problem with nonnegative linear additive demand. *Eur. J. Oper. Res.* **2018**, *269*, 695–698. [[CrossRef](#)]
 58. Bieniek, M. The ubiquitous nature of inventory: Vendor Managed Consignment Inventory in adverse market conditions. *Eur. J. Oper. Res.* **2021**, *291*, 411–420. [[CrossRef](#)]
 59. Zabel, E. Monopoly and Uncertainty. *Rev. Econ. Stud.* **1970**, *37*, 205–219. [[CrossRef](#)]
 60. Wang, Y.; Jiang, L.; Shen, Z.J. Channel performance under consignment contract with revenue sharing. *Manag. Sci.* **2004**, *50*, 34–47. [[CrossRef](#)]
 61. Li, S.; Zhu, Z.; Huang, L. Supply chain coordination and decision-making under consignment contract with revenue sharing. *Int. J. Prod. Econ.* **2009**, *120*, 88–99. [[CrossRef](#)]
 62. Rubio-Herrero, J.; Baykal-Gursoy, M.; Jaskiewicz, A. A price setting newsvendor problem under mean-variance criteria. *Eur. J. Oper. Res.* **2015**, *247*, 575–587. [[CrossRef](#)]
 63. Kharif, O. The Return of the Barter Economy, Swapping Eggs for Toilet Paper. *Bloomberg* **2020**. Available online: <https://www.bloomberg.com/news/articles/2020-03-31/the-return-of-the-barter-economy-swapping-eggs-for-toilet-paper> (accessed on 25 May 2021).
 64. Bearne, S. A barter way of life: Covid crisis fuels swapping of goods and skills. *Guardian* **2020**. Available online: <https://www.theguardian.com/money/2020/oct/31/barter-covid-swapping-goods-skills> (accessed on 25 May 2021).
 65. Bagga, B. How a Barter Strategy Can Solve Excess Capacity and Grow Your Hospitality Business. Hospitality Technology. Smarter Hotels and Restaurants. 2021. Available online: <https://hospitalitytech.com/how-barter-strategy-can-solve-excess-capacity-and-grow-your-hospitality-business> (accessed on 25 January 2021).