Derivation of growth equations

The different forms of the Richards, Weibull and Korf models listed in Table 2 of the paper were derived using the GADA technique described by Cieszewski & Bailey (2000). Below are details of these derivations.

Richards model

Base form:

(S1) $y = a(1 - e^{-bt})^c$

Anamorphic form:

The GADA technique uses a theoretical variable labelled the growth intensity factor, *x*, which quantifies how the growth curve varies in shape or level across different sites. For an anamorphic model, the asymptote parameter, *a*, is replaced by *x*, i.e.,

(S2) $y = x(1 - e^{-bt})^c$

This equation is solved for *x* at point (t_0, y_0) giving,

$$x = y_0/(1 - e^{-bt_0})^c$$

Substituting this into Equation (S2) produces the required anamorphic equation,

(S3) $y = y_0\left(\frac{1-e^{-bt}}{1-e^{-bt_0}}\right)^c$

Inverting this equation produces the following *SI* equation where t_0 is the base age,

$$y_0 = y\left(\frac{1-e^{-bt}}{1-e^{-bt_0}}\right)^{-c}$$

Common-asymptote form:

For the common-asymptote model, the time-scale parameter, *b*, in the base model is replaced by *x*,

(S4) $y = a(1 - e^{-xt})^c$

Solving for *x* at point (t_0, y_0),

$$x = -ln(1 - (y_0/a)^{-c})/t_0$$

Substituting this into Equation (S4) produces the required common-asymptote equation,

(S5) $y = a\left(1 - \left(1 - \left(\frac{y_0}{a}\right)^{1/c}\right)^{t/t_0}\right)^c$

Inverting this equation produces the following *SI* equation,

$$y_0 = a\left(1 - (1 - (y/a)^{-c})^{-t/t_0}\right)^c$$

Weibull model

Base form:

(S6) $$y = a\left(1 - e^{-bt^c}\right)$$

Anamorphic form:

To derive the anamorphic form, replace the parameter, *a*, with *x*,

(S7) $$y = x(1 - e^{-bt})^c$$

Solve for *x* at point (t_0, y_0),

$$x = y_0/\left(1 - e^{-bt_0^c}\right)$$

Substituting this into Equation (S7) produces the required anamorphic equation,

(S8) $$y = y_0 \frac{1-e^{-bt^c}}{1-e^{-bt_0^c}}$$

Inverting this equation produces the following *SI* equation,

$$y_0 = y \frac{1-e^{-bt_0^c}}{1-e^{-bt^c}}$$

Common-asymptote form:

To derive the common-asymptote form, replace the parameter, *b*, with *x*,

(S9) $$y = a\left(1 - e^{-xt^c}\right)$$

Solve for *x* at point (t_0, y_0),

$$x = -\,ln\left(1 - (y_0/a)\right)/t_0^c$$

Substituting this into Equation (S9) produces the required common-asymptote equation,

(S10) $$y = a\left(1 - \left(1 - \frac{y_0}{a}\right)^{(t/t_0)^c}\right)$$

Inverting this equation produces the following *SI* equation,

$$y_0 = a\left(1 - \left(1 - \frac{y}{a}\right)^{-(t/t_0)^c}\right)$$

Korf model

Base form:

(S11) $$y = ae^{-bt^{-c}}$$

Anamorphic form:

To derive the anamorphic form, replace the parameter, *a*, with *x*,

(S12) $$y = xe^{-bt^{-c}}$$

Solve for *x* at point (t_0, y_0),

$$x = y_0/e^{-bt_0^{-c}}$$

Substituting this into Equation (S12) produces the required anamorphic equation,

(S13) $y = y_0 \frac{e^{-bt^{-c}}}{e^{-bt_0^{-c}}}$

Inverting this equation produces the following *SI* equation,

$$y_0 = y \frac{e^{-bt_0^{-c}}}{e^{-bt^{-c}}}$$

Common-asymptote form:

To derive the common-asymptote form, replace the parameter, *b*, with *x*,

(S14) $y = xe^{-xt^{-c}}$

Solve for *x* at point (t_0, y_0),

$$x = -\,ln(y_0/a)/t_0^{-c}$$

Substituting this into Equation (S14) produces the required common-asymptote equation,

(S15) $y = a\left(\frac{y_0}{a}\right)^{(t/t_0)^{-c}}$

Inverting this equation produces the following *SI* equation,

$$y_0 = \left(\frac{y}{a}\right)^{-(t/t_0)^{-c}}$$

Polymorphic form:

To derive the polymorphic form, we start by assuming that *x* is proportional to the exponential of the asymptote, *a*, and inversely related to the time scale parameter, *b*,

(S16) $y = e^x e^{-(b/x)t^{-c}}$

Taking logs and multiplying both sides of the equation by *x* produces the following quadratic equation,

$$x^2 - xlny - bt^{-c} = 0$$

Solving for *x* at point (t_0, y_0) produces,

$$x = \frac{lny_0 + \sqrt{(lny_0)^2 + 4bt_0^{-c}}}{2}$$

Substituting this into Equation (S16) produces the required polymorphic equation,

(S17) $y = e^{R_0/2 - 2b/(R_0 t^c)}$, where, $R_0 = lny_0 + \sqrt{(lny_0)^2 + 4b/t_0^c}$

To produce a *SI* equation for this model, first take logs of (S17) and multiply by R_0 to produce the quadratic equation,

$$R_0^2/2 - R_0 lny - 2b/t^c = 0$$

Solving for R_0 produces,

$$R_0 = lny + \sqrt{(lny)^2 + 4b/t^c}$$

Which leads to,

$$\sqrt{(lny_0)^2 + 4b/t_0^c} = R - \; lny_0, \text{ where, } \; R = lny + \sqrt{(lny)^2 + 4b/t^c}$$

Squaring both sides produces,

$$(lny_0)^2 + 4b/t_0^c = R^2 - 2R\; lny_0 + (lny_0)^2$$

Which simplifies to the required *SI* equation,

$$y_0 = e^{\left(4b/t_0^c - R^2\right)/(-2R)}$$