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Connectivity and Hamiltonicity of Canonical Colouring Graphs of Bipartite and Complete Multipartite Graphs

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Received: 12 February 2018; Accepted: 24 March 2018; Published: 29 March 2018

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Abstract: A k-colouring of a graph G with colours $1, 2, \ldots, k$ is *canonical* with respect to an ordering $\pi = v_1, v_2, \ldots, v_n$ of the vertices of G if adjacent vertices are assigned different colours and, for $1 \leq c \leq k$, whenever colour c is assigned to a vertex v_i, each colour less than c has been assigned to a vertex that precedes v_i in π. The *canonical k-colouring graph of G with respect to π* is the graph $\mathrm{Can}_k^\pi(G)$ with vertex set equal to the set of canonical k-colourings of G with respect to π, with two of these being adjacent if and only if they differ in the colour assigned to exactly one vertex. Connectivity and Hamiltonicity of canonical colouring graphs of bipartite and complete multipartite graphs is studied. It is shown that for complete multipartite graphs, and bipartite graphs there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(G)$ is connected for large enough values of k. It is proved that a canonical colouring graph of a complete multipartite graph usually does not have a Hamilton cycle, and that there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(K_{m,n})$ has a Hamilton path for all $k \geq 3$. The paper concludes with a detailed consideration of $\mathrm{Can}_k^\pi(K_{2,2,\ldots,2})$. For each $k \geq \chi$ and all vertex orderings π, it is proved that $\mathrm{Can}_k^\pi(K_{2,2,\ldots,2})$ is either disconnected or isomorphic to a particular tree.

Keywords: reconfiguration problems; graph colouring; Hamilton cycles; Gray codes

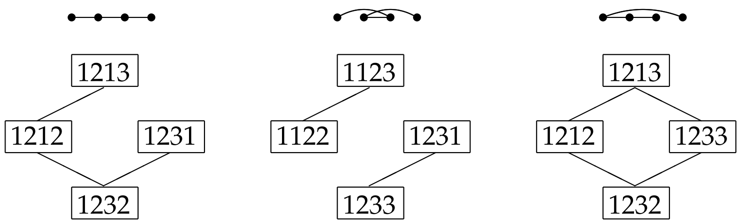
1. Introduction

One definition of a *k-colouring* of a graph G is as a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. Under this definition, k-colourings f_1 and f_2 are different whenever there exists a vertex x such that $f_1(x) \neq f_2(x)$. Each k-colouring f is equivalent to a k-tuple $(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k))$ in which the set of non-empty components is a partition of $V(G)$ into independent sets.

A k-colouring $f : V(G) \to \{1, 2, \ldots, k\}$ is *canonical* with respect to an ordering $\pi = v_1, v_2, \ldots, v_n$ of the vertices of G if, whenever $f(v_i) = c$, every colour less than c has been assigned to some vertex that precedes v_i in π. Thus v_1 is necessarily assigned colour 1, and colour 3 can only be assigned to some vertex after colour 2 has been assigned to a vertex that appears earlier in the sequence π. Note that canonical colourings may be very different than the colourings arising from applying the usual greedy colouring algorithm to G using the vertex ordering π.

Define an equivalence relation $\sim$ on the set of k-colourings of G by $f_1 \sim f_2$ if and only if f_1 and f_2 determine the same partition of $V(G)$ into independent sets. The set of canonical k-colourings of G with respect to π is then the set of representatives of the equivalence classes of $\sim$ that are lexicographically least with respect to π. Thus, canonical k-colourings exist for every $k \geq \chi(G)$ and every proper colouring is equivalent to a canonical colouring.

For an ordering π of the vertices of a graph G, the *canonical k-colouring graph of G*, denoted $\mathrm{Can}_k^{\pi}(G)$, has vertex set equal to the set of canonical k-colourings of G with respect to π, with two of these being adjacent when they differ in the colour assigned to exactly one vertex. While every ordering gives a set of representatives of the possible k-colourings, different orderings can lead to different canonical k-colourings graphs. Examples of the canonical 3-colouring graph of the path on 4 vertices are given in Figure 1 for three different orderings of the vertices of the path. When a canonical colour graph is connected, any given canonical k-colouring can be reconfigured into any other via a sequence of recolourings which each change the colour of exactly one vertex. When it is Hamiltonian, there is a cyclic list that contains all of the k-colourings of G and consecutive elements of the list differ in the colour of exactly one vertex, that is, there is a cyclic Gray code of the k-colourings of G.



(**a**) (**b**) (**c**)

Figure 1. Three different vertex orderings of P_4 with associated $Can_3^{\pi}(P_4)$. In each case the colourings are canonical with respect to the given vertex ordering from left to right.

This paper is organized as follows. Relevant definitions and background information are reviewed in Section 2. A generalization of a lemma of [1] concerning vertex orderings such that $\mathrm{Can}_k^{\pi}(G)$ is disconnected for all $k \geq \chi(G)$ is proved. Connectivity and Hamiltonicity of canonical colouring graphs of unions and joins of graphs are considered in Section 3. The main focus is on the situation where one of the graphs involved is a complete graph or a complement of a complete graph. For $n \geq 1$ and any vertex ordering π, canonical k-colourings of $\overline{K}_n$ correspond exactly to partitions of $\{1, 2, \ldots, n\}$ with at most k cells. Our results give a Gray code listing of these partitions similar to that of Kaye [2]. Since the complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$ is the join of $\overline{K}_{n_1}, \overline{K}_{n_2}, \ldots, \overline{K}_{n_r}$, our results show that there are vertex orderings π for which $\mathrm{Can}_k^{\pi}(K_{n_1,n_2,\ldots,n_r})$ is connected whenever $k \geq r$. In Section 4, we first show that there exists a vertex ordering π such that the canonical k-colouring graph of a bipartite graph is connected whenever $k \geq 1 + |V|/2$, and then give an example showing that this bound is the best possible. We then prove a negative result which implies that complete multipartite graphs with at least two nontrivial parts can not have Hamiltonian canonical colouring graphs, and there cannot be a Hamilton path if there are at least three parts of size that have at least two. This leaves open the possibility that canonical colouring graphs of complete bipartite graphs may have a Hamilton path. We show that there exists an ordering π such that $\mathrm{Can}_k^{\pi}(K_{m,n})$ has a Hamilton path for all $k \geq 3$. In the final section of the paper, we study the canonical k-colouring graph of the complete multipartite graph in which each part has exactly two vertices. We show that, for any vertex ordering π and any integer k at least as large as the number of parts, the canonical k-colouring graph is either disconnected, or isomorphic to a particular tree.

Throughout the paper, proofs of existence results are constructive and lead to algorithms which generate the desired sequences.

2. Background, and a Preliminary Result

For basic definitions in graph theory, we refer to the text of Bondy and Murty [3].

Before briefly surveying some previous research on colour graphs we recall the definition of $col(G)$, *the colouring number of* G. Let $\pi = x_1, x_2, \ldots, x_n$ be an ordering of the vertices of G. Let H_i be the subgraph of G induced by $\{x_1, x_2, \ldots, x_i\}$, for $i = 1, 2, \ldots, n$. Define $D_\pi = \max_{1 \le i \le n} d_{H_i}(x_i)$. Then $col(G) = \min_\pi D_\pi + 1$. Equivalently, $col(G) = 1 + \max \delta(H)$, where the maximum is taken over all subgraphs of G. The quantity $col(G)$ is an upper bound on the number of colours needed if the greedy colouring algorithm is applied to G and vertices are coloured in the order π. Hence $\chi(G) \le col(G) \le \Delta(G) + 1$.

For $k \ge 1$, let $\mathcal{F}_k(G)$ be the set of k-colourings of a graph G. The *k-colouring graph of* G, denoted $\mathcal{C}_k(G)$, has vertex set $\mathcal{F}_k(G)$, with two k-colourings being adjacent if and only if they differ in the colour of exactly one vertex. For example, the 3-colouring graph of a path on four vertices is given in Figure 2. This is an example of a reconfiguration graph in which vertices represent feasible solutions to a problem and there is an edge between two solutions if one can be transformed to the other by some allowable reconfiguration rule. There is a vast literature on the complexity of reconfiguration problems, for example see [4,5]. The graph $\mathcal{C}_k(G)$ is the most studied of the various colour graphs (that is, among the different allowable sets of colourings, and different reconfiguration rules). Connectivity of $\mathcal{C}_k(G)$ arises in random sampling of k-colourings, and approximating the number of k-colourings, for example see [6–8]. Dyer, Flaxman, Frieze and Vigoda proved that there is a least integer $c_0 \le col(G) + 1$ such that k-colouring graph of G is connected for all $k \ge c_0$ [6] (also see [9]). It is NP-complete to decide if the 3-colouring graph of a bipartite graph is connected [10], but polynomial-time to decide if two 3-colourings of a bipartite graph belong to the same component of $\mathcal{C}_3(G)$ [11]. Hamiltonicity of the k-colouring graph was first considered in [12], wherein it was proved that there is always a least integer $k_0 \le col(G) + 2$ such that the k-colouring graph of the graph G is Hamiltonian for all $k \ge k_0$. The number k_0 is known for complete graphs, trees and cycles [12], 2-trees [13], complete bipartite graphs [14], and some complete multipartite graphs [15]. For other results on $C_k(G)$, see [16], and for related results concerning the graph of $L(2,1)$-labellings (colourings with additional conditions), see [17].

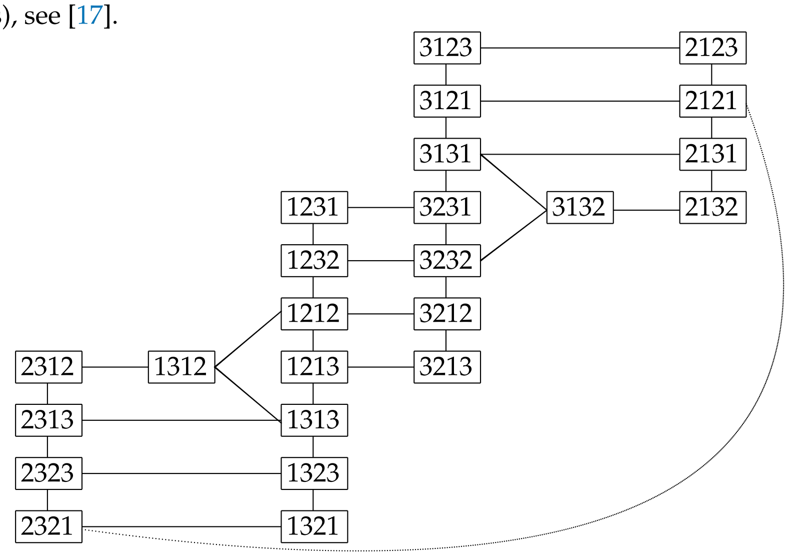


Figure 2. $\mathcal{C}_3(P_4)$, the 3-Colouring Graph of P_4. The vertices are labeled by the colourings of the path.

The *Bell k-colouring graph of* G, denoted $\mathcal{B}_k(G)$, has as vertices the partitions of $V(G)$ into at most k independent sets, with two of these being adjacent when there is a vertex x such that these partitions are equal when restricted to $G - x$. The Bell 3− colouring graph of the path on four vertices is given in Figure 3. Bell k-colouring graphs are staudied in [18], as is the *Stirling ℓ-colouring graph of* G, the subgraph of $\mathcal{B}_{|V|}(G)$ induced by the partitions with exactly ℓ cells. It is proved that $\mathcal{B}_{|V|}(G)$ is Hamiltonian for every graph G except K_n and $K_n - e$, and the quantity $|V|$ is the best possible. It is also proved that the Bell k-colouring graph of a tree with at least four vertices is Hamiltonian for all $k \ge 3$, and the Stirling ℓ-colouring graph of a tree on at least $n \ge 1$ vertices is Hamiltonian for all $\ell \ge 4$.

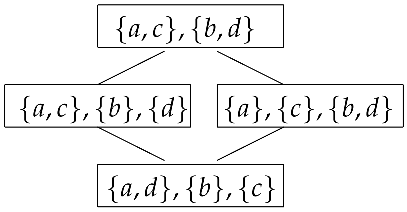


Figure 3. $\mathcal{B}_3(P_4)$, the 3-Bell colouring graph of P_4. The vertices are labeled by the partition of the path *abcd*.

The graph $\mathrm{Can}_k^\pi(G)$ is a spanning subgraph of $\mathcal{B}_k(G)$; the restriction to canonical colourings eliminates some edges of $\mathcal{B}_k(G)$. Thus results asserting connectivity or Hamiltonicity of $\mathrm{Can}_k^\pi(G)$ imply connectivity or Hamiltonicity of $\mathcal{B}_k(G)$, respectively. Since at most n colours can be assigned to the vertices of an n-vertex graph G, it follows that $\mathcal{B}_k(G) = \mathcal{B}_n(G)$ and $\mathrm{Can}_k^\pi(G) = \mathrm{Can}_n^\pi(G)$ for all $k \geq n$.

Canonical k-colouring graphs were first considered in [1]. For every tree T there exists an ordering π of the vertices such that the canonical k-colouring graph of T with respect to π is Hamiltonian for all $k \geq 3$. The canonical 3-colouring graph of the cycle C_n is disconnected for all vertex orderings π, while for each $k \geq 4$ there exists an ordering π for which $\mathrm{Can}_k^\pi(C_n)$ is connected. It is an open problem to find general conditions on k and π such that $\mathrm{Can}_k^\pi(G)$ is connected. Most results are negative assertions about certain vertex orders π. In [1] it was proved that if G is connected, but not complete then there is always a vertex ordering π such that $\mathrm{Can}_k^\pi(G)$ is disconnected for all $k \geq \chi(G)+1$. In particular, the graph $\mathrm{Can}_k^\pi(G)$ is disconnected whenever the first three vertices u, v, w of the vertex ordering π are such that $uv \notin E$ but $uw, vw \in E$. Our first proposition generalizes that statement.

Proposition 1. *Let $\pi = v_1, v_2, \ldots, v_n$ be a vertex ordering of G. If there exists $i \geq 3$ such that v_i is adjacent to each of $v_1, v_2, \ldots, v_{i-1}$, and the subgraph of G induced by $\{v_1, v_2, \ldots, v_i\}$ is not complete, then $\mathrm{Can}_k^\pi(G)$ is disconnected for all $k \geq \chi(G)+1$.*

Proof. Let H_i be the subgraph of G induced by $\{v_1, v_2, \ldots, v_i\}$. Since H_i is not complete, $\chi(H_i) < i$.

Let c_1 be a canonical $\chi(G)$-colouring of G with respect to π. Then $c_1(v_i) = 1 + \max\{c_1(v_1), c_1(v_2), \ldots, c_1(v_{i-1}\}$. Furthermore, if c_2 is an adjacent colouring in $\mathrm{Can}_k^\pi(G)$ then it differs on only one vertex. The colour of v_i cannot change (because we are only considering canonical colourings) so c_2 must differ on a vertex other than v_i. It follows that the vertex v_i is assigned the same colour in any canonical colouring that is joined to c_1 by a path.

Suppose first that c_1 assigns the same colour to two of $v_1, v_2, \ldots, v_{i-1}$, say $c_1(v_a) = c_1(v_b)$ for some $a, b < i$. Then, there is a (non-canonical) $\chi(G)+1$ colouring of G in which v_b is coloured with colour $\chi(G)+1$, and all other vertices, v_j for $j < i$ are assigned the same colour as in c_1. Let c_2 be the equivalent canonical colouring to this with respect to π (defines the same partition of $V(G)$). Then $c_2(v_i) = 1 + c_1(v_i)$. Hence, there is no path in $\mathrm{Can}_k^\pi(G)$ joining c_1 and c_2.

Now assume c_1 assigns distinct colours to each of $v_1, v_2, \ldots, v_{i-1}$. Since H_i is not complete, it has a pair of non-adjacent vertices. There is a $\chi(G)+1$ colouring of G in which these two vertices are assigned the same colour, and all other vertices are assigned the same colour as in c_1. Let c_3 be the canonical version of this colouring. Then $c_3(v_i) = c_1(v_i) - 1$. Hence, there is no path in $\mathrm{Can}_k^\pi(G)$ joining c_1 and c_3.

In both cases, $\mathrm{Can}_k^\pi(G)$ is disconnected. This completes the proof. □

3. Unions and Joins

In this section we explore connectivity and Hamiltonian properties of graphs constructed by the operations of disjoint union and join. Our main focus is the situation where one of the graphs involved is complete, or has no edges.

Recall that the *disjoint union* of disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* of disjoint graphs G_1 and G_2 is the graph $G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{x_1x_2 : x_1 \in V(G_1) \text{ and } x_2 \in V(G_2)\}$. We shall consider unions first, and joins second.

Observe that the canonical k-colouring graph of $\overline{K}_n = K_1 \cup K_1 \cup \cdots \cup K_1$ is the graph of partitions of an n-set into at most k parts. Hence the number of vertices is the sum of Stirling numbers of the second kind, $S(n,1) + S(n,2) + \cdots + S(n,k)$. A Hamilton cycle in this graph corresponds to a cyclic Gray code for set partitions. Many different Gray codes, cyclic and otherwise, for set partitions are known to exist [19]; our method gives a different point of view and leads to a recursive algorithm similar to that of Kaye [2]. A related method that gives Hamilton paths rather than Hamilton cycles is given in Theorem 4.

Theorem 1. *Let π be a vertex ordering such that $\mathrm{Can}_k^{\pi}(G)$ is Hamiltonian. Then, for the vertex ordering π' of $G \cup K_1$ obtained by placing the vertex of K_1 at the end of π, the graph $\mathrm{Can}_k^{\pi'}(G \cup K_1)$ is Hamiltonian.*

Proof. Since $\mathrm{Can}_k^{\pi}(G)$ has at least three vertices, we have $k \geq 2$.

Suppose $k = 2$. Then, G is bipartite and has at least three components. Let X_1 be the component of G containing the first vertex of π. Then $\mathrm{Can}_k^{\pi}(G)$ is isomorphic to the cube of dimension equal to the number of components of $G - X_1$, and $\mathrm{Can}_k^{\pi'}(G \cup K_1)$ is isomorphic to the cube of one higher dimension. Since the t-cube is Hamiltonian for all $t \geq 2$, the statement follows.

Now suppose $k \geq 3$. If c is a canonical k-colouring of $G \cup K_1$ with respect to π', then the restriction of c to G is a canonical k-colouring of G. We will say that the colouring c on $G \cup K_1$ is an *extension* of the colouring on G. Furthermore, each canonical k-colouring of G has at least two extensions to a canonical k-colouring of $G \cup K_1$, and there are exactly two extensions if and only if $G \cong \overline{K}_n$ and only one colour is used on the vertices of G. Notice that the set of canonical k-colourings of $G \cup K_1$ which agree on their restriction to $V(G)$ induces a complete subgraph of $\mathrm{Can}_k^{\pi'}(G \cup K_1)$.

By hypothesis, $\mathrm{Can}_k^{\pi}(G)$ has a Hamilton cycle $c_1, c_2, \ldots, c_t, c_1$. Thus $t \geq 3$, and there exists i such that c_i and c_{i+1} both use at least two colours. Without loss of generality, $i = t$. Thus, the canonical k-colourings c_t and c_1 each have at least three extensions to canonical k-colouring of $G \cup K_1$. For $i = 1, 2, \ldots, t$, let $c_i \cdot \ell$ denote the extension of c_i to a canonical k-colouring of $G \cup K_1$ in which the vertex of K_1 is assigned colour ℓ. Observe that $c_i \cdot 1$ and $c_i \cdot 2$ are adjacent to $c_{i+1} \cdot 1$ and $c_{i+1} \cdot 2$, respectively, $1 \leq i \leq t-1$ and $c_t \cdot 1, c_t \cdot 2$ and $c_t \cdot 3$ are adjacent to $c_1 \cdot 1, c_1 \cdot 2$ and $c_1 \cdot 3$, respectively.

A Hamilton cycle in $\mathrm{Can}_k^{\pi'}(G \cup K_1)$ can be constructed as follows. The first vertex is $c_1 \cdot 1$. Then, for $i = 2, 3, \ldots, t-1$, list all extensions of c_i such that $c_i \cdot 1$ is first and $c_i \cdot 2$ is last if i is even, and $c_i \cdot 2$ is first and $c_i \cdot 1$ is last if i is odd. Observe that any pair of consecutive vertices in the list are adjacent. Let $c_{t-1} \cdot z$ be the last vertex listed according to this procedure. The Hamilton cycle is completed by listing $c_t \cdot z$, then all other extensions of c_t in such a way that $c_t \cdot 3$ is listed last and, finally, $c_1 \cdot 3$ and all extensions of c_1 in such a way that $c_1 \cdot 1$ is listed last (recall that $c_1 \cdot 1$ was the first vertex listed).

This completes the proof. □

Corollary 1. *Let π be a vertex ordering such that $\mathrm{Can}_k^{\pi}(G)$ is Hamiltonian. Then, for the vertex ordering π' of $G \cup \overline{K}_n$ obtained by placing the vertices of $\overline{K}_n$ at the end of π, the graph $\mathrm{Can}_k^{\pi'}(G \cup \overline{K}_n)$ is Hamiltonian.*

The Gray code for set partitions implied by the following is similar to the one found by Kaye [2].

Corollary 2. *For all $n \geq 3$ and $k \geq 2$, and any vertex ordering π, the graph $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ is Hamiltonian.*

We now turn our attention to connectivity of the canonical k-colouring graph of the disjoint union of graphs G_1 and G_2. Since it is an open problem to determine general conditions under which the canonical k-colouring graph of a (connected) graph G is connected, in the results that follow we assume the canonical k-colouring graph of G_1 is connected and give conditions under which a canonical colouring graph of $G_1 \cup G_2$ is connected, no matter how the vertices of G_2 are ordered following the vertex ordering of G_1.

Theorem 2. *Let G_1 and G_2 be disjoint graphs such that $\chi(G_1) \geq 1 + col(G_2)$. Suppose there exists an integer k, and an ordering ϕ of the vertices of G_1, such that $\mathrm{Can}_k^{\phi}(G_1)$ is connected. Then, for any ordering π of the vertices of $G_1 \cup G_2$ obtained by putting an ordering of the vertices of G_2 after ϕ, the graph $\mathrm{Can}_k^{\pi}(G_1 \cup G_2)$ is connected.*

Proof. Let c be some particular canonical colouring of $G_1 \cup G_2$ with $\chi(G_1) = \chi(G_1 \cup G_2)$ colours such that colours $1, 2, \ldots, \chi(G_1)$ appear on the vertices of G_1 (as they must), and colours $1, 2, \ldots, \chi(G_2)$ appear on the vertices of G_2. Let c_2 be the restriction of c to $V(G_2)$.

We complete the proof by showing that any canonical k-colouring of $G_1 \cup G_2$ can be transformed into c by a finite number of steps corresponding to edges in $\mathrm{Can}_k^{\pi}(G_1 \cup G_2)$. Suppose a canonical k-colouring d of $G_1 \cup G_2$ is given. Let M be the largest colour which d assigns to a vertex of G_1. Let H_2 be the subgraph of G_2 induced by the set of vertices on which colours $1, 2, \ldots, M$ appear.

Since $M \geq \chi(G_1) \geq 1 + col(G_2) \geq 1 + col(H_2)$, the (ordinary) M-colouring graph of H_2 is connected [6,9]. Hence there is a sequence of steps corresponding to edges in $\mathrm{Can}_k^{\pi}(G_1 \cup G_2)$ that transforms d to a canonical colouring d' which agrees with c_2 on $V(H_2)$. The following step can then be repeated until d' is transformed into a canonical colouring that agrees with c_2 on $V(G_2)$. If the current colouring does not agree with c_2 on $V(G_2)$, then let x be the last vertex of G_2 which is not coloured $c_2(x)$, and recolour x with $c_2(x)$ (Note that any the colour of any such x is greater than M). The resulting colouring is proper because of the recolouring of H_2 done earlier, and canonical by the maximality of the position of x.

Finally, since $\mathrm{Can}_k^{\phi}(G_1)$ is connected and $\chi(G_1) \geq \chi(G_2)$, the subgraph of $\mathrm{Can}_k^{\pi}(G_1 \cup G_2)$ induced by the set of (canonical) colourings for which the restriction to $V(G_2)$ is c_2 is isomorphic to $\mathrm{Can}_k^{\phi}(G_1)$, and is therefore connected. Hence there is a sequence of steps corresponding to edges in $\mathrm{Can}_k^{\pi}(G_1 \cup G_2)$ that transforms a canonical colouring which agrees with c_2 on $V(G_2)$ into c. This completes the proof. □

The hypothesis of the above theorem can be relaxed slightly to $\chi(G_1) \geq 1 + c_0(G_2)$, where c_0 is the least integer such that k-colouring graph of G is connected for all $k \geq c_0$. By the result of [6], $c_0(G_2) \leq col(G_2)$.

Corollary 3. *Let $k, n \geq 1$ and G be a graph with at least one edge. If there exists a vertex ordering π such that $\mathrm{Can}_k^{\pi}(G)$ is connected, then there exists an order π' for which $\mathrm{Can}_k^{\pi'}(G \cup \overline{K}_n)$ is connected.*

Proof. The colouring number of K_1 equals 1. Apply Theorem 2 inductively. □

We conclude this section by considering the join operation. Observe that in any colouring of $G_1 \vee G_2$, the set of colours that appear on the vertices of G_1 is disjoint from the set of colours that appear on the vertices of G_2. With this observation, the proof of the first proposition below is straightforward, and hence is omitted.

Proposition 2. *Let π be a vertex ordering of the graph G. If π' is the vertex ordering obtained by inserting the vertices of the K_r at the beginning of π, then $\mathrm{Can}_t^{\pi}(G) \cong \mathrm{Can}_{t+r}^{\pi'}(G \vee K_r)$.*

Corollary 4. *If $\mathrm{Can}_t^{\pi}(G)$ is connected (resp. has a Hamilton path, has a Hamilton cycle) then there exists an order π' such that $\mathrm{Can}_{t+r}^{\pi'}(G \vee K_r)$ is connected (resp. has a Hamilton path, has a Hamilton cycle).*

In contrast, by Proposition 1, in almost any ordering π' of the vertices that does not begin with all the vertices of K_r the corresponding $\mathrm{Can}_{t+r}^{\pi'}(G \vee K_r)$ will be disconnected.

Corollary 5. *Let T be a tree with at least three vertices, and $k \geq 4$. For any integer $n > 1$, there exists a vertex ordering π' such that $\mathrm{Can}^{\pi'}_{k+n}(T \vee K_n)$ is Hamiltonian.*

Proof. For any such k, there is a vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(T)$ is Hamiltonian [1]. □

The next corollary implies, among other things, that the canonical c-colouring graph of a wheel on n spokes is connected for all $c \geq 4$.

Corollary 6. *Let $k \geq 4$, $t \geq 3$ and $n \geq 1$. There exists a vertex ordering π' such that $\mathrm{Can}^{\pi'}_{k+n}(C_t \vee K_n)$ is connected.*

Proof. For any such k, there is a vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(C_t)$ is connected [1]. □

Proposition 3. *Let $k, n \geq 1$. Suppose there exists a vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(G)$ is connected. Then there exists an order π' for which $\mathrm{Can}^{\pi'}_{k+i}(G \vee \overline{K}_n)$ is connected for all $i \geq 1$.*

Proof. Let π' be the order obtained from π by inserting one vertex of $\overline{K}_n$ at the beginning of the ordering and all the others at the end. Note that the subgraph of $\mathrm{Can}^{\pi'}_{k+i}(G \vee \overline{K}_n)$ induced by the set of canonical colourings in which every vertex of $\overline{K}_n$ is coloured 1 is isomorphic to $\mathrm{Can}^{\pi}_{k}(G)$. Since, for any canonical colouring c, there is a path in $\mathrm{Can}^{\pi'}_{k+i}(G \vee \overline{K}_n)$ to a canonical colouring in which every vertex of $\overline{K}_n$ is coloured 1 and the colour of every vertex of G is the same as in c, the result follows. □

We note that connected cannot be replaced by Hamiltonian in the above proposition. It follows from Proposition 4 that, for example, there is no vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(K_{2,2})$ has a Hamilton cycle for any $k \geq 3$, and no ordering π' such that $\mathrm{Can}^{\pi'}_{k}(K_{2,2,2})$ has a Hamilton path for any $k \geq 4$.

Corollary 7. *Let H be a complete multipartite graph with p parts. For any $k \geq p$, there exists a vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(H)$ is connected.*

Proof. Suppose one of the maximal independents sets has size s. Take $G = \overline{K}_s$ in Proposition 3, and apply the proposition inductively to construct H and π. □

4. Bipartite Graphs

We now show that, once k is sufficiently large, there is always a vertex ordering such that the canonical k-colouring graph of a bipartite graph is connected. We then show that the bound given is the best possible.

Theorem 3. *Let G be a bipartite graph on n vertices, then there exists an ordering π of the vertices such that $\mathrm{Can}^{\pi}_{t}(G)$ is connected for $t \geq n/2 + 1$.*

Proof. Suppose G has bipartition (A, B), where $|A| \geq |B|$. Choose $a \in A, b \in B$, such that $ab \in E(G)$. Define π to be $a, b, B - b, A - a$. That is vertex a is coloured first, b is coloured second, the rest of B are the third through $(|B| + 1)$st vertices to be coloured, the rest of A are the $(|B| + 2)$nd through nth vertices to receive colours. Label the vertices $v_1, v_2, \ldots v_n$ according to this order.

The standard two colouring $s : V \to \{1, 2\}$ is $s(v_j) = 1$ if $j = 1$ or $j \geq |B| + 2$, and $s(v_j) = 2$ otherwise. The method will be to show that any colouring $c : V \to \{1, 2, \ldots, t\}$ can be obtained from the standard 2-colouring s in a finite number of steps.

First, suppose colour 1 is only used on vertices of A. In this case the colouring c can be transformed into s as follows. Recolour (if necessary) each vertex of A to colour 1 by recolouring from vertex v_n down to $v_{|B|+2}$, and then recolour each vertex of B to colour 2 by recolouring from vertex $v_{|B|+2}$ down to v_2. It is clear that at every stage there is a proper colouring.

If colour 1 is used on vertices in both parts then the number of colours used on B is at most $|B| \leq n/2$. Suppose exactly $r \leq n/2 < t$ colours (including colour 1) are used on vertices in B, and let x_i be the number of the first vertex to receive colour i, $i = 1, \ldots r$. That is $c(v_{x_i}) = i$ and for all $j < i$, $c(v_j) < i$. Clearly $x_1 = 1, x_2 = 2$, and since c is a canonical colouring $x_1 < x_2 < x_3 < \cdots < x_r$. Set $x_{r+1} = |B| + 2$.

We will use the x_i to define an intermediate colouring c' by $c'(v_j) = i$ if $x_i \leq j < x_{i+1} \leq n$, for $j = 1, 2, \ldots, n$. This is a proper colouring because no colour is used on vertices in both parts B and A. It uses $r + 1 \leq t$ colours in total. The colours are used in numerical order, so it is canonical.

The proof is completed by showing that the standard colouring s can be transformed to colouring c' and colouring c' can be transformed to colouring c. Since colouring c' does not use any colour on both parts, the standard colouring s can be transformed to c' by changing the colours on v_1 to v_n in order, if needed. That is change the colour on vertex v_m from $s(v_m)$ to the colour $c'(v_m)$ for $m = 1, \ldots, n$.

Next transform c' to c. Do this by passing through the vertices from v_1 to v_n r times, once for each of the r colours used in c. On the kth pass change vertices to colour k if they are colour k in c. That is, in pass k, step m we will change the colour of vertex v_m, only if $c(v_m) = k$. We need to show that this gives a proper canonical colouring at every step. Let s_{km} be the colouring obtained after the mth step in the kth pass. Then

$$s_{km}(v_j) = \begin{cases} c(v_j) & \text{if } c(v_j) < k, \text{ or if } c(v_j) = k \text{ and } j \leq m, \\ c'(v_j) & \text{otherwise.} \end{cases}$$

To see that each s_{km} is proper, we must show that, $\{v_j | s_{km}(v_j) = i\}$, the set of vertices coloured i, is independent for all colours $i = 1, 2, \ldots, r+1$ and all s_{km}. For $i < k$, the set of vertices coloured i in s_{km} equals the set of vertices coloured i in c. Thus $\{v_j | s_{km}(v_j) = i\}$ is an independent set for $i \leq k - 1$. For $i > k$, the set of vertices coloured i in s_{km} is a subset of the set of vertices coloured i in c' thus $\{v_j | s_{km}(v_j) = i\}$ is an independent set for $i \geq k + 1$. It remains to consider $\{v_j | s_{km}(v_j) = k\}$. The vertices coloured k under s_{km} are $\{v_j | s_{km}(v_j) = k\} = \{v_j | j \leq m, c(v_j) = k\} \cup \{v_j | j > m, c'(v_j) = k\}$.

When $k = 1$ then since $x_1 = 1$ and $x_2 = 2$, we get $\{v_j | s_{km}(v_j) = 1\} \subseteq \{v_j | c(v_j) = 1\}$, for all m so this is an independent set. At the other end, when $x_k \geq |B| + 2$ all vertices coloured k by either colouring c or c' will be in part A. So $\{v_j | s_{km}(v_j) = k\}$ is independent for all m.

If $2 \leq x_k \leq |B| + 1$ then c' only assigns colour k to vertices in part B. No vertex in part A is coloured k until the only vertices coloured k on part B are those coloured k under c. There are two cases.

- If $m \leq |B| + 1$ then all vertices coloured k in s_{km} are in B so the set is independent.
- If $m > |B| + 1$, this means that the only vertices in B that are still coloured k are coloured k under c, that is: $|B| \cap \{v_j | s_{km}(v_j) = k\} = |B| \cap \{v_j | c(v_j) = k\}$. No vertices in A are coloured k under c' so if $v_j \in A$ and $s_{km}(v_j) = k$, then $m > j$ and $s_{km}(v_j) = c(v_j) = k$. Thus $\{v_j | s_{km}(v_j) = k\} \subseteq \{v_j | c(v_j) = k\}$ which is independent.

Finally we show the colourings are canonical. By construction, for all colours, i, $c(v_{x_i}) = c'(v_{x_i}) = s_{km}(v_{x_i}) = i$, and no vertex before v_{x_i} is coloured $i + 1$ or higher in any of the colourings. Thus each of s_{km} is a canonical proper colouring. □

Consider the graph $L_n = K_{n,n} - F$, where F is a perfect matching. In the n-colouring of L_n where the opposite ends of edges in F are assigned the same colour, every vertex has a neighbour of any different colour. Thus, if c is the canonical version of this colouring with respect to a vertex ordering π, then c is an isolated vertex in $\mathrm{Can}_n^\pi(L_n)$. Since L_n has $2n$ vertices, it follows that the lower bound in the above theorem is the best possible.

By Corollary 7, there is always a vertex ordering π such that $\mathrm{Can}_k^\pi(K_{n_1, n_2, \ldots, n_r})$ is connected. We now show that there is no Hamilton cycle, and frequently no Hamilton path, in the canonical

k-colouring graph of a complete multipartite graph. By Corollary 4 it suffices to consider the case where $n_i \geq 2$ for all i. The specific example of $\mathrm{Can}_k^{\pi}(K_{2,2,\ldots,2})$ will be considered in detail in Section 5.

Proposition 4. *Let $G = K_{n_1,n_2,\ldots,n_r}$, where $n_i \geq 2$, for all i. Then, for all vertex orderings π and $k \geq r+1$,*

1. *$\mathrm{Can}_k^{\pi}(G)$ has a cut vertex and hence has no Hamilton cycle;*
2. *if $r \geq 3$ then $\mathrm{Can}_k^{\pi}(G)$ has no Hamilton path.*

Proof. We first prove statement 1. The colouring c where every vertex in the ith part gets colour i is a cut vertex. Note that no colour can be used on vertices in more than one part. Any colouring c_i where a vertex v_i in part i gets colour $r+1$ cannot change to a colouring c_j where a vertex v_j in part j gets coloured $r+1$ without first changing the colour of v_i. If the colour $r+1$ is removed from part i then no higher colour can be used without violating canonicity. So if there is a path from c_i to c_j, it must pass through c.

We now prove statement 2. If π does not start with a maximum clique, then $\mathrm{Can}_k^{\pi}(G)$ is disconnected by Proposition 1. Hence assume the first r vertices of π induce a maximum clique. The argument above shows that the cut vertex c actually partitions the colourings into r cells, corresponding to using the $r+1$ colour in each of the r independent sets. Thus there can be no Hamilton path if there are at least three independent sets with at least two vertices each. □

By Proposition 4, for $m, n \geq 2$ and $k \geq 3$, the graph $\mathrm{Can}_k^{\pi}(K_{m,n})$ has a cut vertex, and hence no Hamilton cycle. On the other hand, for $n \geq 2$, the graph $\mathrm{Can}_k^{\pi}(K_{1,n})$ has a Hamilton cycle for all $k \geq 3$ [1]. The possibility remains that the canonical k-colouring graphs of complete bipartite graphs which are not stars have a Hamilton path. We show next that $\mathrm{Can}_k^{\pi}(K_{n,m})$ in fact has a Hamilton path for all admissible values of m, n, k. To do so, we first give a Gray code (not cyclic) for $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ which has certain properties. The proof is recursive and similar to, but more elaborate than, that of Theorem 1.

Theorem 4. *For all $n \geq 2$ and $k \geq 2$, and any vertex ordering π, the graph $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ has a Hamilton path $x_1, x_2, \ldots, x_t$ such that:*

(i) *the colouring $x_1 = 11\ldots1$, and the colouring x_t uses all k colours.*
(ii) *For each $1 < i < t$, the set of colours used by x_i is identical to the set used by either x_{i-1}, x_{i+1}.*

Proof. The sequences 11 and 11, 12 clearly work for $\mathrm{Can}_1^{\pi}(\overline{K}_2)$ and $\mathrm{Can}_2^{\pi}(\overline{K}_2)$ respectively. We induct first on n and then on k. Note that because colourings are canonical we only consider $n \geq k$.

Let $c_1, c_2, \ldots, c_t$ be a Hamilton path in $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ with properties (i) and (ii). For $i = 1, 2, \ldots, t$, let $c_i \cdot \ell$ denote the extension of c_i to a canonical k-colouring of $\overline{K}_{n+1}$ in which the last vertex is assigned colour ℓ. Observe that $c_i \cdot \ell$ is adjacent to $c_{i+1} \cdot \ell$ whenever both of these are canonical colourings.

First the special case $k = 2$. For $n \geq k = 2$ a Hamilton path in $\mathrm{Can}_k^{\pi}(\overline{K}_{n+1})$ is constructed from the one for $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ as follows: $c_1 \cdot 1, c_1 \cdot 2, c_2 \cdot 2, c_2 \cdot 1, c_3 \cdot 1, c_3 \cdot 2, \ldots c_{2i} \cdot 2, c_{2i} \cdot 1, c_{2i+1} \cdot 1, c_{2i+1} \cdot 2 \cdots$.

For $n \geq k \geq 3$, a Hamilton path in $\mathrm{Can}_k^{\pi}(\overline{K}_{n+1})$ can be constructed from the one for $\mathrm{Can}_k^{\pi}(\overline{K}_n)$ as follows. The first vertices are $c_1 \cdot 1, c_1 \cdot 2, c_2 \cdot 2, c_2 \cdot 1, c_2 \cdot 3, c_3 \cdot 3, c_3 \cdot 2, c_3 \cdot 1$. Starting with $i = 4$, and then repeating for the next unused prefix c_i, suppose $c_i, c_{i+1}, \ldots, c_{i+j}$ is a maximal sequence such that each c_{i+m} uses exactly the same set of colours, and suppose the maximum allowable colour that can be added to each of them is ℓ_i. We construct a Hamilton path on the subgraph induced by $\{c_{i+m} \cdot \ell | m = 0, 1, \ldots, j; 1 \leq \ell \leq \ell_i\}$. These will be pieced together to get the Hamilton path for $\mathrm{Can}_k^{\pi}(\overline{K}_{n+1})$. This path must start with $c_i \cdot 1$ and end with $c_{i+j} \cdot 1$.

Suppose j is odd. Take everything from each prefix c_{i+m} before proceeding to the next prefix. In particular take the Hamilton path starting at $c_{i+2p} \cdot 1$ and ending with $c_{i+2p} \cdot \ell_i$ for $p = 0, 1, \ldots, j/2$ and in the reverse order $c_{i+2p+1} \cdot \ell_i$ and ending with $c_{i+2p+1} \cdot 1$ for $p = 0, 1, \ldots, j/2$.

Suppose j is even. Recall that for $1 \leq m \leq j$, the subgraph induced by $\{c_{i+m} \cdot \ell : 1 \leq \ell \leq \ell_i\}$ is complete, and by assumption $\ell_i \geq 3$. First use any Hamilton path through the subgraph induced by $\{c_i \cdot \ell | 1 \leq \ell \leq \ell_i\} \cup \{c_{i+1} \cdot \ell | 1 \leq \ell \leq \ell_i\}$ which starts at $c_i \cdot 1$, and ends at $c_{i+1} \cdot \ell_i$ and satisfies property (ii). Next, proceed as in the odd case alternating the direction of the Hamilton path, so that again the path through $\{c_i + m \cdot \ell | m = 0, 1, \ldots, j\}$ ends at $c_{i+j} \cdot 1$.

In either case, the set of colours used on $c_{i+m} \cdot \ell$ is identical for all m, ℓ except possibly when $\ell = \ell_i$, and the set of colours used on $c_{i+2p} \cdot \ell_i$ is identical to the set of colours used on $c_{i+2p+1} \cdot \ell_i$ and these colourings are adjacent in the path. That the last colouring in the path uses all k colours follows from the induction hypothesis.

The Hamilton path for $\mathrm{Can}^{\pi}_{n+1}(\overline{K}_{n+1})$ is obtained from the one of $\mathrm{Can}^{\pi}_{n}(\overline{K}_{n+1})$ by appending $c_t \cdot (n+1)$ to the last vertex in $\mathrm{Can}^{\pi}_{n}(\overline{K}_{n+1})$, which will be of the form $c_t \cdot \ell$. $\square$

The properties for the Hamilton paths required in the above proof are similar to those studied by various authors in the context of Gray codes for set partitions. In [19] the authors give Gray codes for the set of restricted growth functions, which is the set of non-negative integer sequences $\{a_1 a_2 \ldots a_n : a_{i+1} \leq \max\{a_1, a_2, \ldots, a_i\} + 1\}$. While these Gray codes start with $11\ldots1$ and end with $123\ldots n$, they do not have the property that at least two sequences in a row use the same set of integers (see for example Figure 5 in [19]). The set of bounded restricted growth functions is $R_b(n) = \{a_1 a_2 \ldots a_n : a_{i+1} \leq \max\{a_1, a_2, \ldots, a_i\} + 1 \text{ and } a_i \leq b\}$. Ruskey and Savage also considered Gray codes on $R_b(n)$, but restrict their attention to strict and weak Gray codes which have the further property that successive elements can differ by only 1 (if strict) or 2 (weak) in the one position in which they differ. They show that such codes cannot exist. In the Gray codes considered here, successive sequences can differ in only one position, but the elements can differ by any amount. In other words, Theorem 4 says there is a (non strict, non weak) Gray Code for the set of bounded restricted growth functions, $R_b(n)$, that satisfies properties (i) and (ii).

Theorem 5. *There exists a vertex ordering π such that $\mathrm{Can}^{\pi}_{k}(K_{n,m})$ has a Hamilton path for $n, m \geq 2$, $k \geq 3$.*

Proof. Let $K_{n,m}$ have bipartition (A, B), where $A = \{a_1, \ldots a_n\}$, $B = \{b_1, \ldots b_m\}$. Let the vertex ordering $\pi = a_1 b_1 a_2 a_3 \ldots a_n b_2 b_3 \ldots b_m$.

By Theorem 4 there is a Hamilton path, $x_1, x_2, \ldots x_t$, with properties (i) and (ii), in the canonical k-colouring graph of the subgraph induced by the restriction of π to its first $n+1$ vertices, $a_1 b_1 a_2 a_3 \ldots a_n$ (since $\{a_1, a_2, \ldots, a_n\}$ is an independent set, and b_1 will always be assigned the same colour in any canonical colouring). For each such colouring x_i, let $G(x_i)$ be the subgraph of $\mathrm{Can}^{\pi}_{k}(K_{n,m})$ consisting of the canonical colourings which are extensions of x_i. Note that each subgraph $G(x_i)$ is isomorphic to a graph $\mathrm{Can}^{\pi}_{k-r}(\overline{K}_m)$, corresponding to the colourings of the vertices $b_1 b_2 b_3 \ldots b_m$ in the $k-r$ colours not used on $a_1, a_2, a_3 \ldots a_n$ (starting with 2 which was the colour used on b_1), and also that $V(\mathrm{Can}^{\pi}_{k}(K_{n,m})) = \cup_{i \leq i \leq t} V(G(x_i))$.

The Hamilton path in $\mathrm{Can}^{\pi}_{k}(K_{n,m})$ will be constructed by piecing together Hamilton paths from the $G(x_i)$ in the order $i = 1, 2, \ldots, t$. In order to be able to piece these paths together, the first colouring in the Hamilton path of $G(x_{i+1})$ must be identical to the last colouring in the Hamilton path of $G(x_i)$. Note that if x_i and x_{i+1} use different colours then the only colouring that $G(x_i)$ and $G(x_{i+1})$ will have in common is $22\ldots2$. For each $G(x_i)$ there is a Hamilton path that satisfies the conditions of Theorem 4, in this case one end is $22\ldots2$ and the other uses all the colours of x_i.

Suppose that $x_i, x_{i+1}, \ldots x_{i+j}$ is a maximal sequence which use the same set of colours, and further that neither $i \neq 1$ nor $i + j \neq t$. The Hamilton path from $G(x_i)$ that is used must start with $22\ldots2$ and the one from $G(x_{i+j})$ must end with $22\ldots2$. If j is odd, this is accomplished by taking the Hamilton path starting at $22\ldots2$ for $G(x_{i+2p})$, for $p = 0, 1, \ldots, \lfloor j/2 \rfloor$, and ending with $22\ldots2$ for $G(x_{i+2p+1})$ for $p = 0, 1, \ldots, \lfloor j/2 \rfloor$. If j is even, then first use a Hamilton path through the subgraph induced by $V(G(x_i)) \cup V(G(x_{i+1}))$ which starts with $22\ldots2$ and ends in a colouring that uses all the colours.

Then proceed as in the even case alternating the direction of the Hamilton path, so that the Hamilton path through $G(x_{i+j})$ can end with $22\ldots2$.

We finish the argument by reiterating the conditions that must hold for the construction to succeed. The Hamilton path $x_1, x_2, \ldots, x_t$ in the subgraph induced by the canonical k-colourings of the first $n+1$ vertices of π needs the property that for each $i \neq 1$, the set of colours used for x_i is identical to the set used on either x_{i-1}, or x_{i+1}. In addition, for each x_i, there should be a Hamilton path in $G(x_i)$ that starts with $22\ldots2$. These are precisely the conditions guaranteed by our choice of the Hamilton path $x_1, x_2, \ldots, x_t$. $\square$

5. $\mathrm{Can}_k^{\pi}(T_{2n,n})$

For $n \geq 1$, let $T_{2n,n}$ be the complete n-partite graph on $2n$ vertices in which each independent set is size two. Then $T_{2,1} \cong \overline{K}_2$, $T_{4,2} \cong K_{2,2} \cong C_4$, $T_{6,3} \cong K_{2,2,2}$, and so on.

The purpose of this section is to study the canonical k-colouring graphs of $T_{2n,n}$. The results proved in this section are summarized in Theorem 6 below. In the cases where the canonical colour graph is connected, we describe it completely.

Theorem 6. *Let $n \geq 1$. Then*

1. $\mathrm{Can}_n^{\pi}(T_{2n,n}) \cong K_1$ *for any vertex ordering π.*
2. *If $k \geq 2n$, then $\mathrm{Can}_k^{\pi}(T_{2n,n}) \cong \mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ for any vertex ordering π.*
3. *If $n < k$ and the subgraph of $T_{2n,n}$ induced by the first n vertices in the vertex ordering π is not complete, then $\mathrm{Can}_k^{\pi}(T_{2n,n})$ is disconnected.*
4. *If $n < k$ and the subgraph of $T_{2n,n}$ induced by the first n vertices in the vertex ordering π is complete, then $\mathrm{Can}_k^{\pi}(T_{2n,n})$ is a tree. Further, if $\mathrm{Can}_k^{\pi}(T_{2n,n})$ and $\mathrm{Can}_k^{\phi}(T_{2n,n})$ are both trees, then $\mathrm{Can}_k^{\pi}(T_{2n,n}) \cong \mathrm{Can}_k^{\phi}(T_{2n,n})$.*
5. $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ *never has a Hamilton cycle and has a Hamilton path only when $n = 2$, $k = 2$.*

Statements 1 and 2 are clear. Statement 3 is immediate by Proposition 1. The proof of statement 4 is partitioned into a sequence of propositions. First, we consider the graphs $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$, for vertex orderings π that start with a maximal clique. The graphs $\mathrm{Can}_k^{\pi}(T_{2n,n})$, with $n < k < 2n$, will be considered later. According to statement 2 we need not consider the situations in which $k > 2n$.

Proposition 5. *Let $n \geq 1$. If the subgraph of $T_{2n,n}$ induced by the first n vertices in the sequence π is complete, then $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ is a tree on 2^n vertices. Further, if the subgraph of $T_{2n,n}$ induced by the first n vertices in the sequence ϕ is complete, then $\mathrm{Can}_{2n}^{\pi}(T_{2n,n}) \cong \mathrm{Can}_{2n}^{\phi}(T_{2n,n})$.*

Proof. In any colouring of $T_{2n,n}$, a pair of independent vertices either has the same colour, or different colours. In the latter case, each vertex in the pair is the only vertex to be assigned that colour. Suppose that the last n vertices of π are $x_1, x_2, \ldots, x_n$. A canonical $2n$-colouring with respect to π can be encoded as a binary sequence $b_1 b_2 \ldots b_n$ of length n in which the i-th element is 0 if vertex x_i is assigned the same colour as its unique non-neighbour (which is one of the first n vertices of π), and 1 if it is assigned the first colour not used on a vertex earlier in the sequence. Thus, $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ has precisely 2^n vertices.

We claim that an element b_i of the binary sequence can change (from 0 to 1, or 1 to 0) if and only if $b_j = 0$ for all $j > i$. Suppose that x_i is the only vertex of its colour, that is, it has a different colour than its unique non-neighbour, w, and $b_i = 1$. If there exists $j > i$ such that x_j also has a different colour than its unique non-neighbour, then the colouring arising from assigning the colour of w to x_i is not canonical (because the colour of x_i, which is smaller than the colour of x_j, would not be used on any vertex). Similarly, if x_i has the same colour as its unique non-neighbour, then it can only be assigned a different colour if there is no $j > i$ such that x_j has a different colour than its unique non-neighbour. This proves the claim.

We now show that, for any such sequence π, the graph $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ is a tree. According to the discussion above, the vertices of $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ can be taken to be the binary sequences of length n, with two sequences being adjacent if and only if they differ in exactly one position, and all entries to the right of that position are zero. Since any binary sequence can be reached from $00\dots0$ by introducing 1s from left to right, the graph $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ is connected. The proof is complete once we show that the sum of the vertex degrees equals $2(2^n-1)$. The degree of $00\dots0$ is n. Any other binary sequence contains at least one 1. If the rightmost 1 is in position i then the degree of $b_1b_2\cdots b_n$ is $n-i+1$ and the number of such sequences is 2^{i-1}. Hence, the sum of the vertex degrees is

$$\begin{aligned} n+\textstyle\sum_{i=1}^{n} 2^{i-1}(n-i+1) &= n+(n+1)\textstyle\sum_{i=1}^{n} 2^{i-1}-\sum_{i=1}^{n} 2^{i-1}i \\ &= n+(n+1)(2^n-1)-((n+1)2^n-(2^{n+1}-1)) \\ &= 2\cdot 2^n-2. \end{aligned}$$

Since the description of $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ uses no properties of π other than that the subgraph of $T_{2n,n}$ induced by the first n vertices of π is complete, it is clear that any two trees arising from such sequences are isomorphic. This can also be proved by induction on n by using the observation that the subtree induced by the set of sequences in which the first entry is 0 is isomorphic to $\mathrm{Can}_{2(n-1)}^{\pi}(T_{2(n-1),n-1})$, as is the subtree induced by the set of sequences in which the first entry is 1. $\square$

The argument above shows that, for $n>1$, the leaves of $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ correspond to precisely the binary sequences in which $b_n=1$. Thus, $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ has exactly $2^{n-1}\geq 2$ leaves, and hence never has a Hamilton cycle. There is a Hamilton path only when $n\leq 2$ (recall that $T_{2,1}\cong \overline{K}_2$, and $T_{4,2}\cong K_{2,2}\cong C_4$).

For an ordering π such that the subgraph induced by the first n vertices is complete, the tree $\mathrm{Can}_{6}^{\pi}(T_{6,3})$ is shown in Figure 4. For any such ordering, the tree $\mathrm{Can}_{8}^{\pi}(T_{8,4})$ is constructed from two copies of this tree, one arising from concatenating a 1 on the left of each sequence and the other arising from concatenating a 0 on the left of each sequence, and then joining the vertices 0000 and 1000.

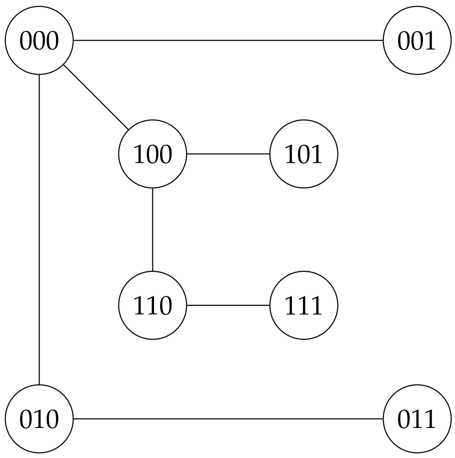


Figure 4. The tree $\mathrm{Can}_{6}^{\pi}(T_{6,3})$

It remains to consider the graphs $\mathrm{Can}_{k}^{\pi}(T_{2n,n})$ for $n<k<2n$ and sequences π for which the first n vertices is complete.

Proposition 6. *Let $n\geq 1$ and $n<k<2n$. If the subgraph of $T_{2n,n}$ induced by the first n vertices in the sequence π is complete, then $\mathrm{Can}_{k}^{\pi}(T_{2n,n})$ is a tree on $\binom{n-1}{t}+\binom{n-1}{t-1}+\cdots+\binom{n-1}{0}$ vertices. Further, if the subgraph of $T_{2n,n}$ induced by the first n vertices in the sequence ϕ is complete, then $\mathrm{Can}_{k}^{\pi}(T_{2n,n})\cong \mathrm{Can}_{2n}^{\phi}(T_{2n,n})$.*

Proof. Observe that $\mathrm{Can}_k^{\pi}(T_{2n,n})$ is the subgraph of $\mathrm{Can}_{2n}^{\pi}(T_{2n,n})$ induced by the sequences with at most $t = k - n$ ones. There are $\nu = \binom{n-1}{k} + \binom{n-1}{k-1} + \cdots + \binom{n-1}{0}$ such sequences. Hence $\mathrm{Can}_k^{\pi}(T_{2n,n})$ has exactly ν vertices.

As before, since any binary sequence with at most t ones can be reached from $00\ldots0$ by introducing ones from left to right, the graph $\mathrm{Can}_k^{\pi}(T_{2n,n})$ is connected, and therefore is a tree. In addition, as before, the description of $\mathrm{Can}_k^{\pi}(T_{2n,n})$ uses no properties of π other than that of the subgraph of $T_{2n,n}$ induced by the first n vertices of π is complete. Thus, once again it is clear that any two trees arising from such sequences are isomorphic. □

For $n > 1$ and $n < k \leq 2n$, the leaves of the tree $\mathrm{Can}_k^{\pi}(T_{2n,n})$ are the binary sequences with exactly k ones and a zero in the last position, or with at most k ones and a one in the last position. Hence there cannot be a Hamilton cycle, and there is a Hamilton path only when $n = 2$ and $k = 2$.

The proof of Theorem 6 is now complete.

6. Conclusions

In this paper we have continued the study of reconfiguration of canonical colourings. Our main results are that for all bipartite graphs and complete multipartite graphs there exists a vertex ordering π such that $\mathrm{Can}_k^{\pi}(G)$ is connected for large enough values of k. In addition, we have shown that a canonical colouring graph of a complete multipartite graph usually does not have a Hamilton cycle, but that there exists a vertex ordering π such that $\mathrm{Can}_k^{\pi}(K_{m,n})$ has a Hamilton path for all $k \geq 3$. The paper also gave a detailed consideration of $\mathrm{Can}_k^{\pi}(K_{2,2,\ldots,2})$. For each $k \geq \chi$ and all vertex orderings π, $\mathrm{Can}_k^{\pi}(K_{2,2,\ldots,2})$ is either disconnected or isomorphic to a particular tree.

Furthermore, the technical nature of these results leads us to believe that additional results about reconfiguration of canonical colourings will require significant effort. In addition, we posit that unlike for the k-colouring graph or the Bell k-colouring graph, there will be no criteria that ensure connectivity for all base graphs.

Acknowledgments: Research of the first author supported by Simons Foundation Award #281291. Research of the second author is supported by NSERC.

Author Contributions: The authors contributed equally to this work.

Conflicts of Interest: The authors declare no conflict of interest.

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