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Inapproximability of Rank, Clique, Boolean, and Maximum Induced Matching-Widths under Small Set Expansion Hypothesis

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Abstract: Wu et al. (2014) showed that under the small set expansion hypothesis (SSEH) there is no polynomial time approximation algorithm with any constant approximation factor for several graph width parameters, including tree-width, path-width, and cut-width (Wu et al. 2014). In this paper, we extend this line of research by exploring other graph width parameters: We obtain similar approximation hardness results under the SSEH for rank-width and maximum induced matching-width, while at the same time we show the approximation hardness of carving-width, clique-width, NLC-width, and boolean-width. We also give a simpler proof of the approximation hardness of tree-width, path-width, and cut-widththan that of Wu et al.

Keywords: graph width parameter; inapproximability; small set expansion hypothesis

1. Introduction

There are many graph width parameters, such as cut, path, tree, band, branch, carving, clique, NLC, rank, boolean, maximum induced matching-widths, and the approximability and inapproximability of some of these width parameters have been investigated extensively. For example, regarding the approximability of tree-width tw, it is known that there are polynomial time approximation algorithms with ratio $O(\sqrt{\log tw(G)})$ [1]. Regarding inapproximability, tree-width cannot be approximated within any additive constant c unless P = NP [2]. Regarding rank-width rw, for every *fixed* k, there is a polynomial time algorithm that reports $rw(G) > k$, or outputs a rank decomposition of width at most $3k - 1$ [3]. Recently, it has been shown that maximum induced matching-width cannot be approximated within any constant factor in polynomial time unless NP = ZPP [4]. For several graph parameters, there are still large gaps between approximability and inapproximability results: it is a major concern as to whether there are constant factor approximation algorithms for those graph width parameters. Indeed, it is a long-standing open problem as to whether tree-width can be approximated within a constant factor.

Raghavendra and Steurer introduced a complexity assumption referred to as the small set expansion hypothesis (SSEH) that is deeply related to the unique games conjecture (UGC) [5], and since then several inapproximability results under SSEH have been reported. For example, in [6] Raghavendra et al. showed that under SSEH there are no constant factor approximation algorithms for the balanced separator and minimum linear arrangement problems (a similar result was already known for the balanced separator problem under UGC [7]). Recently, Manurangsi showed inapproximability results for maximum biclique problems, minimum k-cut, and densest at-least-k-subgraph [8]. In [9], Wu et al. (2014) showed under SSEH that there are no constant factor approximation algorithms for cut, path, tree-widths, minimum fill-in (it has recently been shown that minimum fill-in has no polynomial time approximation scheme unless P = NP and that assuming ETH, there is some positive ε such that no algorithm can find a $(1+\varepsilon)$ approximation in time $2^{O(n^{1-\delta})}$ for any positive constant

δ [10]), one-shot black pebbling costs, and other problems. Those were the first results showing the hardness of constant factor approximation for these graph parameters. However, the hardness result of tree-width in [9] does not necessarily mean that the long-standing open problem of tree-width is solved, because there is no consensus on the correctness of UGC and SSEH at this time [11] and it was shown in [12] that both unique games and small set expansion admit a subexponential time approximation algorithm. Regardless of the veracity of these two assumptions, the results in [6,9] stimulate the study of approximation hardness for graph parameters, and it is widely acknowledged that UGC and SSEH have played important roles in the study of approximation algorithms.

The above width parameters have widespread applications from both theoretical and practical viewpoints (e.g., [13]). Efficiently computing these width parameters becomes especially relevant when considering the fact that many NP-hard problems admit efficient graph algorithms for instances whose width parameters have a small value. The reader is invited to refer to the literature on *fixed parameter tractability* for further information (e.g., Downey and Fellow [14], Cygan et al. [15], Flum and Grohe [16]). Inspired by [9], in this paper, we extend the research in [9] to other graph parameters. That is, we demonstrate in a unified manner that under SSEH there are no constant factor polynomial time approximation algorithms for cut, path, tree, branch, carving, NLC, rank, clique, boolean, and maximum induced matching-widths (see Figure 1 in Section 4).

2. Definitions and Known Results

2.1. Graphs, Expansion, Matrices

In this subsection, we recall some definitions and notation of graphs, expansion, and matrices. Let G be a simple graph. We use operators V and E to refer to the vertex and edge sets of G as $V(G)$ and $E(G)$, respectively. We will frequently write $G = (V, E)$ instead of $G = (V(G), E(G))$. For a subset X of $V(G)$ ($E(G)$, resp.), $\overline{X}$ denotes $V(G)\backslash X$ ($E(G)\backslash X$, resp.). Let X and Y be subsets of V. $N(X)$ denotes the neighbors of X (i.e., $N(X) = \{u \in \overline{X} \mid \exists v \in X \text{ s.t. } \{u, v\} \in E\}$). $E(X, Y)$ denotes $\{\{u, v\} \in E : u \in X, v \in Y\}$. For each $v \in V$, $d_G(v)$ (or simply $d(v)$) denotes the degree of v in G. $\Delta(G)$ denotes the maximum degree of G. $G[X]$ denotes the induced subgraph of G induced by X. The *vertex boundary width* of order i of G, denoted by $b_v(i, G)$, is defined as $b_v(i, G) := \min_{S \subseteq V, |S| = i} |N(S)|$. Similarly, the *edge boundary width* of order i, denoted by $b_e(i, G)$, is defined as $b_e(i, G) := \min_{S \subseteq V, |S| = i} |E(S, \overline{S})|$.

Let S be a subset of V. The *edge expansion* $\Phi(S)$ is defined as $\Phi_G(S) = \frac{|E(S,\overline{S})|}{\min\{\mathsf{vol}(S), \mathsf{vol}(\overline{S})\}}$, where $\mathsf{vol}(S) = \sum_{v \in S} d(v)$. Moreover, $\mu(S)$ denotes $\frac{\mathsf{vol}(S)}{\mathsf{vol}(V)}$. Since, in this paper, we mainly consider d-regular graphs, $\mathsf{vol}(S)$ and $\mu(S)$ can be regarded as $d|S|$ and $\frac{|S|}{|V|}$, respectively.

Let M be the adjacency matrix of G. For $X, Y \subseteq V$ such that $X \cap Y = \emptyset$, $M[X, Y]$ means a matrix satisfying the following. The rows and columns are labeled by X and Y, respectively, and each entry (x, y) with $x \in X$ and $y \in Y$ is 1 if $\{x, y\} \in E(G)$, 0 otherwise. We denote the rank over $GF(2)$ of $M[X, Y]$ as $rank(M[X, Y])$. For the rank over $GF(2)$, the following is known.

Lemma 1 (Lemma 4.3 in [17])**.** *Let A be a matrix over $GF(2)$ such that A has at least p non-zero entries and each row and each column in A has at most q non-zero entries. Then, $rank(A) \geq \frac{p}{q^2}$ holds.*

2.2. Graph Width Parameters

In this subsection, we briefly review the definitions of graph width parameters. We only give the definitions of graph width parameters that will be needed in our proofs. Definitions of the other graph width parameters can be found as follows: For the definitions of path-width, tree-width, branch-width, and band-width, see e.g., [18]. For the definitions of carving-width, clique-width, and boolean-width, see e.g., [19–21], respectively. Note that the decision problems related to graph width parameters considered in this paper are all minimization problems.

As some of graph width parameters are based on decomposition trees, let us first review the notion of a decomposition tree. Given a tree T, let us denote the set of leaves of T by $L(T)$. For an edge e in T, $(T\backslash e)_1$ and $(T\backslash e)_2$ denote the two subtrees obtained from T by removing e. Given a graph $G = (V, E)$ and a tree T such that $|L(T)| = |V|$, let f_T be a bijection from $L(T)$ to V. For each edge e in T, we denote the subset of V mapped by f_T from the leaves in $(T\backslash e)_i$ as $(V\backslash e)_i$ for $i \in \{1, 2\}$. That is, $(V\backslash e)_i = \{f_T(v) : v \in L((T\backslash e)_i)\}$. Note that $(V\backslash e)_i$ obviously depends on T and f_T. In this paper, we will refer to a pair (T, f_T) satisfying the following conditions as a *decomposition tree* of $G = (V, E)$:

- T is a subcubic tree with $|V|$ leaves, where a tree is *subcubic* if every vertex in T has degree 1 or 3;
- f_T is a bijection from $L(T)$ of T to V.

We denote the set of tree decompositions of G by $\mathscr{D}_G$.

Cut-width: For a graph $G = (V, E)$, let $\pi : V \to \{1, \ldots, |V|\}$ be an ordering of V. Let $S_\pi(i) := \{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(i)\}$, $\partial_e(\pi, i) := E(S_\pi(i), \overline{S_\pi(i)})$. Then, $cutw(G) := \min_{\pi} \max_{1 \le i \le n} |\partial_e(\pi, i)|$.

Rank-width: For a graph $G = (V, E)$, let T be a subcubic tree and f_T be a bijection from $L(T)$ to V. Then, $rw(G) := \min_{(T, f_T) \in \mathscr{D}_G} \max_{e \in E(T)} rank\big(M\big[(V\backslash e)_1, (V\backslash e)_2\big]\big)$.

Maximum induced matching-width For a graph $G = (V, E)$ and a subset A of V, we denote the size of a maximum induced matching in the bipartite graph $(A, \overline{A}; E(A, \overline{A}))$ by $mim(A)$. Then, $mimw(G) := \min_{(T, f_T) \in \mathscr{D}_G} \max_{e \in E(T)} mim((V \setminus e)_1))$.

3. SSE Hypothesis

In this section, we briefly review the small set expansion hypothesis (SSEH), which is deeply related to the unique games conjecture (UGC). Research on the unique games conjecture and semidefinite programming has led to significant progress in the field of approximation algorithms in the past decade. In [9], Wu et al. provided the following very useful strong form of SSEH.

SSE hypothesis (strong form) [Conjecture 2.23 and Remark 2.25 in [9]] There is a constant c such that for every integer $q > 1$ and arbitrarily small $\varepsilon > 0$, the following problem is NP-hard:

Problem 1. *Given a regular graph $G = (V, E)$, distinguish between the following two cases:*

Yes *There exist q disjoint sets $S_1, \ldots, S_q \subseteq V$ such that $\Phi_G(S_i) \le 2\varepsilon$ and $|S_i| = \frac{|V|}{q}$ holds for all $1 \le i \le q$,*
No *For every $\frac{|V|}{10} \le |S| \le \frac{9|V|}{10}$, $\Phi_G(S) \ge c\sqrt{\varepsilon}$ holds.*

4. Method for Showing Inapproximability

In this section, we explain the useful method which is used implicitly in [9] to prove the inapproximability of width parameters in a unified setting. Through the remainder of this section, gw denotes a graph width parameter such that determining gw is a minimization problem and P denotes any polynomial time computable parameter of G, typically $|V|$ or $|E|$. That is, the graph parameters gw and P are functions from graphs to the natural numbers such that isomorphic graphs are mapped to the same number.

To show the approximation hardness of gw, it is sufficient to prove that there are constants c_Y and c_N such that for any graph G,

- $gw(G) \leq c_Y \varepsilon P$ holds if G is a YES instance in Problem 1 (i.e., completeness), and
- $gw(G) \geq c_N \sqrt{\varepsilon} P$ holds if G is a NO instance in Problem 1 (i.e., soundness),

where ε is the same as in Problem 1. We will refer to such $c_Y \varepsilon P$ ($c_N \sqrt{\varepsilon} P$, resp.) as *upper threshold* (*lower threshold*, resp.).

Suppose that $gw(G)$ can be approximated within a constant factor in polynomial time. That is, there is a constant $\rho > 0$ for which there is an approximation algorithm A such that for any graph G, A outputs a value $A(G)$ satisfying $gw(G) \leq A(G) \leq \rho \cdot gw(G)$. Then, take ε such that $\frac{1}{\rho} > \frac{c_Y \sqrt{\varepsilon}}{c_N}$. That is, $\varepsilon < \left(\frac{c_N}{c_Y} \cdot \frac{1}{\rho} \right)^2$. Then, the following Algorithm 1, which uses the approximation algorithm A as a subroutine, solves Problem 1.

Algorithm 1 DeciInstByAlg$A(G)$

Input: a graph G
Output: YES/NO

```
compute an approximate solution A(G) of gw(G);
if A(G) < c_N √ε P then
    output "G is a YES instance in Problem 1"
else
    output "G is a NO instance in Problem 1"
end
```

The correctness of Algorithm 1 can be explained as follows. In the case of "$A(G) < c_N \sqrt{\varepsilon} P$", from $gw(G) \leq A(G) < c_N \sqrt{\varepsilon} P$, we can guarantee that G is not a NO instance by the soundness condition. In the case of "$A(G) \geq c_N \sqrt{\varepsilon} P$", from $gw(G) \geq \frac{A(G)}{\rho} \geq \frac{c_N \sqrt{\varepsilon}}{\rho} P > c_N \sqrt{\varepsilon} \cdot \frac{c_Y \sqrt{\varepsilon}}{c_N} P \geq c_Y \varepsilon P$, we can conclude that G is not a YES instance by the completeness condition.

It is worth mentioning that for graph width parameters gw_i ($1 \leq i \leq 3$) such that $gw_1(G) \preceq gw_2(G) \preceq gw_3(G)$ for any graph G, the approximation hardness of gw_i ($1 \leq i \leq 3$) can be shown by just showing both the completeness for gw_3 and soundness for gw_1, where $gw_i(G) \preceq gw_{i+1}(G)$ means that there exists a constant c such that $gw_i(G) \leq c \cdot gw_{i+1}(G)$ for any G. Figure 1 illustrates the unified setting: two width parameters gw_u and gw_l such that gw_u is arranged above gw_l are linked by a line if $gw_l \preceq gw_u$ holds. In the figure, $lclw$ means *linear clique-width*, and the left (right, resp.) side illustrates a scheme of proofs for the inapproximability in [9] (this paper, resp.). The relations among width parameters illustrated in Figure 1 can be confirmed from the inequalities in Appendix A.

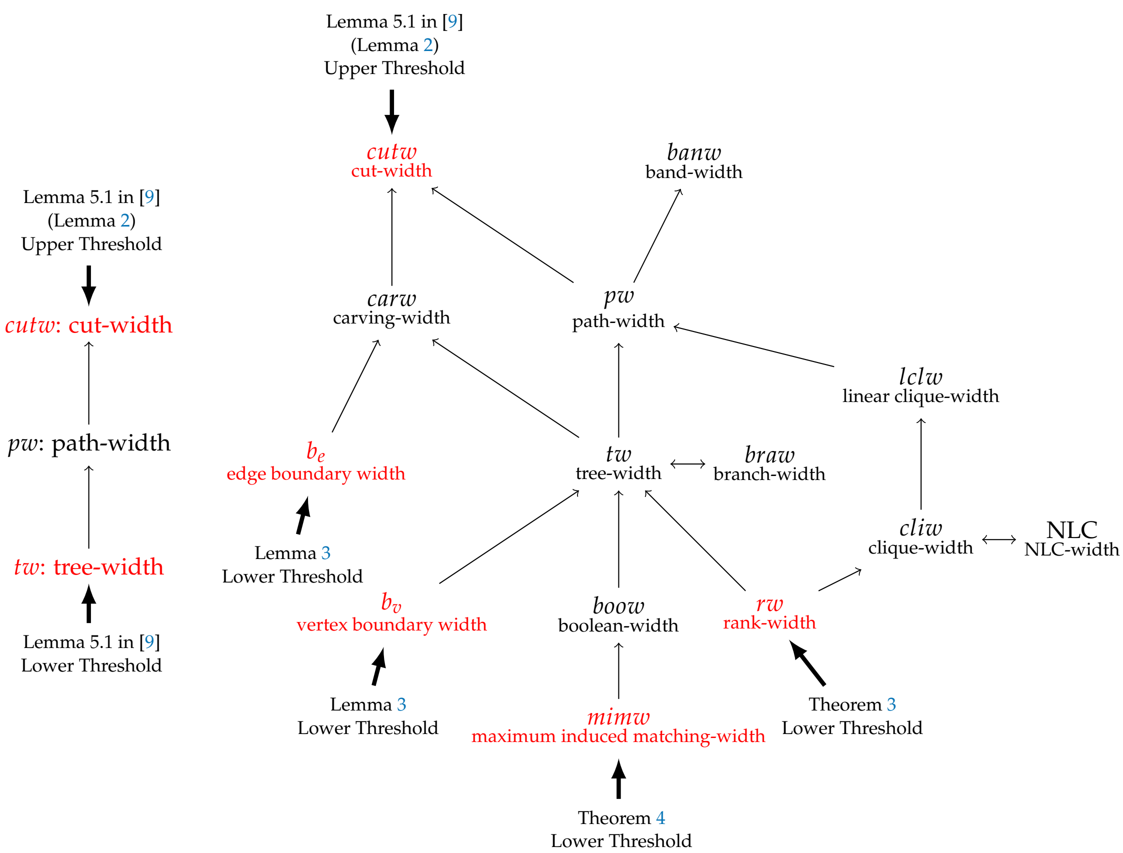

Figure 1. Scheme showing how to prove the inapproximability. ($gw_1 \to gw_2$ means that $gw_1 \preceq gw_2$).

5. Hardness Results Derived from Inapproximability of Tree-Width

In this section, we exhibit approximation hardness results for several graph width parameters which can be derived from the approximation hardness of tree-width together with known results.

Theorem 1. *Assume that tree-width cannot be approximated within any constant factor in polynomial time. Then, {branch, carving, clique}-widths cannot be approximated within any constant factor either.*

Proof. The approximation hardness of branch-width follows from the fact that $braw(G) - 1 \leq tw(G) \leq \lfloor \frac{3}{2} \times braw(G) \rfloor - 1$ [18].

For carving-width, it is known that from a graph G we can construct a graph G' in polynomial time such that $tw(G) \leq tw(G') \leq tw(G) + 1$ and $\Delta(G') \leq 3$ [22]. It is also known that $\frac{carw(G)}{\Delta(G)} \leq tw(G) \leq 2 \times carw(G)$ [23–25]. By combining both results, we can conclude that if carving-width can be approximated within a constant factor, then tree-width can be approximated within a constant factor as well.

The hardness of clique-width can be shown by the fact that $\frac{tw(G)+1}{4} \leq cliw(L(G)) \leq 2tw(G) + 2$ [26], where $L(G)$ means the line graph of G. □

6. Results

6.1. Simpler Proof of the Inapproximability of {Cut, Path, Tree}-Widths

As mentioned in Section 1, in [9], Wu et al. demonstrated that under SSEH there are no constant factor approximation algorithms for cut-width, path-width, and tree-width. In this subsection, we show the same result in a different way from that of Wu et al. The difference is as follows. Recall first the relations $cutw(G) \geq pw(G) \geq tw(G)$. To show the approximation, hardness, completeness, and soundness should be proved (see Section 4). To prove completeness, Wu et al. gave an upper threshold of cut-width, and we just use their upper threshold, so there is no difference in this part. To prove soundness, they gave a lower threshold of tree-width. To obtain the lower threshold, they used the lower bound of "1/2-vertex separator", which is a well-known lower bound of tree-width. Instead of the 1/2-vertex separator, we employ the vertex boundary width, which is also known as a lower bound of tree-width. The most obvious difference is that they used an auxiliary graph G' produced from an input graph G to show a lower bound of 1/2-vertex separator of G', while in our case, we do not need an auxiliary graph. In this sense, our proof is simpler than that of Wu et al. (2014). To be self-contained, we review the proof of the completeness of Theorem 4.1 in [9].

Lemma 2 (Completeness of Theorem 4.1 in [9]). *Let $q = \frac{1}{\varepsilon}$. Let $G = (V, E)$ be d-regular and a YES instance in Problem 1, where d is an universal constant. Then, $cutw(G) \leq c_c \varepsilon |E|$ holds for some universal constant c_c.*

Proof. Since G is a YES instance, we have

$$|E(S_i, V \backslash S_i)| = \Phi_G(S_i) \times d|S_i| = \Phi_G(S_i) \times d\frac{|V|}{q} = \Phi_G(S_i) \times \frac{2|E|}{q} \leq \frac{4\varepsilon|E|}{q}.$$

Hence, the number of edges whose endpoints do not belong to the same partition S_i is upper-bounded as follows:

$$\frac{1}{2}\sum_{i=1}^{q} |E(S_i, V \backslash S_i)| \leq \frac{1}{2}\sum_{i=1}^{q} \frac{4\varepsilon|E|}{q} \leq 2\varepsilon|E|.$$

From $\frac{1}{q} = \varepsilon$, we have $|S_i| = \varepsilon|V|$. Hence, $|E(S_i, S_i)|$ is at most $\frac{d|S_i|}{2} = \frac{d\varepsilon|V|}{2} = \varepsilon|E|$. Thus, by considering an ordering in which u comes before v for any vertices of $u \in S_i$ and $v \in S_j$ with $i < j$, we have $cutw(G) \leq 3\varepsilon|E|$ (i.e., $c_c = 3$). □

We now show the soundness in Lemma 3.

Lemma 3. *Let $G = (V, E)$ be a NO instance stated in Problem 1. Then, for all $\frac{|V|}{4} \leq i \leq \frac{|V|}{2}$,*

- $\min_i b_e(i, G) \geq \frac{c}{2}\sqrt{\varepsilon}|E|$, *and*
- $\min_i b_v(i, G) \geq \frac{c}{2d}\sqrt{\epsilon}|E|$

hold, where c is the constant in the NO instance in Problem 1.

Proof. For each $\frac{|V|}{4} \leq i \leq \frac{|V|}{2}$, let S^i be a set such that $b_e(i, G) = |E(S^i, \overline{S^i})|$ and $|S^i| = i$. Then, for each i, we have $b_e(i, G) = |E(S^i, \overline{S^i})| = \Phi_G(S^i) \times d|S^i| \geq c\sqrt{\epsilon} \times d \times i \geq c\sqrt{\epsilon} \times d \times \frac{|V|}{4} = \frac{c}{2}\sqrt{\epsilon} \times |E|$.

Let S^i be a set such that $b_v(i, G) = |N(S^i)|$ and $|S^i| = i$. From the above and the fact that $|E(S, \overline{S})| \leq |N(S)| \times d$, we have

$$\begin{aligned} \min_{\frac{|V|}{4} \leq i \leq \frac{|V|}{2}} b_v(i, G) &= \min_{\frac{|V|}{4} \leq i \leq \frac{|V|}{2}} |N(S^i)| \geq \min_{\frac{|V|}{4} \leq i \leq \frac{|V|}{2}} \frac{|E(S^i, \overline{S^i})|}{d} \\ &\geq \frac{1}{d} \times \min_{\frac{|V|}{4} \leq i \leq \frac{|V|}{2}} |E(S^i, \overline{S^i})| \geq \frac{1}{d} \times \frac{c}{2}\sqrt{\epsilon}|E|. \end{aligned}$$

□

Theorem 2. *Under SSEH, it is NP-hard to approximate $\{cutw, pw, tw, carw\}$-widths of a graph to within a constant factor in polynomial time.*

Proof. Combining Lemmas 2 and 3 and the relations: for each $1 \le j \le |V|$,

- $\min_{j/2 \le i \le j} b_v(i, G) - 1 \le tw(G) \le pw(G) \le \min\{cutw(G), banw(G)\}$ [27],
- $\min_{j/2 \le i \le j} b_v(i, G) \le \min_{j/2 \le i \le j} b_e(i, G) \le carw(G) \le cutw(G)$ [28],

we have the theorem. □

6.2. Inapproximability of Rank, Clique, Boolean, and Mim-Widths

Theorem 3. *Under SSEH, it is NP-hard to approximate rank and clique-widths of a graph to within a constant factor in polynomial time.*

Proof. Let $G = (V, E)$ be a YES instance stated in Problem 1, and M an adjacency matrix of G. Then, from the fact that $rw(G) \le tw(G) + 1 \le pw(G) + 1$ [29] and Lemma 2, we have $rw(G) \le 2 \times pw(G) \le c \times \varepsilon|E|$ for some constant c, from which follows the completeness.

Now, let $G = (V, E)$ be a NO instance stated in Problem 1. Hence, for any $S \subseteq V$ such that $\frac{|V|}{10} \le |S| \le \frac{9|V|}{10}$, $\Phi_G(S) \ge c\sqrt{\varepsilon}$ holds, where c is the constant in the NO instance in Problem 1. To prove the soundness, we will show that $rw(G) \ge c'\sqrt{\epsilon}|E|$ for some constant c'. Let (T, f_T) be an optimal decomposition tree of G, (i.e., $\max_{e \in E(T)} rank(M[(V \backslash e)_1, (V \backslash e)_2]) = rw(G)$). Since T is a subcubic tree, T has an edge e such that $\min\{|(V \backslash e)_1|, |(V \backslash e)_2|\} \ge \frac{|V|-1}{3}$. For graphs with $|V| \ge 10$, we have $\frac{|V|-1}{3} \ge 0.9 \cdot \frac{|V|}{3}$. Denote $(V \backslash e)_1$ by A and $(V \backslash e)_2$ by B. As $rw(G) \ge rank(M[A, B])$, it is sufficient to show that $rank(M[A, B]) \ge c'\sqrt{\epsilon}|E|$ for some c'.

Since $\min\{\mathrm{vol}(A), \mathrm{vol}(B)\} = d \times \min\{|A|, |B|\} \ge d \times \frac{|V|-1}{3} \ge 0.9 \times d \times \frac{|V|}{3}$ holds. Thus, $|E(A, \overline{A})| = \min\{\mathrm{vol}(A), \mathrm{vol}(B)\} \times \Phi_G(A) \ge 0.9d\frac{|V|}{3} \times c\sqrt{\varepsilon} = 0.9 \times \frac{2|E|}{3} \times c\sqrt{\varepsilon}$. Since the number of nonzero elements (i.e., 1's) of M equals $|E(A, B)|$, the number of nonzero elements is at least $c\sqrt{\varepsilon} \times 0.9 \times \frac{2|E|}{3}$. Meanwhile, the number of nonzero elements in each row and/or column of M is at most d, because G is d-regular. Hence, by Lemma 1, we have $rank(M[A, B]) \ge \frac{2c}{3d^2} \times 0.9 \times \sqrt{\epsilon}|E|$ as desired (i.e., $c' = \frac{2c}{3d^2} \times 0.9$).

The approximation hardness of clique-width follows from the relations $cliw(G) \le pw(G) + 2$ [30] and $rw(G) \le cliw(G)$ [31] (see also inequalities (A4) and (A3) in Appendix A). As stated in Theorem 1, the hardness can also be obtained from the approximation hardness of tree-width. □

Theorem 4. *Under SSEH, it is NP-hard to approximate mim and boolean-widths of a graph to within a constant factor in polynomial time.*

Proof. The proof is quite similar to that of Theorem 3. It is known that $mimw(G) \le boow(G)$ [32] and $boow(G) \le tw(G) + 1$ [33] (see also inequalities (A6) and (A7) in Appendix A). From the relations, it is sufficient to show that, given a NO instance $G = (V, E)$, $mimw(G) \ge c'\sqrt{\epsilon}|E|$ for some constant c'. Let (T, f_T) be an optimal decomposition tree of G, and e be an edge such that $\min\{|(V \backslash e)_1|, |(V \backslash e)_2|\} \ge \frac{|V|-1}{3}$. Note that since T is a subcubic tree, there exists such edge e. Without loss of generality, we may assume that $|(V \backslash e)_1| \le |(V \backslash e)_2|$. Let A denote $(V \backslash e)_1$. As $\frac{|V|-1}{3} \le |A| \le \frac{2|V|+1}{3}$, $\Phi_G(A) \ge c\sqrt{\varepsilon}$ holds. Since $\min\{\mathrm{vol}(A), \mathrm{vol}(\overline{A})\} = d \times \min\{|A|, |\overline{A}|\} = d \times |A| \ge d \times \frac{|V|-1}{3}$ holds, we have $|E(A, \overline{A})| = \min\{\mathrm{vol}(A), \mathrm{vol}(\overline{A})\} \times \Phi_G(A) \ge d\frac{|V|-1}{3} \times c\sqrt{\varepsilon} = c\sqrt{\varepsilon}\frac{2|E|-d}{3} \ge c\sqrt{\varepsilon}\frac{|E|}{3}$.

Pick any edge e from $E(A, \overline{A})$. Let D_1 denote the set of edges e' such that e and e' have a common end vertex and D_2 denote the set of edges e'' such that e'' and e' have a common end vertex for some $e' \in D_1$. Then, remove the edges e, D_1, and D_2. Since G is a d regular graph, we can iterate this process at least $\frac{E(A, \overline{A})}{2 \cdot d^2}$ times. The edges picked in each iteration consists of an induced matching in the bipartite graph $(A, \overline{A}; E(A, \overline{A}))$. As $mim(A) \ge \frac{c\sqrt{\varepsilon}|E|}{3 \cdot 2d^2}$ and (T, f_T) is an optimal decomposition tree of G, we have $mimw(G) \ge \frac{c\sqrt{\varepsilon}|E|}{3 \cdot 2d^2}$. □

7. Future Research

In this paper, we showed in a unified manner that under SSEH it is NP-hard to approximate various graph width parameters to within any constant factor in polynomial time. Such width parameters include rank-width, clique-width, boolean-width, and maximum induced matching-width. However, there are several graph parameters for which it is not known whether there are constant factor approximation algorithms. For example, it would be interesting to investigate the constant approximability for path-distance-width [34,35].

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Appendix A. Relations among Graph Parameters

For the graph parameters described in the previous subsection, the following relations are known.

- For each $1 \leq j \leq |V|$,

$$\min_{j/2 \leq i \leq j} b_v(i, G) - 1 \leq tw(G) \leq pw(G) \leq \min\{cutw(G), banw(G)\}. \text{ (see Lemma 9 in [27])} \quad \text{(A1)}$$

- For each $1 \leq j \leq |V|$,

$$\min_{j/2 \leq i \leq j} b_v(i, G) \leq \min_{j/2 \leq i \leq j} b_e(i, G) \leq carw(G) \leq cutw(G) \text{ (see Theorem 1 in [28])}, \quad \text{(A2)}$$

- $$rw(G) \leq \min\{tw(G) + 1, cliw(G)\} \text{ (see Corollary 5 in [29], Proposition 6.3 in [31])}, \quad \text{(A3)}$$

- $$lclw(G) \leq pw(G) + 2 \text{ (see Section 5 in [30])}, \quad \text{(A4)}$$

- $$\mathrm{NLC}(G) \leq cliw(G) \leq 2\mathrm{NLC}(G) \text{ (see [36,37])}, \quad \text{(A5)}$$

  where $\mathrm{NLC}(G)$ denotes the NLC-width of G (see [38] for the definition of NLC-width and details).

- $$boow(G) \leq tw(G) + 1 \text{ (see Figure 1 in [33])}. \quad \text{(A6)}$$

- $$mimw(G) \leq boow(G) \leq mimw(G) \log_2(|V(G)|) \text{ (see Theorem 4.2.10 in [32])}. \quad \text{(A7)}$$

Note that relation (A5) implies that clique-width can be approximated within a constant factor if and only if NLC-width can be approximated within a constant factor.

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