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# A Multi-Attribute Decision Making Process with Immediate Probabilistic Interactive Averaging Aggregation Operators of T-Spherical Fuzzy Sets and Its Application in the Selection of Solar Cells

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**Abstract:** The objective of this paper is to present new interactive averaging aggregation operators by assigning associate probabilities for T-spherical fuzzy sets (T-SFSs). T-SFS is a generalization of several existing theories such as intuitionistic fuzzy sets and picture fuzzy sets to handle imprecise information. Under such an environment, we developed a series of averaging interactive aggregation operators under the features that each element is represented with T-spherical fuzzy numbers. Various properties of the proposed operators are also investigated. Further, to rank the different T-SFSs, we exhibit the new score functions and state their some properties. To demonstrate the presented algorithm, a decision-making process algorithm is presented with T-SFS features. To save non-renewable resources and to the protect environment, the use of renewable resources is important. Solar energy is one of the best renewable energy resources and is also environment-friendly and thus the selection of solar cells is typically a multi-attribute decision-making problem. Therefore, the applicability of the developed algorithm is demonstrated with a numerical example in the selection of the solar cells and comparison of their performance with the several existing approaches.

**Keywords:** multi-attribute decision making; aggregation operators; interactive aggregation operators; t-spherical fuzzy sets

## 1. Introduction

The concept of fuzzy sets (FSs) was proposed by Zadeh [1] in 1965. FSs referred the degree of membership of an object to a non-empty set. A generalization of FSs called intuitionistic fuzzy sets (IFSs) was developed by Atanassov [2,3]. In IFSs, there are two functions namely membership function and non-membership function. Membership function tells the degree of membership while non-membership function denotes the degree of non-membership of a non-empty set object. An IFS has a condition that the sum of membership and non-membership values must belong to a (0, 1) interval. Yager [4] relaxed this condition with the condition that the sum of squares of membership and non-membership values belong to (0, 1) and named it as Pythagorean fuzzy sets (PyFSs).

IFSs and PyFSs fail when abstinence is involved in information, such as in voting, where there are four conditions: yes, no, abstinence, and refusal. To deal with these situations, Cuong [5,6] proposed the concept of picture fuzzy sets (PFSs). Like IFSs, PFSs also have a condition that the sum of membership, abstinence, and non-membership values belong to (0, 1) interval. A generalization of PFSs called spherical fuzzy sets (SFSs) was proposed by Mahmood et al. [7]. In SFSs, a condition is relaxed with the condition that the square of sum of membership, abstinence, and non-membership values belong to (0, 1) interval [7]. Mahmood et al. introduced the concept of T-spherical fuzzy set (T-SFSs) and in T-SFSs there is no restriction overall. The nature of restrictions of these mentioned frameworks are described in Table 1 below.

**Table 1.** Different fuzzy structures and their restrictions.

| Structure | Characteristic Functions | Restriction on Characteristic Function |
|---|---|---|
| FS | $m$ | $0 \le m \le 1$ |
| IFS | $(m,\ n)$ | $0 \le m + n \le 1$ |
| PyFS | $(m,\ n)$ | $0 \le m^2 + n^2 \le 1$ |
| PFS | $(m,\ i,\ n)$ | $0 \le m + i + n \le 1$ |
| SFS | $(m, i,\ n)$ | $0 \le m^2 + i^2 + n^2 \le 1$ |
| T-SFS | $(m, i,\ n)$ | $0 \le m^t + i^t + n^t \le 1, t \in \mathbb{Z}^+$ |

*Abbreviations:* FS: Fuzzy Set; IFS: Intuitionistic fuzzy set; PyFS: Pythagorean fuzzy set; PFS: Picture fuzzy set; SFS: Spherical fuzzy set; T-SFS: T-spherical fuzzy set

In Table 1, some restrictions on different fuzzy structures are discussed. The aggregation operators proposed for IFSs or PyFSs fail when abstinence is involved in data and also fail when sum or square sum of membership and non-membership function exceeds 1, respectively. PFS fail to handle data when the sum of membership, abstinence, and non-membership values exceeds 1. Similarly, SFSs fail to handle data when the sum of squares of membership, abstinence, and non-membership values exceeds 1.

Many authors used these tools of uncertainty to deal with decision-making problems. Xu [8] proposed aggregation operators for IFSs. Zhao et al. [9] proposed generalized aggregation operators for IFSs. Garg [10] introduced interactive geometric aggregation operators for IFSs and applied these operators to a muti-attribute decision-making (MADM) problem. Garg [11] presented the improved operations laws, which is based on an interactive averaging aggregation operator for solving the MADM problems. Liu [12] solved the MADM problem using a power Heronian aggregation operator for IFSs. Dogan et al. [13] investigated a problem of a corridor selection for locating autonomous vehicles using the interval-valued intuitionistic fuzzy AHP and TOPSIS method. Deveci et al. [14] solved the public bread factory site selection problem using GIS-based interval type-2 fuzzy set. Zhang [15] introduced interval-valued intuitionistic fuzzy behavioral MADM method. Garg [16] proposed a Pythagorean aggregation operator using Einstein operations, which solves a decision-making problem. Garg [17] proposed geometric aggregation operators for PyFSs and studied their application in the MADM problem. Khaligh et al. [18] proposed a multi-attribute expansion planning model for integrated gas-electricity system. Sun et al. [19] introduced an integrated decision-making model for a transformer condition assessment. Khalil et al. [20] proposed some new operations on interval-valued PFSs. Wang et al. [21] proposed Muirhead mean operators for PFSs. Wei [22] proposed a picture fuzzy Hamacher aggregation operator and used it to solve the MADM problem. Garg et al. [23] proposed interactive geometric operators for T-SFSs and used them to solve the MADM problem. Garg [24,25] presented the concept of neutrality operational laws for PFNs and the AOs for solving the group decision making problems. Apart from these, many other researchers have presented the different kinds of the aggregation operators and their application to decision making process. For more details, we refer to other studies [26–41].

Sirbiladze et al. [42] proposed associated immediate probability intuitionistic fuzzy aggregation (Associated IP-IFA) operators. With the help of these operators, the MADM problem is solved. Garg [28] presented the concept of immediate probability, which is based on aggregation operators for solving decision-making problems under the Pythagorean fuzzy set environment. The associated immediate probability averaging aggregation operators have an advantage in that they reflect the interaction among all subsets of states of nature. However, the proposed associated IP-IFA operators have some issues that the decision makers were bound for giving values to membership and non-membership grades, i.e., there sum must belong to unit interval. Another main shortcoming in these operators is that the existing approaches fail under some conditions, e.g., considering intuitionistic fuzzy numbers (IFNs) $(0, n_A)$, $(m_B, n_B)$, and $(m_C, 0)$, then the associated immediate probability intuitionistic fuzzy averaging aggregation operators, which aggregate to $(\text{some value},\ 0)$ that seems meaningless. The score function proposed in [7] has some issues and the abstinence and refusal degree are not accurate.

To overcome these shortcomings, we propose new interactive averaging aggregation operators by utilizing the tool of uncertainty called T-spherical fuzzy sets (T-SFSs). T-SFS is the most generalized fuzzy structure and it can handle information in any other fuzzy structure. To overcome its first shortcoming, we proposed some existing operators for T-SFSs, which include a T-spherical fuzzy ordered weighted averaging aggregation (T-SFOWA) operator, an immediate probability T-spherical fuzzy ordered weighted averaging (IP-T-SFOWA) operator, a T-spherical fuzzy Choquet averaging (IP-T-SFCA) operator, and an associated immediate probability T-spherical fuzzy ordered weighted averaging (associated IP-T-SFOWA) operators. To overcome the second shortcoming, some interactive averaging aggregation operators are proposed for T-SFSs, including the T-spherical fuzzy ordered weighted interactive averaging aggregation (T-SFOWIA) operator, thr immediate probability T-spherical fuzzy ordered weighted interactive averaging (IP-T-SFOWIA) operator, the T-spherical fuzzy Choquet interactive averaging operator (IP-T-SFCIA), and the associated immediate probability T-spherical fuzzy ordered weighted interactive averaging (associated IP-T-SFOWIA) operators.

In the world, the major source of generating electricity is fossil fuel. Fossil fuels are a non-renewable resource and the burning of fossil fuels contributes to global warming. Hydroelectricity is a very costly method for generating electricity because water is a limited resource. Generating electricity via energy is a much better way than hydroelectricity and thermal electricity. Solar energy is a clean fuel source and also provides electricity at a very cheap rate. A solar cell converts light energy into electrical energy via the photovoltaic effect. The selection of the solar cell is a typical MADM problem. In our study, the selection of the solar cell problem is discussed. Thus, the objectives of this manuscript are summarized as:

(1) To develop new interactive aggregation operators for T-SFSs, which provide more accurate aggregated values.
(2) The proposed operators can handle the information given in any fuzzy structures like IFSs, PyFSs, PFSs, and SFSs.
(3) To establish a new MADM method by utilizing imprecise and uncertain information.
(4) To illustrate the approach with numerical examples and to compare their performance with the existing approaches.

The manuscript is organized as follows. Section 2 introduces basic definitions. In Section 3, a new score function is proposed to help compare two or more aggregated values. In Section 4, some new aggregation operators for T-SFSs are proposed and their advantages are illustrated by a comparative analysis. Section 5 develops an application for MADM problem and reliability is checked with the help of a numerical example. In Section 6, the advantages of the proposed work are discussed and a comparison with the existing methods is done. Section 7 concludes the paper.

## 2. Preliminaries

In this section, we define basic definitions and operations that help developed the proposed work. Unless otherwise stated, $X$ is used as universal set.

**Definition 1.** [7] A T-SFS on $X$ *is defined as:*

$$T = \{(x, m(x), i(x),\ n(x)) \mid x \in X\} \tag{1}$$

*where* $m, i, n : X \to [0,1]$ *are membership, abstinence, and non-membership functions with a condition that* $0 \leq m^t(x) + i^t(x) + n^t(x) \leq 1$. *The refusal degree of T-SFS is defined as* $r(x) = \sqrt[t]{1 - (m^t(x) + i^t(x) + n^t(x))}$ *and the triplet* $(m, i, n)$ *is called the T-spherical fuzzy number (T-SFN).*

**Remark 1.**

1. *For* $t = 2$, *the definition 1 is reduced to SFSs*
2. *For* $t = 1$, *the definition 1 is reduced to PFSs*
3. *For* $t = 2$ *and* $i = 0$, *the definition 1 is reduced to PyFSs*
4. *For* $t = 1$ *and* $i = 0$, *the definition 1 is reduced to IFSs*
5. *For* $t = 1$, $i = 0$, *and* $n = 0$ *the definition 1 is reduced to FSs*

**Remark 2.** *For any two IFNs* $T_1 = (m_1, n_1)$ *and* $T_2 = (m_2, n_2)$, $T_1 \geq T_2$ *if* $m_1 \geq m_2$ *and* $n_1 \leq n_2$.

**Definition 2.** [26] *For any collection of IFNs* $T_j = \left(m_j, n_j\right)$ $(j = 1, 2, \ldots, k)$, *the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator is defined as:*

$$IFOWA(T_1, \ldots\ldots, T_k) = \oplus_{j=1}^{k} \left(w_j T_{\sigma(j)}\right) = \left(1 - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{w_j}\right) \tag{2}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with conditions all weight vectors must belong to* $[0,1]$ *and the sum of all weights is equal to 1 and* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $T_{\sigma(j-1)} \geq T_{\sigma(j)}$.

**Definition 3.** [27] *For any collection of IFNs,* $T_j = \left(m_j, n_j\right)$ $(j = 1, 2, \ldots, k)$, *the immediate probability intuitionistic fuzzy ordered weighted averaging (IFOWA) operator is defined as:*

$$IP - IFOWA_P(T_1, \ldots\ldots, T_k) = \oplus_{j=1}^{k} \left(\lambda'_j T_{\sigma(j)}\right) = \left(1 - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_j}, \prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_j}\right) \tag{3}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with a condition that all weight vectors belong to* $[0,1]$ *and the sum of all weights must be equal to 1.* $\lambda_j$ *is probability for each* $T_j$, $\lambda_j$ *is associated probability of* $T_{\sigma(j)}$, *and* $\lambda'_j = \frac{(w_j \lambda_j)}{\sum_{j=1}^{k} w_j \lambda_j}$. $\sigma = (\sigma(1), \ldots\ldots, \sigma(k))$ *is the permutation such that* $T_{\sigma(j-1)} \geq T_{\sigma(j)}$.

**Definition 4.** [41] *A fuzzy measure* $\Theta : 2^X \to [0,1]$ *on a finite set* $X$ *is defined as:*

*i* $\Theta(\phi) = 0; \Theta(X) = 1$

*ii* *For all* $X_1, X_2 \subseteq X$, *if* $X_1 \subseteq X_2$ *then* $\Theta(X_1) \leq \Theta(X_2)$

*The possible orderings of elements of* $X$ *are presented by the permutation of* $X$ *with k elementsm which forms a group,* $X_n$.

**Definition 5.** [41] *The probability function $P_\rho$ on X defined by:*

$$
\begin{gathered}
P_\rho\left(x_{\rho(1)}\right) = \Theta\left(\left\{x_{\rho(1)}\right\}\right), \ldots\ldots\ldots, \\
P_\rho\left(x_{\rho(j)}\right) = \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots\ldots, x_{\rho(j)}\right\}\right) - \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots\ldots, x_{\rho(j-1)}\right\}\right), \ldots\ldots, \\
P_\rho\left(x_{\rho(k)}\right) = 1 - \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots\ldots, x_{\rho(k-1)}\right\}\right), \\
\Theta\left(\left\{x_{\rho(0)}\right\}\right) \equiv 0.
\end{gathered}
$$

*where $\rho = (\rho(1), \rho(2), \ldots\ldots, \rho(k)) \in X_n$ are called associated probabilities and $\left\{P_\rho\right\}_{\rho \in X_n}$ is associated probability class of $\Theta$.*

**Definition 6.** [41] *For any collection of IFNs, $T_j = \left(m_j, n_j\right)$ $(j = 1, 2, \ldots\ldots, k)$ on a set of states of nature $X = \{x_1, \ldots\ldots, x_k\}$, then the intuitionistic fuzzy Choquet averaging (IFCA) operator is defined as:*

$$
\begin{aligned}
IFCA_\Theta(T_1, \ldots\ldots, T_k) &= \oplus_{j=1}^{k}\left(\lambda_j T_{\sigma(j)}\right) \\
&= \left(1 - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda_j}, \prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda_j}\right),
\end{aligned} \tag{4}
$$

*where $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$ and $\Theta$ is fuzzy measure. $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ is the permutation such that $T_{\sigma(j-1)} \geq T_{\sigma(j)}$.*

**Definition 7.** [41] *For any collection of IFNs, $T_j = \left(m_j, n_j\right)$ $(j = 1, 2, \ldots\ldots, k)$, the associated immediate probability intuitionistic fuzzy ordered weighted averaging (associated IP-IFOWA) operators are defined as:*

$$
\begin{aligned}
&Ass.IP - IFOWA_\vee(T_1, \ldots\ldots, T_k) = \bigvee_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right] \\
&= \left(1 - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right)
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
&Ass.IP - IFOWA_\wedge(T_1, \ldots\ldots, T_k) = \bigwedge_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right] \\
&= \left(1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right),
\end{aligned} \tag{6}
$$

*where $w = (w_1, \ldots\ldots, w_k)^T$ is a weight vector with a condition that all weight vectors belong to $[0, 1]$ and the sum of all weights must be equal to 1, $\sigma = (\sigma(1), \ldots\ldots, \sigma(k))$ is the permutation such that $T_{\sigma(j-1)} \geq T_{\sigma(j)}$. For each associated probability, $P_\rho : \lambda'_{\rho(j)} = \frac{(w_j \lambda_{\rho(j)})}{\sum_{j=1}^{k} w_j \lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho\left(T_{\sigma(j)}\right)$, is an associated immediate probability.*

## 3. New Score Function of T-SFSs

This section shows that existing score functions for T-SFSs have some shortcomings and a new score function is proposed to overcome these shortcomings.

**Definition 8.** [7] *For any T-SFN, $T = (m, i, n)$ the score function is defined as:*

$$SCF(T) = m^t(x) - n^t(x), \tag{7}$$

*and the accuracy of T-SFN is defined as:*

$$ACF(T) = m^t(x) + i^t(x) + n^t(x), \tag{8}$$

A T-SFN is said to be superior to other T-SFN if its score is greater. If the score of two T-SFNs is the same then the superiority of these numbers is checked by accuracy; the number with the greater accuracy is superior to the other. If the accuracy of the two T-SFNs becomes equal then these T-SFNs will be similar.

The score function defined above is not valid because abstinence is not involved, so a new score function is proposed with the help of the curve function $h(x) = \frac{e^x}{e^x+1}$, which has the following properties:

(i) $h(x)$ is strictly increasing in domain of all real numbers;
(ii) the range of $h(x)$ is always between $(0,\ 1)$ ;
(iii) $h(x) + h(-x) = 0$ and $h(0) = 0.5$.

**Definition 9.** *For any T-SFN $T = (m, i, n)$, the new score function is defined as:*

$$SC(T) = m^t - i^t - n^t + \left(\frac{e^{m^t - i^t - n^t}}{e^{m^t - i^t - n^t} + 1} - \frac{1}{2}\right) r^t, \quad (9)$$

*where* $r = \sqrt[t]{1 - (m^t + i^t + n^t)}$.

A T-SFN $T_1 = (m_1, i_1, n_1)$ is said to be superior than another T-SFN $T_2 = (m_2, i_2, n_2)$ if the score of $T_1$ is greater than $T_2$. If the score of both numbers is equal then the superiority is checked by the comparison of their refusal degree. If the number $T_1$ is superior, then its refusal degree is smaller than $T_2$ and numbers will be similar if their refusal degree is same.

**Remark 3.** *The following special cases are concluded from the proposed score value, defined in Equation (9), as:*

*(a) Equation (9) reduces the score value for SFSs when $t = 2$;*
*(b) Equation (9) become valid for PFSs if $t = 1$;*
*(c) Equation (9) become valid for PyFSs if $t = 2$ and $i = 0$;*
*(d) Equation (9) reduces the score value for IFSs if $t = 1$ and $i = 0$.*

## 4. T-Spherical Fuzzy Averaging Operators

In this section, some operations for T-SFSs are defined. With the help of these operations, some T-spherical fuzzy aggregation operators are proposed. This section is divided into three subsections. In the first subsection, averaging aggregation operators are proposed and basic properties of these operators are discussed. In the second subsection, interactive averaging aggregation operators and basic properties are proposed. In the third subsection, the superiority of the interactive averaging aggregation operators over averaging aggregation operators is explained with the help of an example.

**Definition 10.** *Some operations for any two T-SFNs $T_1 = (m_1, i_1, n_1)$ and $T_2 = (m_2, i_2, n_2)$ are defined as:*

(i) $\mathrm{T}_1 \oplus \mathrm{T}_2 = \left(\sqrt[t]{1 - \left(1 - \mathrm{m}_1^t\right)\left(1 - \mathrm{m}_2^t\right)},\ \mathrm{i}_1\mathrm{i}_2,\ \mathrm{n}_1\mathrm{n}_2\right)$;
(ii) $\tau\mathrm{T}_1 = \left(\sqrt[t]{1 - \left(1 - \mathrm{m}_1^t\right)^\tau},\ (\mathrm{i}_1\mathrm{i}_2)^\tau,\ (\mathrm{n}_1\mathrm{n}_2)^\tau\right),\ \tau > 0.$

**Definition 11.** *Some interactive operations for any two T-SFNs $T_1 = (m_1, i_1, n_1)$ and $T_2 = (m_2, i_2, n_2)$ are defined as:*

(i) $$\mathrm{T}_1 \oplus \mathrm{T}_2 = \left(\begin{array}{c} \sqrt[t]{1 - \left(1 - \mathrm{m}_1^t\right)\left(1 - \mathrm{m}_2^t\right)},\ \sqrt[t]{1 - \left(1 - \mathrm{i}_1^t\right)\left(1 - \mathrm{i}_2^t\right)}, \\ \sqrt[t]{\left(1 - \mathrm{m}_1^t\right)\left(1 - \mathrm{m}_2^t\right) - \left(1 - \left(\mathrm{m}_1^t + \mathrm{i}_1^t + \mathrm{n}_1^t\right)\right)\left(1 - \left(\mathrm{m}_2^t + \mathrm{i}_2^t + \mathrm{n}_2^t\right)\right) - \mathrm{i}_1^t\mathrm{i}_2^t} \end{array}\right);$$

(ii) $$\tau\mathrm{T}_1 = \left(\sqrt[t]{1 - \left(1 - \mathrm{m}_1^t\right)^\tau},\ \sqrt[t]{1 - \left(1 - \mathrm{i}_1^t\right)^\tau},\ \sqrt[t]{\left(1 - \mathrm{m}_1^t\right)^\tau - \left(1 - \left(\mathrm{m}_1^t + \mathrm{i}_1^t + \mathrm{n}_1^t\right)\right)^\tau - \left(\mathrm{i}_1^t\right)^\tau}\right),\ \tau > 0.$$

**Remark 4.**

(a) *For $t = 2$, the above operations defined in Definitions 10 and 11 become valid for SFSs;*
(b) *For $t = 1$, the above operations defined in Definitions 10 and 11 become valid for PFSs;*
(c) *For $t = 2$ and $i = 0$, the above operations defined in Definitions 10 and 11 become valid for PyFSs;*
(d) *For $t = 1$ and $i = 0$, the above operations defined in Definitions 10 and 11 become valid for IFSs.*

### 4.1. T-Spherical Fuzzy Averaging Aggregation Operators

In this subsection, some averaging aggregation operators (e.g., T-SFOWA, IP-T-SFOWA, T-SGCA, and associated IP-T-SFOWA operators) are proposed with some of their basic properties.

**Definition 12.** *Consider the collection of T-SFNs, $T_j = \left(m_j, i_j, n_j\right)$, then the T-SFOWA operator is defined as:*

$$T - SFOWA_w(T_1, T_2, \ldots\ldots, T_k) = \oplus_{j=1}^{k}\left(w_j T_{\sigma(j)}\right), \tag{10}$$

*where $w = (w_1, \ldots\ldots, w_k)^T$ is a weight vector with a condition that all weight vectors must belong to $[0,1]$ and the sum of all weights is equal to 1 and $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ is the permutation such that $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.*

**Theorem 1.** *Consider a collection of T-SFNs $T_j = \left(m_j, i_j, n_j\right)$, then:*

$$\begin{gathered} T - SFOWA_w(T_1, T_2, \ldots\ldots, T_k) \\ = \left(\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \prod_{j=1}^{k} i^{w_j}_{\sigma(j)},\ \prod_{j=1}^{k} n^{w_j}_{\sigma(j)}\right), \end{gathered}$$

**Proof.** The above result is proven by using mathematical induction. □

For $k = 1$,

$$\begin{aligned} T - SFOWA_w(T_1, T_2, \ldots\ldots, T_k) &= \left(\sqrt[t]{1 - \left(1 - m^t_1\right)},\ i_1, n_1\right) \\ &= (m_1, i_1, n_1) \end{aligned}$$

Thus, the results hold for $k = 1$. Now, consider that the results hold for $k = l$:

$$\begin{gathered} T - SFOWA_w(T_1, T_2, \ldots\ldots, T_l) \\ = \left(\sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \prod_{j=1}^{l} i^{w_j}_{\sigma(j)},\ \prod_{j=1}^{l} n^{w_j}_{\sigma(j)}\right) \end{gathered}$$

Then, to prove that result, hold for $k = l + 1$:

$$\begin{gathered} T - SFOWA_w(T_1, T_2, \ldots\ldots, T_{l+1}) \\ = \left(\sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \prod_{j=1}^{l} i^{w_j}_{\sigma(j)},\ \prod_{j=1}^{l} n^{w_j}_{\sigma(j)}\right) \oplus \left(\sqrt[t]{1 - \left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ i^{w_j}_{\sigma(j)},\ n^{w_j}_{\sigma(j)}\right) \\ = \left(\sqrt[t]{1 - \prod_{j=1}^{l+1}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \prod_{j=1}^{l+1} i^{w_j}_{\sigma(j)},\ \prod_{j=1}^{l+1} n^{w_j}_{\sigma(j)}\right) \end{gathered}$$

This proves that the results hold for all $k \in Z^{+}$.

**Property 1.** *If $T_j = T_0 = (m_0, i_0, n_0)$ for all $j = 1, 2, \ldots\ldots, k$ then $T - SFOWA_w(T_1, T_2, \ldots\ldots, T_k) = T_0$.*

**Proof.** As $T_j = T_0 = (m_0, i_0, n_0)$. □

Then

$$T-SFOWA_w(T_1,T_2,\ldots\ldots,T_k)=\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}},\ \prod_{j=1}^{k} i_{\sigma(j)}^{w_j},\prod_{j=1}^{k} n_{\sigma(j)}^{w_j}\right)$$
$$=\left(\sqrt[t]{1-\left(1-m_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k} w_j}},\ \left(i_{\sigma(j)}\right)^{\sum_{j=1}^{k} w_j},\left(n_{\sigma(j)}\right)^{\sum_{j=1}^{k} w_j}\right)$$
$$=(m_0,\ i_0,\ n_0)=T_0$$

**Property 2.** *For a collection of any two different T-SFNs* $T_j=\left(m_j,i_j,n_j\right)$ *and* $T'_j=\left(m'_j,i'_j,n'_j\right)$ $(j=1,2,\ldots\ldots,k)$ *such that* $m_j\le m'_j$, $i_j\le i'_j$ *and* $n_j\ge n'_j$ *for all* $j$, *then*

$$T-SFOWA_w(T_1,T_2,\ldots\ldots,T_k)\le T-SFOWA_w\left(T'_1,T'_2,\ldots\ldots,T'_k\right)$$

**Proof.** As $m_j\le m'_j$, $i_j\le i'_j$ and $n_j\ge n'_j$ for all $j$. □

This implies that

$$\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}}\le\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{w_j}}$$
$$\prod_{j=1}^{k} i_{\sigma(j)}^{w_j}\le\prod_{j=1}^{k}\left(i'_{\sigma(j)}\right)^{w_j}$$
$$\prod_{j=1}^{k} n_{\sigma(j)}^{w_j}\ge\prod_{j=1}^{k}\left(n'_{\sigma(j)}\right)^{w_j}$$
$$\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}},\prod_{j=1}^{k} i_{\sigma(j)},\prod_{j=1}^{k} n_{\sigma(j)}\right)$$
$$\le\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{w_j}},\prod_{j=1}^{k}\left(i'_{\sigma(j)}\right)^{w_j},\prod_{j=1}^{k}\left(n'_{\sigma(j)}\right)^{w_j}\right)$$

**Property 3.** *Consider a collection of T-SFNs* $T_j=\left(m_j,i_j,n_j\right)$ *for all* $j=1,2,\ldots\ldots,k$ *such that* $T_L=\min_j\{T_j\}$ *and* $T_U=\max_j\{T_j\}$ *. Then:*

$$T_L\le T-SFOWA_w(T_1,T_2,\ldots\ldots,T_k)\le T_U$$

The proof is straightforward.

**Definition 13.** *Consider a collection of T-SFNs,* $T_j=\left(m_j,i_j,n_j\right)$, *then IP-T-SFOWA operator is defined as:*

$$IP-T-SFOWA_P(T_1,T_2,\ldots\ldots,T_k)=\oplus_{j=1}^{k}\left(\lambda'_j T_{\sigma(j)}\right),\tag{11}$$

*where* $w=(w_1,\ldots\ldots,w_k)^T$ *is a weight vector with a condition that all weight vectors belong to* $[0,1]$ *and the sum of all weights must be equal to 1.* $\lambda_j$ *is the probability for each* $T_j$. $\lambda_j$ *is the associated probability of* $T_{\sigma(j)}$ *and* $\lambda'_j=\frac{(w_j\lambda_j)}{\sum_{j=1}^{k} w_j\lambda_j}$ *and* $\sigma=(\sigma(1),\ldots\ldots,\sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right)\ge SC\left(T_{\sigma(j)}\right)$.

**Theorem 2.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$*, then:*

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_k) = \left(\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \prod_{j=1}^{k} i^{\lambda'_j}_{\sigma(j)}, \prod_{j=1}^{k} n^{\lambda'_j}_{\sigma(j)}\right),$$

**Proof.** The above result is proven by using mathematical induction. □

For $k = 1$,

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_k) = \left(\sqrt[t]{1 - \left(1 - m^t_1\right)},\ i_1, n_1\right)$$
$$= (m_1, i_1, n_1)$$

Thus, the results hold for $k = 1$. Now, consider that the results hold for $k = l$:

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_l)$$
$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \prod_{j=1}^{l} i^{\lambda'_j}_{\sigma(j)}, \prod_{j=1}^{l} n^{\lambda'_j}_{\sigma(j)}\right)$$

Then, to prove that result, hold for $k = l + 1$:

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_{l+1})$$
$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \prod_{j=1}^{l} i^{\lambda'_j}_{\sigma(j)}, \prod_{j=1}^{l} n^{\lambda'_j}_{\sigma(j)}\right) \oplus \left(\sqrt[t]{1 - \left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ i^{\lambda'_j}_{\sigma(j)}, n^{\lambda'_j}_{\sigma(j)}\right)$$
$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l+1}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \prod_{j=1}^{l+1} i^{\lambda'_j}_{\sigma(j)}, \prod_{j=1}^{l+1} n^{\lambda'_j}_{\sigma(j)}\right)$$

This proves that the results hold for all $k \in Z^+$.

**Property 4.** *If* $T_j = T_0 = (m_0, i_0, n_0)$ *for all* $j = 1, 2, \ldots\ldots, k$ *then:*

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_k) = T_0.$$

**Proof.** As $T_j = T_0 = (m_0, i_0, n_0)$. □

Then:

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left(\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \prod_{j=1}^{k} i^{\lambda'_j}_{\sigma(j)}, \prod_{j=1}^{k} n^{\lambda'_j}_{\sigma(j)}\right)$$
$$= \left(\sqrt[t]{1 - \left(1 - m^t_{\sigma(j)}\right)^{\sum_{j=1}^{k}\lambda'_j}},\ i^{\sum_{j=1}^{k}\lambda'_j}_{\sigma(j)},\ n^{\sum_{j=1}^{k}\lambda'_j}_{\sigma(j)}\right)$$
$$= (m_0,\ i_0,\ n_0) = T_0$$

**Property 5.** *For a collection of any two different T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$ *and* $T'_j = \left(m'_j, i'_j, n'_j\right)$ $(j = 1, 2, \ldots\ldots, k)$ *such that* $m_j \leq m'_j$, $i_j \leq i'_j$ *and* $n_j \geq n'_j$ *for all* $j$*. Then:*

$$IP - T - SFOWA_P(T_1, T_2, \ldots\ldots, T_k)$$
$$\leq IP - T - SFOWA_P\left(T'_1, T'_2, \ldots\ldots, T'_k\right)$$

**Proof.** As $m_j \leq m'_j$, $i_j \leq i'_j$ and $n_j \geq n'_j$ for all $j$. □

This implies that:

$$\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_j}} \leq \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}}$$
$$\prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_j} \leq \prod_{j=1}^{k}\left(i'_{\sigma(j)}\right)^{\lambda'_j}$$
$$\prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_j} \geq \prod_{j=1}^{k}\left(n'_{\sigma(j)}\right)^{\lambda'_j}$$
$$\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_j}}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda'_j}, \prod_{j=1}^{k} n_{\sigma(j)}^{\lambda'_j}\right)$$
$$\leq \left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}}, \prod_{j=1}^{k}\left(i'_{\sigma(j)}\right)^{\lambda'_j}, \prod_{j=1}^{k}\left(n'_{\sigma(j)}\right)^{\lambda'_j}\right)$$

**Property 6.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$ *for all* $j = 1, 2, \ldots\ldots, k$ *such that* $T_L = \min_j\{T_j\}$ *and* $T_U = \max_j\{T_j\}$ *. Then*

$$T_L \leq IP-T-SFOWA_P(T_1, T_2, \ldots\ldots, T_k) \leq T_U$$

The proof is straightforward.

**Definition 14.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$ *on a set of states of nature* $X = \{x_1, \ldots\ldots, x_k\}$*, then T-SFCA operator with respect to fuzzy measure* $\Theta$ *is defined as*

$$T-SFCA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) = \oplus_{j=1}^{k}\left(\lambda_j T_{\sigma(j)}\right), \tag{12}$$

*where* $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$ *and* $\sigma$ *is the permutation.* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 3.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$*, then*

$$T-SFCA_{\Theta}(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda_j}}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j}, \prod_{j=1}^{k} n_{\sigma(j)}^{\lambda_j}\right)$$

**Proof.** This can be proven by following Theorem 1. □

Further, it is observed that the T-SFCA operator satisfies the properties as defined in Properties 1–3, so we omitted their proofs.

**Definition 15.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$*, the associated IP-T-SFOWA operators is defined as:*

$$Ass.\ IP-T-SFOWA_{\vee}(T_1, T_2, \ldots\ldots, T_k) = \bigvee_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right], \tag{13}$$

*and*

$$Ass.IP-T-SFOWA_{\wedge}(T_1, T_2, \ldots\ldots, T_k) = \bigwedge_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right], \tag{14}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with a condition that all weight vectors belong to* $[0, 1]$ *and the sum of all weights must be equal to 1.* $\sigma = (\sigma(1), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

*For each associated probability,* $P_\rho : \lambda'_{\rho(j)} = \frac{(w_j\lambda_{\rho(j)})}{\sum_{j=1}^{k} w_j\lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho\left(T_{\sigma(j)}\right)$ *is an associated immediate probability and* $\vee$ = *maximum and* $\wedge$ = *minimum.*

**Theorem 4.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$, *then:*

$$\begin{aligned} &Ass.\ IP - T - SFOWA_{\vee}(T_1, T_2, \ldots\ldots, T_k) \\ &= \left( \sqrt[t]{1 - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right) \end{aligned}$$

*and*

$$\begin{aligned} &Ass.IP - T - SFOWA_{\wedge}(T_1, T_2, \ldots\ldots, T_k) \\ &= \left( \sqrt[t]{1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right) \end{aligned}$$

**Proof.** This can be proven by following Theorem 1. □

Further, it is observed that associated $\mathrm{IP} - \mathrm{T} - \mathrm{SFOWA}_{\vee}$ and associated $\mathrm{IP} - \mathrm{T} - \mathrm{SFOWA}_{\wedge}$ operator also satisfies the properties as defined in Properties 1–3, so we omitted the proofs.

**Definition 16.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$ *on a set of states of nature* $X = \{x_1, \ldots\ldots, x_k\}$, *then T-spherical fuzzy conjugate Choquet averaging (T-SFCCA) operator with respect to fuzzy measure* $\Theta$ *is defined as:*

$$T - SFCCA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) = \left(\oplus_{j=1}^{k}\left(\lambda_j\left(T_j\right)^c\right)\right)^c, \tag{15}$$

*where* $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$, *and* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 5.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$, *then:*

$$\begin{aligned} &T - SFCCA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) \\ &= \left(\prod_{j=1}^{k}\left(m_{\sigma(j)}\right)^{\lambda_j}, \prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda_j}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n^t_{\sigma(j)}\right)^{\lambda_j}}\right) \end{aligned}$$

**Proof.** This can be proven by following Theorem 1. □

*4.2. T-Spherical Fuzzy Interactive Aggregation Operators*

In this subsection, some interactive averaging aggregation operators (e.g., T-SFOWIA, IP-T-SFOWIA, T-SGCIA, and associated IP-T-SFOWIA operators) along with some of their basic properties are proposed.

**Definition 17.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$, *then the T-SFOWIA operator is defined as:*

$$T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_k) = \oplus_{j=1}^{k}\left(w_j T_{\sigma(j)}\right), \tag{16}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with a conditions that all weight vectors must belong to* $[0,1]$ *and the sum of all weights is equal to 1 and* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 6.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$*, then*

$$T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_k) = \left(\begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{w_j}} \end{array}\right),$$

**Proof.** The above result is proven using mathematical induction. □

For $k = 1$,

$$T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_k) = \left(\sqrt[t]{1 - \left(1 - m^t_1\right)}, \sqrt[t]{1 - \left(1 - i^t_1\right)}, \sqrt[t]{\left(1 - m^t_1\right) - \left(1 - m^t_1 - i^t_1 - n^t_1\right) - i^t_1}\right) = (m_1, i_1, n_1)$$

Thus, the results hold for $k = 1$. Now, consider that the results hold for $k = l$:

$$T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_l) = \left(\begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - i^t_{\sigma(j)}\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l}\left(i^t_{\sigma(j)}\right)^{w_j}} \end{array}\right)$$

Then, to prove that result hold for $k = l + 1$:

$$\begin{aligned} &T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_{l+1}) \\ &= \left(\begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \sqrt[t]{1 - \prod_{j=1}^{l}\left(1 - i^t_{\sigma(j)}\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l}\left(i^t_{\sigma(j)}\right)^{w_j}} \end{array}\right) \\ &\oplus \left(\begin{array}{c} \sqrt[t]{1 - \left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \sqrt[t]{1 - \left(1 - i^t_{\sigma(j)}\right)^{w_j}}, \\ \sqrt[t]{\left(1 - m^t_{\sigma(j)}\right)^{w_j} - \left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{w_j} - \left(i^t_{\sigma(j)}\right)^{w_j}} \end{array}\right) \\ &= \left(\begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{l+1}\left(1 - m^t_{\sigma(j)}\right)^{w_j}},\ \sqrt[t]{1 - \prod_{j=1}^{l+1}\left(1 - i^t_{\sigma(j)}\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l+1}\left(1 - m^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l+1}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{w_j} - \prod_{j=1}^{l+1}\left(i^t_{\sigma(j)}\right)^{w_j}} \end{array}\right) \end{aligned}$$

This proves that the results hold for all $k \in Z^+$.

**Definition 18.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$*, then the IP-T-SFOWIA operator is defined as:*

$$IP - T - SFOWIA_P(T_1, T_2, \ldots\ldots, T_k) = \oplus_{j=1}^{k}\left(\lambda'_j T_{\sigma(j)}\right), \tag{17}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with a condition that all weight vectors belong to* $[0,1]$ *and the sum of all weights must be equal to 1.* $\lambda_j$ *is probability for each* $T_j$, $\lambda_j$ *is associated probability of* $T_{\sigma(j)}$, *and* $\lambda'_j = \frac{(w_j\lambda_j)}{\sum_{j=1}^{k} w_j\lambda_j}$ *and* $\sigma = (\sigma(1), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 7.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$, *then:*

$$IP - T - SFOWIA_w(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left( \begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j}},\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{\lambda'_j}}, \\ \sqrt[t]{\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_j}} \end{array} \right)$$

**Proof.** This can be proven by following Theorem 6. □

**Definition 19.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$ *on a set of states of nature* $X = \{x_1, \ldots\ldots, x_k\}$, *then T-SFCIA operator with respect to fuzzy measure* $\Theta$ *is defined as*

$$T - SFCIA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) = \oplus_{j=1}^{k}\left(\lambda_j T_{\sigma(j)}\right), \tag{18}$$

*where* $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$ *and* $\sigma$ *is the permutation.* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 8.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$, *then:*

$$T - SFCIA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) =$$
$$\left( \begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda_j}},\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{\lambda_j}}, \\ \sqrt[t]{\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda_j}} \end{array} \right)$$

**Proof.** This can be proven by following Theorem 6. □

**Definition 20.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$, *then associated IP-T-SFOWIA operators is defined as:*

$$Ass.IP - T - SFOWIA_{\vee}(T_1, T_2, \ldots\ldots, T_k) = \bigvee_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right], \tag{19}$$

*and*

$$Ass.IP - T - SFOWIA_{\wedge}(T_1, T_2, \ldots\ldots, T_k) = \bigwedge_{\rho \in X_n}\left[\oplus_{j=1}^{k}\left(\lambda'_{\rho(j)} T_{\sigma(j)}\right)\right], \tag{20}$$

*where* $w = (w_1, \ldots\ldots, w_k)^T$ *is a weight vector with a condition that all weight vectors belong to* $[0,1]$ *and the sum of all weights must be equal to 1.* $\sigma = (\sigma(1), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$. *For each associated probability* $P_\rho : \lambda'_{\rho(j)} = \frac{(w_j\lambda_{\rho(j)})}{\sum_{j=1}^{k} w_j\lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho\left(T_{\sigma(j)}\right)$ *is an associated immediate probability and* $\vee$ = *maximum and* $\wedge$ = *minimum.*

**Theorem 9.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$*, then*

$$Ass.IP - T - SFOWIA_{\vee}(T_1, T_2, \ldots\ldots, T_k) = \left(\begin{array}{c} \sqrt[t]{1 - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \sqrt[t]{1 - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\ \sqrt[t]{\min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)} \end{array}\right)$$

*and*

$$Ass.IP - T - SFOWIA_{\wedge}(T_1, T_2, \ldots\ldots, T_k) = \left(\begin{array}{c} \sqrt[t]{1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \sqrt[t]{1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\ \sqrt[t]{\max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)} \end{array}\right)$$

**Proof.** This can be proven by following Theorem 21. □

**Definition 21.** *Consider a collection of T-SFNs,* $T_j = \left(m_j, i_j, n_j\right)$ *on a set of states of nature* $X = \{x_1, \ldots\ldots, x_k\}$*, then T-spherical fuzzy conjugate Choquet interactive averaging (T-SFCCIA) operator with respect to fuzzy measure* $\Theta$ *is defined as:*

$$T - SFCCIA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) = \left(\oplus_{j=1}^{k}\left(\lambda_j\left(T_j\right)^c\right)\right)^c, \tag{21}$$

*where* $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots\ldots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$ *and* $\sigma = (\sigma(1), \sigma(2), \ldots\ldots, \sigma(k))$ *is the permutation such that* $SC\left(T_{\sigma(j-1)}\right) \geq SC\left(T_{\sigma(j)}\right)$.

**Theorem 10.** *Consider a collection of T-SFNs* $T_j = \left(m_j, i_j, n_j\right)$*, then:*

$$T - SFCCIA_{\Theta}(T_1, T_2, \ldots\ldots, T_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1 - n^t_{\sigma(j)}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(1 - m^t_{\sigma(j)} - i^t_{\sigma(j)} - n^t_{\sigma(j)}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i^t_{\sigma(j)}\right)^{\lambda_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n^t_{\sigma(j)}\right)^{\lambda_j}} \end{array}\right)$$

**Proof.** This can be proven by following Theorem 6. □

Moreover, it is observed that all the above defined operators also satisfied the properties as defined in Properties 1–3.

**Remark 5.** *If the fuzzy measure and probability of T-SFSs become equal and furthermore probabilities of all T-SFNs become equal, then the associated IP-T-SFOWIA operator reduces to the T-SFOWA operator.*

*4.3. Comparison between Aggregation Operators and Interactive Aggregation Operators*

In this section, the superiority of interactive averaging aggregation operators over averaging aggregation operators is explained with the help of an example. It is also explained that under some conditions, the averaging aggregation operators fail while interactive averaging aggregation operators overcome this shortcoming.

**Example 1.** *Consider T-SFNs,* $g_1 = (0.63,\ 0.0,\ 0.0)$, $g_2 = (0.68,\ 0.25,\ 0.81)$, *and* $g_3 = (0.0,\ 0.51,\ 0.93)$ *having a weight vector* $w = \{0.25,\ 0.40,\ 0.35\}$. *Fuzzy measures will be:*

$$\Theta(\phi) = 0,\ \Theta(\{g_1\}) = 0.125, \Theta(\{g_2\}) = 0.200, \Theta(\{g_3\}) = 0.175, \Theta(\{g_1, g_2\}) = 0.325,$$
$$\Theta(\{g_1, g_3\}) = 0.300, \Theta(\{g_2, g_3\}) = 0.375, \text{ and } \Theta(\{g_1,\ g_2,\ g_3\}) = 1.$$

Immediate probabilities for all possible permutations and the associated immediate probabilities for all possible permutations are given in Table 2.

**Table 2.** Immediate probabilities and associated immediate probability.

| | **Immediate Probabilities** | | |
|---|---|---|---|
| | $\lambda_1'$ | $\lambda_2'$ | $\lambda_3'$ |
| $\sigma = (g_1, g_2, g_3)$ | 0.125 | 0.200 | 0.675 |
| $\sigma = (g_1, g_3, g_2)$ | 0.125 | 0.700 | 0.175 |
| $\sigma = (g_2, g_1, g_3)$ | 0.200 | 0.200 | 0.600 |
| $\sigma = (g_2, g_3, g_1)$ | 0.700 | 0.200 | 0.100 |
| $\sigma = (g_3, g_1, g_2)$ | 0.125 | 0.700 | 0.175 |
| $\sigma = (g_3, g_2, g_1)$ | 0.700 | 0.125 | 0.175 |
| | **Associated Immediate Probability** | | |
| | $\lambda'_{\rho(1)}$ | $\lambda'_{\rho(2)}$ | $\lambda'_{\rho(3)}$ |
| $\sigma = (g_1, g_2, g_3)$ | 0.0899 | 0.2302 | 0.6798 |
| $\sigma = (g_1, g_3, g_2)$ | 0.0839 | 0.7517 | 0.1644 |
| $\sigma = (g_2, g_1, g_3)$ | 0.1470 | 0.2353 | 0.6176 |
| $\sigma = (g_2, g_3, g_1)$ | 0.6034 | 0.2759 | 0.1207 |
| $\sigma = (g_3, g_1, g_2)$ | 0.0839 | 0.7517 | 0.1644 |
| $\sigma = (g_3, g_2, g_1)$ | 0.6114 | 0.1747 | 0.2140 |

As for $t = 1$, $0.68 + 0.25 + 0.81 = 1.74 \notin [0,\ 1]$,
As for $t = 2$, $0.68^2 + 0.25^2 + 0.81^2 = 1.181 \notin [0,\ 1]$
As for $t = 3$, $0.68^3 + 0.25^3 + 0.81^3 = 0.861 \in [0,\ 1]$
Similarly, for $t = 3$, $T_1$ and $T_3$ are T-SFNs.
Then, some averaging aggregation operators defined in Definitions 12–15 will be:

$$T - SFOWA_w(g_1, g_2, g_3) = (0.5846,\ 0.0,\ 0.0)$$

$$IP - T - SFOWA_P(g_1, g_2, g_3) = (0.5452,\ 0.0,\ 0.0)$$

$$T - SFCA_\Theta(g_1, g_2, g_3) = (0.4752,\ 0.0,\ 0.0)$$

$$Associated\ IP - T - SFOWA_\vee(g_1, g_2, g_3) = (0.6423,\ 0.0,\ 0.0)$$

$$Associated\ IP - T - SFOWA_\wedge(g_1, g_2, g_3) = (0.4742,\ 0.0,\ 0.0)$$

The above aggregation results seem meaningless as these averaging operators do not aggregate abstinence and non-membership value because one of the abstinence and non-membership value of given data is zero. So the results obtained through these averaging aggregation operators are not valid. To overcome this shortcoming we used interactive averaging aggregation operators. The results obtained by using interactive operators defined in Definitions 17–20 will be

$$T - SFOWIA_w(g_1, g_2, g_3) = (0.5846,\ 0.3793,\ 0.8617)$$

$$IP - T - SFOWIA_P(g_1, g_2, g_3) = (0.5452,\ 0.4205,\ 0.8973)$$

$$T - SFCIA_\Theta(g_1, g_2, g_3) = (0.4752,\ 0.4554,\ 0.9260)$$

$$Associated\ IP - T - SFOWIA_\vee(g_1, g_2, g_3) = (0.6423,\ 0.4571,\ 0.7420)$$

$$Associated\ IP - T - SFOWIA_\wedge(g_1, g_2, g_3) = (0.4742,\ 0.2772,\ 0.9279)$$

The proposed interactive operators aggregate all membership, abstinence, and non-membership values. This shows the superiority of interactive aggregation operators and the results obtained using these interactive operators are more reliable.

## 5. Algorithm for MADM based on the Proposed Operators

In this section, an algorithm was developed to solve the MADM problem using the proposed averaging aggregation and interactive aggregation operators, as well as an MADM example solved by the algorithm.

Consider a set of alternatives $D = \{d_1, d_2, \ldots\ldots, d_l\}$ and set of attributes $G = \{g_1, g_2, \ldots\ldots, g_k\}$ having a weight vector $w = \{w_1, w_2, \ldots\ldots, w_k\}$, set of probabilities associated with them is $P = \{\lambda_1, \lambda_2, \ldots\ldots, \lambda_k\}$, and the associated immediate probabilities are $P_{\rho(j)} = \{\lambda'_{\rho(1)}, \lambda'_{\rho(2)}, \ldots\ldots, \lambda'_{\rho(k)}\}$. The weight vector and set of probabilities have the same condition that the sum of weights and probabilities must equal to 1 and weight and probabilities belong to the closed unit interval. The fuzzy measure $\Theta$ has been calculated for all subsets of $\{d_1, d_2, \ldots\ldots, d_l\}$. Then, to find the finest alternative among the feasible one, we summarized the following steps based on the proposed aggregation operators.

Step 1. Rate the given alternatives under the different set of attributes by an expert in terms of T-spherical fuzzy numbers and summarize in the decision matrix as follows:

$$T = \begin{pmatrix} (m_{11}, i_{11}, n_{11}) & (m_{12}, i_{12}, n_{12}) & \ldots & \ldots & (m_{1k},\ i_{ik},\ n_{ik}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (m_{l1}, i_{l1}, n_{l1}) & (m_{l2}, i_{l2}, n_{l2}) & \ldots & \ldots & (m_{lk},\ i_{lk},\ n_{lk}) \end{pmatrix}$$

Step 2. Normalize the data, if required by converting the cost type ratings into the benefit type by using the following equation:

$$r_{lk} = \begin{cases} (m_{lk},\ i_{lk},\ n_{lk});\ for\ benefit\ type\ attributes \\ (n_{lk},\ i_{lk},\ m_{lk});\ for\ cost\ type\ attributes \end{cases}$$

and obtain the normalized decision matrix $R = (r_{lk})$.

Step 3. Find the value of t for which information matrix R lie in T-spherical fuzzy environment, i.e., find the smallest value of $t$ that satisfies the condition $m_{lk}^t + i_{lk}^t + n_{lk}^t \leq 1$ for all $l, k$.

Step 4. Utilize the normalized data and the value of $t$, aggregate all the numbers into the collective ones by using the aggregation operators such as IP-T-SFOWA, T-SFCA, associated IP-T-SFOWA, etc. The resultant number is denoted by $T_k = (m_k,\ i_k,\ n_k)$ for $k = 1, 2, \ldots, l$.

Step 5. Compute the score value of the obtained number $T_k = (m_k,\ i_k,\ n_k)$ by using equation

$$SC(T_k) = m_k^t - i_k^t - n_k^t + \left( \frac{e^{m_k^t - i_k^t - n_k^t}}{e^{m_k^t - i_k^t - n_k^t} + 1} - \frac{1}{2} \right) r_k^t,$$

where $r_k^t = 1 - m_k^t - i_k^t - n_k^t$

Step 6. Rank the alternatives based on the score values and select the best one.

*Numerical Example 2*

To save the non-renewable energy resources and the environment, the use of renewable energy plays a significant role in the production of electricity. Solar cells are the best renewable resource when it comes to energy. There are several types of solar cells, but few of them are studied in our application. Solar cells made with inorganic semiconductors like crystalline silicon solar cell, a solar cell with advanced III-V thin layer, amorphous silicon solar cell, cadmium telluride solar cell, etc., are expensive and their use has been confined to a few technological options. On the other hand, solar cells made with organic semiconductors like a dye-sensitized solar cell, can be processed on large surfaces at a relatively low temperature but they also have some serious problems, i.e., the degradation of their compounds (plastics) and lack of less efficiency (5–11%).

A MARCO company is situated in Islamabad, Pakistan. This factory manufactures PVC pipes and plastic water tanks. Due to the load shedding of electricity, the company is unable to meet the demand. To overcome the deficit of supply and demand, the company wants to generate electricity using solar energy, for which they have to select the best solar cell that increases production or efficiency, minimizes cost, and at the same time confers high maturity and reliability. They have a set of alternatives $D = \{d_1, d_2, d_3, d_4, d_5\}$ where:

$d_1$ : Amorphous silicon solar cell;
$d_2$ : dye-sensitized solar cell;
$d_3$ : cadmium telluride solar cell;
$d_4$ : solar cell with advanced III-V thin layer with tracking systems for solar concentration;
$d_5$ : crystalline silicon solar cell.

Experts have evaluated these alternatives under the consideration of the following attributes: $G = \{g_1, g_2, g_3\}$

$g_1$ : Cost;
$g_2$ : efficiency in energy conversion;
$g_3$ : eeat tolerance.

The experts give information in T-spherical fuzzy numbers after evaluation, as in Table 3.

**Table 3.** Decision matrix.

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $d_1$ | $(0.51, 0.42, 0.87)$ | $(0.54, 0.21, 0.44)$ | $(0.47, 0.36, 0.81)$ |
| $d_2$ | $(0.53, 0.33, 0.84)$ | $(0.00, 0.23, 0.47)$ | $(0.67, 0.11, 0.55)$ |
| $d_3$ | $(0.39, 0.26, 0.77)$ | $(0.73, 0.38, 0.59)$ | $(0.64, 0.41, 0.52)$ |
| $d_4$ | $(0.00, 0.34, 0.93)$ | $(0.91, 0.27, 0.56)$ | $(0.66, 0.19, 0.79)$ |
| $d_5$ | $(0.22, 0.46, 0.78)$ | $(0.69, 0.52, 0.42)$ | $(0.59, 0.41, 0.72)$ |

Assume that the $g_1$ is the cost type attribute. We normalize the given information by converting the cost type into benefit type and obtain the normalized decision matrix given in Table 4.

**Table 4.** Normalized decision matrix.

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $d_1$ | $(0.87, 0.42, 0.51)$ | $(0.54, 0.21, 0.44)$ | $(0.47, 0.36, 0.81)$ |
| $d_2$ | $(0.84, 0.33, 0.53)$ | $(0.00, 0.23, 0.47)$ | $(0.67, 0.11, 0.55)$ |
| $d_3$ | $(0.77, 0.26, 0.39)$ | $(0.73, 0.38, 0.59)$ | $(0.64, 0.41, 0.52)$ |
| $d_4$ | $(0.93, 0.34, 0.00)$ | $(0.91, 0.27, 0.56)$ | $(0.66, 0.19, 0.79)$ |
| $d_5$ | $(0.78, 0.46, 0.22)$ | $(0.69, 0.52, 0.42)$ | $(0.59, 0.41, 0.72)$ |

As for $t = 1$, $0.87 + 0.42 + 0.51 = 1.8 \notin [0,\ 1]$,
As for $t = 2$, $0.87^2 + 0.42^2 + 0.51^2 = 1.19 \notin [0,\ 1]$
As for $t = 3$, $0.87^3 + 0.42^3 + 0.51^3 = 0.865 \in [0,\ 1]$

Similarly, for $t = 3$ all the information given in Table 4 are T-SFNs. The interaction of states of nature and weights of given attributes is in Table 5.

**Table 5.** Interaction of states of nature and weights.

| | $g_1$ | $g_2$ | $g_3$ | **Risk Importance** |
|---|---|---|---|---|
| $g_1$ | - | 0.150 | 0.100 | 0.250 |
| $g_2$ | 0.150 | - | 0.250 | 0.400 |
| $g_3$ | 0.100 | 0.250 | - | 0.350 |

With the help of Table 5, the interaction between attributes will be $I_{g_1} = 0.250$, $I_{g_2} = 0.400$, $I_{g_3} = 0.350$, $I_{g_1 g_2} = 0.150$, $I_{g_1 g_3} = 0.100$, and $I_{g_2 g_3} = 0.250$. The fuzzy measure can be calculated with the help of the interaction using the following relationship:

$$\Theta\left(\left\{g_j\right\}\right) = I_{g_j} - \frac{1}{2} \sum_{g \in G\{g_j\}} I_{g_j g}$$

$$\Theta\left(\left\{g_j, g_k\right\}\right) = I_{g_j} + I_{g_k} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_j g} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_k g}$$

$$j, k = 1, 2, 3, 4.\ j \neq k$$

$$\Theta(\phi) = 0, \Theta(G) = 1.$$

The fuzzy measures will be $\Theta(\phi) = 0$, $\Theta(\{g_1\}) = 0.125$, $\Theta(\{g_2\}) = 0.200$, $\Theta(\{g_3\}) = 0.175$, $\Theta(\{g_1, g_2\}) = 0.325$, $\Theta(\{g_1, g_3\}) = 0.300$, $\Theta(\{g_2, g_3\}) = 0.375$, and $\Theta(G) = 1$. The immediate probabilities for every possible permutation are summarized in Table 6. An associated immediate probability for every possible permutation is summarized in Table 7.

**Table 6.** Immediate probability.

| | $\lambda_1'$ | $\lambda_2'$ | $\lambda_3'$ |
|---|---|---|---|
| $\sigma = (g_1, g_2, g_3)$ | 0.125 | 0.200 | 0.675 |
| $\sigma = (g_1, g_3, g_2)$ | 0.125 | 0.700 | 0.175 |
| $\sigma = (g_2, g_1, g_3)$ | 0.200 | 0.200 | 0.600 |
| $\sigma = (g_2, g_3, g_1)$ | 0.700 | 0.200 | 0.100 |
| $\sigma = (g_3, g_1, g_2)$ | 0.125 | 0.700 | 0.175 |
| $\sigma = (g_3, g_2, g_1)$ | 0.700 | 0.125 | 0.175 |

**Table 7.** Associated immediate probability.

| | $\lambda_{\rho(1)}'$ | $\lambda_{\rho(2)}'$ | $\lambda_{\rho(3)}'$ |
|---|---|---|---|
| $\sigma = (g_1, g_2, g_3)$ | 0.0899 | 0.2302 | 0.6798 |
| $\sigma = (g_1, g_3, g_2)$ | 0.0839 | 0.7517 | 0.1644 |
| $\sigma = (g_2, g_1, g_3)$ | 0.1470 | 0.2353 | 0.6176 |
| $\sigma = (g_2, g_3, g_1)$ | 0.6034 | 0.2759 | 0.1207 |
| $\sigma = (g_3, g_1, g_2)$ | 0.0839 | 0.7517 | 0.1644 |
| $\sigma = (g_3, g_2, g_1)$ | 0.6114 | 0.1747 | 0.2140 |

The aggregated values by proposed operators (Definitions 12–15) are shown in Table 8.

**Table 8.** Aggregated values using aggregation operator.

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ |
|---|---|---|---|---|---|
| $T-SFOWA_w$ | $\begin{pmatrix} 0.6790, \\ 0.3016, \\ 0.5652 \end{pmatrix}$ | $\begin{pmatrix} 0.6658, \\ 0.1944, \\ 0.5117 \end{pmatrix}$ | $\begin{pmatrix} 0.7154, \\ 0.4963, \\ 0.5090 \end{pmatrix}$ | $\begin{pmatrix} 0.8718 \\ 0.2529, \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.6911, \\ 0.4641, \\ 0.4315 \end{pmatrix}$ |
| $IP-T-SFOWA_P$ | $\begin{pmatrix} 0.6074, \\ 0.3002, \\ 0.6088 \end{pmatrix}$ | $\begin{pmatrix} 0.6328, \\ 0.1662, \\ 0.5163 \end{pmatrix}$ | $\begin{pmatrix} 0.6967, \\ 0.3764, \\ 0.5257 \end{pmatrix}$ | $\begin{pmatrix} 0.8397, \\ 0.2330, \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.6630, \\ 0.4545, \\ 0.5076 \end{pmatrix}$ |
| $T-SFCA_\Theta$ | $\begin{pmatrix} 0.5994, \\ 0.3295, \\ 0.6766 \end{pmatrix}$ | $\begin{pmatrix} 0.6679, \\ 0.1462, \\ 0.5305 \end{pmatrix}$ | $\begin{pmatrix} 0.6812, \\ 0.3815, \\ 0.5145 \end{pmatrix}$ | $\begin{pmatrix} 0.7988, \\ 0.2192, \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.6466, \\ 0.4362, \\ 0.5574 \end{pmatrix}$ |
| $Ass.IP-T-SFOWA_\vee$ | $\begin{pmatrix} 0.7982, \\ 0.2432, \\ 0.4925 \end{pmatrix}$ | $\begin{pmatrix} 0.7748, \\ 0.1439, \\ 0.4872 \end{pmatrix}$ | $\begin{pmatrix} 0.7474, \\ 0.3050, \\ 0.4459 \end{pmatrix}$ | $\begin{pmatrix} 0.9111, \\ 0.2171, \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.7423, \\ 0.4376, \\ 0.3034 \end{pmatrix}$ |
| Associated $IP-T-SFOWA_\wedge$ | $\begin{pmatrix} 0.5748, \\ 0.3600, \\ 0.6752 \end{pmatrix}$ | $\begin{pmatrix} 0.5008, \\ 0.2616, \\ 0.5287 \end{pmatrix}$ | $\begin{pmatrix} 0.6787, \\ 0.3867, \\ 0.5581 \end{pmatrix}$ | $\begin{pmatrix} 0.7954, \\ 0.2974, \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.6411, \\ 0.4950, \\ 0.5717 \end{pmatrix}$ |

This seems meaningless, however, because the averaging aggregation operators cannot aggregate the non-membership value of $d_4$, so the valid aggregate of $d_4$ is not obtained.

The score values of aggregated operators are listed in Table 9 and the corresponding score values the ranking of alternatives is shown in the Table 10.

**Table 9.** Score value.

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ |
|---|---|---|---|---|---|
| $T-SFOWA_w$ | 0.1176 | 0.1754 | 0.1226 | 0.6967 | 0.1687 |
| $IP-T-SFOWA_P$ | −0.0324 | 0.1279 | 0.1556 | 0.6352 | 0.0749 |
| $T-SFCA_\Theta$ | −0.1445 | 0.1655 | 0.1397 | 0.5579 | 0.0159 |
| $Ass.IP-T-SFOWA_\vee$ | 0.4077 | 0.3822 | 0.3353 | 0.7877 | 0.3326 |
| Associated $IP-T-SFOWA_\wedge$ | −0.1832 | −0.0472 | 0.0901 | 0.5320 | −0.0493 |

**Table 10.** Rankings order of the alternatives.

| Operators | Rankings |
|---|---|
| $T-SFOWA_w$ | $d_4 \geq d_2 \geq d_5 \geq d_3 \geq d_1$ |
| $IP-T-SFOWA_P$ | $d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$ |
| $T-SFCA_\Theta$ | $d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$ |
| $Ass.IP-T-SFOWA_\vee$ | $d_4 \geq d_1 \geq d_2 \geq d_3 \geq d_5$ |
| Associated $IP-T-SFOWA_\wedge$ | $d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$ |

The ranking results in Table 10 are not accurate because the non-membership value of $d_4$ in averaging aggregation operators has not been aggregated.

Now, the aggregated values of all proposed interactive aggregation operators are shown in Table 11, the corresponding score values are listed in Table 12, and the ranking of alternatives summarized in Table 13.

**Table 11.** Aggregated values by interactive aggregation operators.

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ |
|---|---|---|---|---|---|
| $T-SFOWIA_w$ | $\begin{pmatrix}0.6790,\\0.3389,\\0.6681\end{pmatrix}$ | $\begin{pmatrix}0.6658,\\0.2433,\\0.5577\end{pmatrix}$ | $\begin{pmatrix}0.7154,\\0.3700,\\0.5365\end{pmatrix}$ | $\begin{pmatrix}0.8718\\0.2722,\\0.5942\end{pmatrix}$ | $\begin{pmatrix}0.6911,\\0.4721,\\0.5452\end{pmatrix}$ |
| $IP-T-SFOWIA_P$ | $\begin{pmatrix}0.6074,\\0.3315,\\0.7050\end{pmatrix}$ | $\begin{pmatrix}0.6328,\\0.2138,\\0.5496\end{pmatrix}$ | $\begin{pmatrix}0.6967,\\0.3857,\\0.5452\end{pmatrix}$ | $\begin{pmatrix}0.8397,\\0.2508,\\0.6502\end{pmatrix}$ | $\begin{pmatrix}0.6630,\\0.4639,\\0.6019\end{pmatrix}$ |
| $T-SFCIA_\Theta$ | $\begin{pmatrix}0.5994,\\0.3498,\\0.7428\end{pmatrix}$ | $\begin{pmatrix}0.6679,\\0.1991,\\0.5552\end{pmatrix}$ | $\begin{pmatrix}0.6812,\\0.3911,\\0.5303\end{pmatrix}$ | $\begin{pmatrix}0.7988,\\0.2384,\\0.6881\end{pmatrix}$ | $\begin{pmatrix}0.6466,\\0.4436,\\0.6443\end{pmatrix}$ |
| $Ass.IP-T-SFOWIA_\vee$ | $\begin{pmatrix}0.7982,\\0.3853,\\0.5960\end{pmatrix}$ | $\begin{pmatrix}0.7748,\\0.2936,\\0.5204\end{pmatrix}$ | $\begin{pmatrix}0.7474,\\0.3940,\\0.4757\end{pmatrix}$ | $\begin{pmatrix}0.9111,\\0.3109,\\0.4613\end{pmatrix}$ | $\begin{pmatrix}0.7423,\\0.5008,\\0.4089\end{pmatrix}$ |
| Associated $IP-T-SFOWIA_\wedge$ | $\begin{pmatrix}0.5748,\\0.2763,\\0.7477\end{pmatrix}$ | $\begin{pmatrix}0.5008,\\0.1912,\\0.5637\end{pmatrix}$ | $\begin{pmatrix}0.6787,\\0.3249,\\0.5729\end{pmatrix}$ | $\begin{pmatrix}0.7954,\\0.2338,\\0.6963\end{pmatrix}$ | $\begin{pmatrix}0.6411,\\0.4456,\\0.6486\end{pmatrix}$ |

**Table 12.** Score values.

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ |
|---|---|---|---|---|---|
| $T-SFOWIA_w$ | −0.0262 | 0.1211 | 0.1783 | 0.4442 | 0.0691 |
| $IP-T-SFOWIA_P$ | −0.1846 | 0.0887 | 0.1318 | 0.3012 | −0.0291 |
| $T-SFCIA_\Theta$ | −0.2568 | 0.1344 | 0.1199 | 0.1768 | −0.0922 |
| $Ass.IP-T-SFOWIA_\vee$ | 0.2528 | 0.3262 | 0.2744 | 0.6456 | 0.2363 |
| Associated $IP-T-SFOWIA_\wedge$ | −0.2723 | −0.0709 | 0.1008 | 0.1584 | −0.1070 |

**Table 13.** Rankings ordering.

| Operators | Rankings |
|---|---|
| $T-SFOWIA_w$ | $d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$ |
| $IP-T-SFOWIA_P$ | $d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$ |
| $T-SFCIA_\Theta$ | $d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$ |
| $Ass.IP-T-SFOWIA_\vee$ | $d_4 \geq d_2 \geq d_3 \geq d_1 \geq d_5$ |
| Associated $IP-T-SFOWIA_\wedge$ | $d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$ |

It is observed from Tables 10–13 that T-SFOWA and T-SFOWIA operators do not reflect interactions between some states of nature. While the T-SFCA and T-SFCIA operators reflect interactions between some states of nature, associated IP-T-SFOWA and associated IP-TSFOWIA operators reflect interactions among all states of nature.

## 6. Advantages

In this section, the advantages of the proposed operators over existing operators are discussed and some conditions are also discussed under which the proposed operators become valid for existing operators.

Consider associated IP-T-SFOWIA operators

$$Ass.IP-T-SFOWIA_\vee(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left( \begin{array}{c} \sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\ \sqrt[t]{\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)} \end{array} \right) \quad (22)$$

and

$$
\begin{gathered}
Ass.IP-T-SFOWIA_{\wedge}(T_1,T_2,\ldots\ldots,T_k) \\
=\left(\begin{array}{c}
\sqrt[t]{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)},\ \sqrt[t]{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\
\sqrt[t]{\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}
\end{array}\right)
\end{gathered} \tag{23}
$$

(1) For $t=2$, Equations (22) and (23) reduce the associated immediate probability spherical fuzzy ordered weighted interaction averaging (Associated IP-SFOWIA) operator

$$
\begin{gathered}
Ass.IP-SFOWIA_{\vee}(T_1,T_2,\ldots\ldots,T_k) \\
=\left(\begin{array}{c}
\sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)},\ \sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\
\sqrt{\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}
\end{array}\right)
\end{gathered}
$$

and

$$
\begin{gathered}
Ass.IP-SFOWIA_{\wedge}(T_1,T_2,\ldots\ldots,T_k) \\
=\left(\begin{array}{c}
\sqrt{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)},\ \sqrt{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}, \\
\sqrt{\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}
\end{array}\right)
\end{gathered}
$$

(2) For $t=1$, Equations (1) and (2) reduces the associated immediate probability picture fuzzy ordered weighted interaction averaging (Associated IP-PFOWIA) operator

$$
\begin{gathered}
Ass.IP-PFOWIA_{\wedge}(T_1,T_2,\ldots\ldots,T_k) \\
=\left(\begin{array}{c}
1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\ 1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \\
\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)
\end{array}\right)
\end{gathered}
$$

and

$$
\begin{gathered}
Ass.IP-PFOWIA_{\wedge}(T_1,T_2,\ldots\ldots,T_k) \\
=\left(\begin{array}{c}
1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\ 1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \\
\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^t_{\sigma(j)}-i^t_{\sigma(j)}-n^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i^t_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)
\end{array}\right)
\end{gathered}
$$

(3) For $t=2$ and $i=0$ Equations (22) and (23) reduces the associated immediate probability Pythagorean fuzzy ordered weighted interaction averaging (Associated IP-PyFOWIA) operator

$$
\begin{gathered}
Ass.IP-PyFOWIA_{\vee}(T_1,T_2,\ldots\ldots,T_k) \\
=\sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^2_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)},\ \sqrt{\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^2_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m^2_{\sigma(j)}-n^2_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)}
\end{gathered}
$$

and

$$Ass.IP - PyFOWIA_{\wedge}(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left( \sqrt{1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}, \sqrt{\max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right) - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^2 - n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)} \right)$$

(4) For $t = 1$ and $i = 0$, Equations (22) and (23) reduces the associated IP-IFOWIA operator

$$Ass.IP - IFOWIA_{\vee}(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left(1 - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \min_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right)$$

and

$$Ass.IP - IFOWIA_{\wedge}(T_1, T_2, \ldots\ldots, T_k)$$
$$= \left(1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right), \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right) - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\right)$$

Similarly, we can reduce the T-SFOWIA, IP-T-SFOWIA, T-SFCIA, and T-SFCCIA operators. Another advantage of the proposed operators is that they aggregate information where the existing operator fails.

Next, we investigate a comparison analysis between the proposed method and existing work. The existing operators have some limitations that the existing operators cannot handle the information given in PyFSs, PFSs, SFSs, and T-SFSs. The proposed operators are mostly generalized and so they can handle the information given in IFSs, PyFSs, PFSs, SFSs, and T-SFSs. Here with the help of an example discussed in a previous work [42], it is shown that the proposed operators can solve the information given in IFSs.

**Example 3.** *Consider a normalized decision matrix in which information is given in IFNs (Table 14).*

**Table 14.** Decision Matrix for Example 3.

| | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|
| $g_1$ | (0.60, 0.30) | (0.50, 0.20) | (0.60, 0.35) |
| $g_2$ | (0.60, 0.30) | (0.50, 0.20) | (0.20, 0.00) |
| $g_3$ | (0.31, 0.00) | (0.50, 0.20) | (0.60, 0.35) |
| $g_4$ | (0.20, 0.00) | (0.50, 0.20) | (0.60, 0.30) |
| $g_5$ | (0.70, 0.30) | (0.40, 0.20) | (0.80, 0.10) |
| $g_6$ | (0.60, 0.30) | (0.80, 0.20) | (0.50, 0.20) |

The given information can be written as T-spherical fuzzy information and is henceforth summarized in Table 15.

**Table 15.** Decision Matrix in T-SF information.

| | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|
| $g_1$ | (0.60, 0.00, 0.30) | (0.50, 0.00, 0.20) | (0.60, 0.00, 0.35) |
| $g_2$ | (0.60, 0.00, 0.30) | (0.50, 0.00, 0.20) | (0.20, 0.00, 0.00) |
| $g_3$ | (0.31, 0.00, 0.00) | (0.50, 0.00, 0.20) | (0.60, 0.00, 0.35) |
| $g_4$ | (0.20, 0.00, 0.00) | (0.50, 0.00, 0.20) | (0.60, 0.00, 0.30) |
| $g_5$ | (0.70, 0.00, 0.30) | (0.40, 0.00, 0.20) | (0.80, 0.00, 0.10) |
| $g_6$ | (0.60, 0.00, 0.30) | (0.80, 0.00, 0.20) | (0.50, 0.00, 0.20) |

The weight vector for attributes will be $w = \{0.25,\ 0.40,\ 0.35\}$ and fuzzy measures are as defined in [41]:

$$\Theta(\phi) = 0,\ \Theta(\{d_1\}) = 0.175, \Theta(\{d_2\}) = 0.125, \Theta(\{d_3\}) = 0.100, \Theta(\{d_1, d_2\}) = 0.500,$$
$$\Theta(\{d_1, d_3\}) = 0.425, \Theta(\{d_2, d_3\}) = 0.475, \text{ and } \Theta(\{d_1,\ d_2,\ d_3\}) = 1.$$

Immediate probabilities and associated immediate probabilities for all possible orders are computed in Tables 16 and 17, respectively.

**Table 16.** Immediate probabilities.

| | $\lambda'_1$ | $\lambda'_2$ | $\lambda'_3$ |
|---|---|---|---|
| $\sigma = (d_1, d_2, d_3)$ | 0.175 | 0.325 | 0.500 |
| $\sigma = (d_1, d_3, d_2)$ | 0.175 | 0.575 | 0.250 |
| $\sigma = (d_2, d_1, d_3)$ | 0.375 | 0.125 | 0.500 |
| $\sigma = (d_2, d_3, d_1)$ | 0.525 | 0.175 | 0.350 |
| $\sigma = (d_3, d_1, d_2)$ | 0.325 | 0.575 | 0.100 |
| $\sigma = (d_3, d_2, d_1)$ | 0.525 | 0.375 | 0.100 |

**Table 17.** Associated immediate probability.

| | $\lambda'_{\rho(1)}$ | $\lambda'_{\rho(2)}$ | $\lambda'_{\rho(3)}$ |
|---|---|---|---|
| $\sigma = (d_1, d_2, d_3)$ | 0.1885 | 0.3500 | 0.4615 |
| $\sigma = (d_1, d_3, d_2)$ | 0.1815 | 0.5963 | 0.2222 |
| $\sigma = (d_2, d_1, d_3)$ | 0.4038 | 0.1346 | 0.4615 |
| $\sigma = (d_2, d_3, d_1)$ | 0.5526 | 0.1316 | 0.3158 |
| $\sigma = (d_3, d_1, d_2)$ | 0.3297 | 0.5833 | 0.0870 |
| $\sigma = (d_3, d_2, d_1)$ | 0.5326 | 0.3804 | 0.0870 |

As $0.6 + 0.0 + 0.3 = 0.9 \in [0,\ 1]$ for $t = 1$, all values lie in T-SFSs. Thus, here $t = 1$ is taken. Then the aggregated values of all aggregation operators defined in Definitions 15–24 for $t = 1$ are summarized in Table 18.

**Table 18.** Aggregated values.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| T-SFOWA | (0.57, 0.00, 0.27) | (0.47, 0.00, 0.00) | (0.48, 0.00, 0.00) | (0.45, 0.00, 0.00) | (0.66, 0.00, 0.19) | (0.66, 0.00, 0.23) |
| T-SFCA | (0.57, 0.00, 0.26) | (0.54, 0.00, 0.00) | (0.42, 0.00, 0.00) | (0.51, 0.00, 0.00) | (0.72, 0.00, 0.16) | (0.73, 0.00, 0.23) |
| Associated IP-T-SFOWA$_\vee$ | (0.588, 0.00, 0.244) | (0.537, 0.00, 0.00) | (0.521, 0.00, 0.00) | (0.528, 0.00, 0.00) | (0.725, 0.00, 0.154) | (0.718, 0.00, 0.211) |
| Associated IP-T-SFOWA$_\wedge$ | (0.543, 0.00, 0.305) | (0.404, 0.00, 0.00) | (0.418, 0.00, 0.00) | (0.385, 0.00, 0.00) | (0.584, 0.00, 0.228) | |

The score values of the aggregated values are given in Table 19 and the ranking of all alternatives through score values or accuracy function is represented in Table 20.

**Table 19.** Score Values.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| T-SFOWA | 0.3119 | 0.5312 | 0.5412 | 0.5108 | 0.4873 | 0.4416 |
| T-SFCA | 0.3231 | 0.6006 | 0.4800 | 0.5712 | 0.5764 | 0.5049 |
| Associated IP-T-SFOW $A_\vee$ | 0.3583 | 0.5977 | 0.5820 | 0.5889 | 0.5878 | 0.5158 |
| Associated IP-T-SFOW $A_\wedge$ | 0.2470 | 0.4634 | 0.4779 | 0.4435 | 0.3054 | 0.3678 |

**Table 20.** Rankings of Alternatives.

| **Operators** | **Rankings** |
|---|---|
| T-SFOWA | $g_3 > g_2 > g_4 > g_5 > g_6 > g_1$ |
| T-SFCA | $g_2 > g_5 > g_4 > g_6 > g_3 > g_1$ |
| Associated IP-T-SFOW $A_\vee$ | $g_2 > g_4 > g_5 > g_3 > g_6 > g_1$ |
| Associated IP-T-SFOW $A_\wedge$ | $g_3 > g_2 > g_4 > g_6 > g_5 > g_1$ |

From the above example, it is clear that the results obtained from the proposed operators are similar to the existing operators. This proves that the proposed operators are generalizations of existing operators.

## 7. Conclusions

In this manuscript, we developed an extension of the existing immediate probability, Choquet averaging, and associated immediate probability averaging operators by utilizing the concept of T-SFSs. Herein, we pointed out that the existing operators have limitations and decision makers are not free to make a decision freely—they fail to work when the information is given in PyFSs, PFSs, SFSs, and T-SFSs. To overcome this shortcoming, averaging aggregation operators are defined in the most generalized tool of uncertainty called T-SFSs. However, they fail under certain conditions. To overcome this defect, some interactive averaging operators are defined and a comparison between these proposed operators is developed with the help of an example. The existing score values have shortcomings: they do not involve abstinence so a new score function is proposed in which all degrees are involved. With the help of this new score function, the different aggregated values are compared. Further, we developed the reliability of an application of the MADM problem. The advantages of proposed work are also discussed. The comparative study of existing and proposed operators is also developed with the help of an example. In future research, it would be interesting to develop some immediate probability geometric and power aggregation operators for the diverse fuzzy environment [43–47].

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