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Entangled-Based Quantum Wavelength-Division-Multiplexing and Multiple-Access Networks

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Abstract: This paper investigates the mathematical model of the quantum wavelength-division-multiplexing (WDM) network based on the entanglement distribution with the least required wavelengths and passive devices. By adequately utilizing wavelength multiplexers, demultiplexers, and star couplers, N wavelengths are enough to distribute the entanglement among each pair of N users. Moreover, the number of devices employed is reduced by substituting a waveguide grating router for multiplexers and demultiplexers. Furthermore, this study examines implementing the BBM92 quantum key distribution in an entangled-based quantum WDM network. The proposed scheme in this paper may be applied to potential applications such as teleportation in entangled-based quantum WDM networks.

Keywords: entanglement; quantum network; wavelength division multiplexing

1. Introduction

The entangled state introduced by the EPR (Einstein, Podolsky, and Rosen) paradox in 1935 [1] threatened the completeness of quantum mechanics. Nevertheless, a violation of a Bell-type inequality based on the local hidden variables theory [2,3] confirmed the counter-intuitive property that arose from the theory of quantum mechanics, e.g., non-locality [4,5]. Meanwhile, the entangled states have become one of the key quantum mechanical resources in upcoming quantum technologies, such as quantum communications [6,7], quantum computations [8], quantum teleportations [9], and quantum metrologies [10]. An extraordinary download speed via quantum teleportations [11] and unconditional security without trusting any devices [5] are some of the consequences of utilizing entanglement resources toward a futuristic quantum internet [12–14].

The effective distribution of these resources, entangled states, for different usages functioning as quantum communications [6,15], distributed computing [16], and so on requires reliable quantum networks [17]. Conventionally, long-haul communication needs amplifiers. Due to the no-cloning theorem arising from quantum mechanic formalism, perfect amplification is impossible in quantum networks. Instead, quantum repeaters comprising entanglement swapping [18,19] and quantum memories must be utilized for entanglement distribution and distillation over long distances. However, quantum repeaters have a long way to go to become cost-effective and reliable commercial devices. In this regard, a three-node quantum entangled-based network of remote solid-state qubits has been realized [20]. Distributing multipartite entanglement over noisy quantum networks has recently been studied in [21]. On the other hand, the local access network can distribute the entanglement among local users utilizing a broadband entangled photon source (EPS) [22,23] without relying on quantum repeaters. An untrusted entanglement provider is responsible for assigning photon pairs produced by an EPS to many subscribers. Several studies and practical realizations, such as entanglement-based QKD networks or remote



Citation: Bathaee, M.; Salehi, J.A. Entangled-Based Quantum Wavelength-Division-Multiplexing and Multiple-Access Networks. *Entropy* **2023**, *25*, 1658. <https://doi.org/10.3390/e25121658>

Academic Editor: Rosario Lo Franco

Received: 17 October 2023

Revised: 29 November 2023

Accepted: 12 December 2023

Published: 14 December 2023



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state preparation (RSP) with passive and active devices, have already been studied and implemented [24–31].

Based on the strategy of quantum state sharing among all end users, any quantum home network can be categorized into two well-known classes: a prepare and measure-based network and entanglement distribution across a quantum network. While the latter is of uppermost importance in a future-proof quantum internet, it is also necessary to have loophole-free device-independent security in local networks due to imperfections in real employed devices [32]. Using the WDM scheme studied in [15], the entanglement distribution in a metropolitan optical network consists of tree-type access networks. Different experiments have been performed in the point-to-multipoint network architecture for entanglement distribution, which utilized wavelength demultiplexers and beam splitters [33,34]. Preserving non-locality in a noisy quantum network has been recently studied in the noisy intermediate-scale quantum (NISQ) era [35]. For example, ref. [36] used variational quantum optimization to investigate the criteria for distributing entanglement states and non-locality in the network using entanglement swapping. Future work can explore similar local access entangled-based WDM networks in the presence of noise, which is out of the scope of the current paper.

This paper studies entangled-based quantum WDM (QWDM) networks utilizing passive devices for access networking in the same terminology presented in [37]. We show to the best of our knowledge for the first time how, by an intelligent utilization of a star coupler and a waveguide grating router, an untrusted entanglement provider can distribute entanglement among $2N$ users just by selecting $2N$ different wavelengths from a broadband entangled source. Figure 1 illustrates the building block of our scheme for four subscribers. As depicted in Figure 1, a quantum signal composed of a wide-band entangled photon source passes through a 3×3 star coupler and gets involved in three input ports of a 4×4 waveguide grating router (WGR). Therefore, each subscriber accesses three different wavelengths out of four WGR working wavelengths. Because the four-wavelength scheme provides two distinct state pairs of entangled photons, this scheme guarantees the sharing of at least one entangled photon state between each pair of users. This work focuses on distributing entanglement in a local quantum network without utilizing entanglement swapping as experimentally investigated in previous works, such as [33,34]. This paper aims to theoretically investigate the evolution of a broadband entangled state as it passes through various WDM network components. This investigation will help to determine the necessary number of wavelengths and the rate of entanglement distribution, which are essential factors in characterizing a future quantum network.

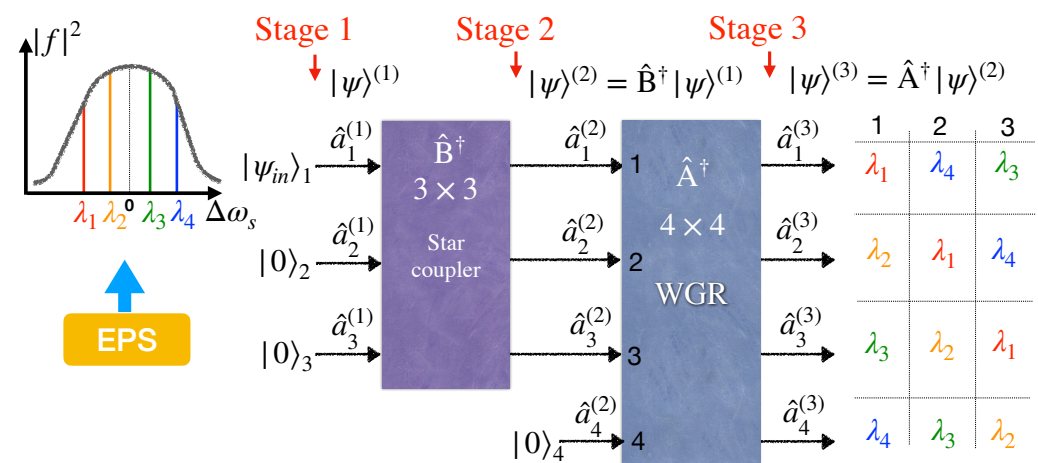


Figure 1. Schematic configuration of four quantum entanglement distribution network users. EPS is an entangled photon source. WGR stands for waveguide grating router.

This paper is organized as follows. Section 2 explores the entangled-based quantum WDM (QWDM) network with the least required number of wavelengths and devices. This section first studies the network utilizing a waveguide grating router (WGR) and a star coupler for a four-user network. Then, the scheme is generalized for N users. In Section 3, a combination of a demultiplexer and multiplexers is used in place of the WGR, and several star couplers with a lower splitting loss are utilized. Then, in Section 4, our proposed schemes are compared to the previous related works. The well-known quantum key distribution protocol BBM92 [38] as a benchmark application for entanglement versions of the quantum WDM network is presented in Section 5. Finally, the material of this paper is concluded in Section 6.

2. Entanglement-Based Quantum Network Based on WDM

This section studies an entanglement distribution scheme between $2N$ users via a combination of an $(N + 1) \times (N + 1)$ star coupler and a $2N \times 2N$ WGR by using only $2N$ different wavelengths existing in a broadband polarization-entangled photon source [22,23]. To start, we describe the entangled photon source used. Next, we investigate the entanglement of the WDM network for four and multiple users.

2.1. Entangled Photon Source

A broadband polarization-entangled photon source can be implemented practically and commercially by a periodically poled second-order nonlinear crystal, such as periodically poled silica fiber (PPSF) [22,23]. PPSF is well known as a compensation-free broadband entangled photon source (EPS), suitable for multi-channel quantum communications. In this structure, the entangled photon pair is generated based on the type-II spontaneous parametric down-conversion (SPDC) caused by the second-order nonlinearity response of the medium to the electric input field (pump). The second-order nonlinear term of polarization in the medium can be written as $P_i^{NL} = \sum \chi_{ijk}^{(2)} : E_j(\omega)E_k(\omega)$ [39], where $\chi^{(2)}$ is the second-order susceptibility. This term is responsible for creating two entangled photons called historically signal and idler after properly pumping the system, satisfying the conservation of energy ($\omega_p = \omega_s + \omega_i$) and the phase-matching condition ($\beta_p = \beta_s + \beta_i + 2\pi/\Lambda$), where ω and β are the angular frequency and waveguide propagation constant, respectively, and Λ is the quasi-phase-matching period that appears whenever a periodically poled crystal is used for efficient entanglement generation via the SPDC procedure. The quantum Hamiltonian of this term is given by

$$\hat{H}_{\text{PDC}} \propto \int_{-\frac{L}{2}}^{\frac{L}{2}} \chi^{(2)} \hat{E}_p^{(+)}(z, t) \hat{E}_s^{(-)}(z, t) \hat{E}_i^{(-)}(z, t) dz + h.c., \quad (1)$$

where

$$\hat{E}_X^{(+)} = \hat{E}_X^{(-)\dagger} = A \sum_{m=H,V} \int d\omega_X \exp[i(\beta_X(\omega_X)z - \omega_X t)] \hat{a}_m(\omega_X) \quad X \in \{p, s, i\}, \quad (2)$$

is the positive frequency part of the quantized electric field of the pump ($X = p$), signal ($X = s$), and idler ($X = i$). The parameter A [39] includes all the necessary constants. The length of the PPSF along the $\hat{z}$ (propagation) direction is denoted by L , and $h.c.$ stands for the hermitian conjugate. Furthermore, H and V represent the electric field's horizontal and vertical polarization modes. Because the incoming pump field is strong, the pump operator can be treated as a classical field. Indeed, in the SPDC procedure, the pump field amplifies the signal and idler modes' fluctuations in the vacuum states. Therefore, in the Schrödinger picture, the evolved state by the Hamiltonian of Equation (1) becomes

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \int dt \hat{H}_{\text{PDC}}(t)} |0\rangle. \quad (3)$$

After some mathematical manipulations and keeping just the related terms [39], due to the negligible group birefringence in the PPSF [22], the biphoton output state of the EPS is expressed as

$$\begin{aligned} |\psi\rangle_{\text{PDC}} &= \frac{1}{\sqrt{2}} \int d\omega_s \int d\omega_i f(\omega_s, \omega_i) (\hat{a}_H^\dagger(\omega_s) \hat{a}_V^\dagger(\omega_i) + \hat{a}_V^\dagger(\omega_s) \hat{a}_H^\dagger(\omega_i)) |0_s\rangle |0_i\rangle \\ &= \frac{1}{\sqrt{2}} \int d\omega_s \int d\omega_i f(\omega_s, \omega_i) (|H, \omega_s\rangle |V, \omega_i\rangle + |V, \omega_s\rangle |H, \omega_i\rangle), \end{aligned} \quad (4)$$

where the joint spectral amplitude (JSA), i.e., $f(\omega_s, \omega_i)$, has been driven by [22] as

$$\begin{aligned} f(\omega_s, \omega_i) &\propto \text{sinc}\left(\frac{L}{2} \Delta\beta_f(\Delta\omega_s)\right), \\ \Delta\beta_f &\approx \beta_2 \Delta\omega_s^2, \quad \beta_2 \approx \frac{1}{2} \left(\frac{d^2\beta_H}{d\omega^2} + \frac{d^2\beta_V}{d\omega^2} \right) |_{\omega=\omega_0}. \end{aligned} \quad (5)$$

In Equation (5), $\Delta\omega_s = -\Delta\omega_i = \omega_s - \omega_0$, where the central spectrum is $\omega_0 = \omega_p/2$. It is important to note that the signal and idler frequencies of the entangled pair are equally separated from the central spectrum ω_0 of the JSA, $f(\omega_s, \omega_i)$. It is worth mentioning that the PPSF developed in [23] is coupled to a single-mode fiber, which implies that all entangled photons produced are directed in the single path (fiber), which is an important factor in utilizing this entangled source in optical fiber-based quantum communications. For instance, the star coupler based on the 2×2 fiber coupler can be easily connected to the EPS implemented by the PPSF.

2.2. Four-User Entanglement Distribution WDM Network Based on WGR

Figure 1 depicts our proposed scheme for entanglement distribution among four users via a waveguide grating router. A 3×3 star coupler (indicated by $\hat{B}^\dagger$) splits a broadband entangled signal equally into three parts as inputs of a 4×4 WGR (indicated by $\hat{A}^\dagger$). Thereby, the entangled signal produced by the EPS in stage 1 reaches stage 2 after passing the star coupler and finally distributes to users in stage 3 via the WGR as follows.

• Stage 1: Inputs

Considering Equations (4) and (5), the input state $|\psi_{in}\rangle_1$ shown in Figure 1 can be rewritten according to the creation operator $\hat{a}_1^{\dagger(1)}$, in stage 1 indicated by the superscript (1), so the input signal of the 3×3 star coupler is as follows

$$\begin{aligned} |\psi\rangle^{(1)} &= |\psi_{in}\rangle_1 |0\rangle_2 |0\rangle_3 \\ &= \frac{1}{\sqrt{2}} \iint d\omega_s d\omega_i f(\omega_s, \omega_i) (\hat{a}_{1H}^{\dagger(1)}(\omega_s) \hat{a}_{1V}^{\dagger(1)}(\omega_i) + \hat{a}_{1V}^{\dagger(1)}(\omega_s) \hat{a}_{1H}^{\dagger(1)}(\omega_i)) |0\rangle \\ &= \frac{1}{\sqrt{2}} \iint d\Delta\omega_s d\Delta\omega_i f(\Delta\omega_s, \Delta\omega_i) (\hat{a}_{1H}^{\dagger(1)}(\omega_0 + \Delta\omega_s) \hat{a}_{1V}^{\dagger(1)}(\omega_0 + \Delta\omega_i) \\ &\quad + \hat{a}_{1V}^{\dagger(1)}(\omega_0 + \Delta\omega_s) \hat{a}_{1H}^{\dagger(1)}(\omega_0 + \Delta\omega_i)) |0\rangle_1 |0\rangle_2 |0\rangle_3, \end{aligned} \quad (6)$$

where $\omega_s = \omega_0 + \Delta\omega_s$ and $\omega_i = \omega_0 + \Delta\omega_i$ and subscripts H and V specify the horizontal and vertical polarizations of the signal and idler modes. The phase-matching and conservation of energy ($\omega_s + \omega_i = 2\omega_0$) conditions imply $f(\Delta\omega_s, \Delta\omega_i)$ becomes dominant at $\Delta\omega_i = -\Delta\omega_s$. In the last line of Equation (6), $|0\rangle_2 |0\rangle_3$ corresponds to the unused inputs of the star coupler, and subindices 1, 2, and 3 are the input port numbers of the 3×3 star coupler. Note that the first port of the star coupler receives signals containing two different mode frequencies, signal and idler, i.e., $|0\rangle_1 = |0\rangle_s |0\rangle_i$.

• Stage 2: Outputs of Star Coupler

In stage 2, the mode operators are identified by the superscript (2). In this stage, the system's state experiences a 3×3 star coupler. The 3×3 polarization-insensitive and frequency-independent star coupler $\hat{B}^\dagger$ relates the input–output modes as

$$\begin{aligned}
\hat{a}_{1P}^{(1)}(\omega) &= B_{11}^* \hat{a}_{1P}^{(2)}(\omega) + B_{21}^* \hat{a}_{2P}^{(2)}(\omega) + B_{31}^* \hat{a}_{3P}^{(2)}(\omega), \\
\hat{a}_{2P}^{(1)}(\omega) &= B_{12}^* \hat{a}_{1P}^{(2)}(\omega) + B_{22}^* \hat{a}_{2P}^{(2)}(\omega) + B_{32}^* \hat{a}_{3P}^{(2)}(\omega), \\
\hat{a}_{3P}^{(1)}(\omega) &= B_{13}^* \hat{a}_{1P}^{(2)}(\omega) + B_{23}^* \hat{a}_{2P}^{(2)}(\omega) + B_{33}^* \hat{a}_{3P}^{(2)}(\omega),
\end{aligned} \quad (7)$$

where $P \in \{H, V\}$. Star couplers can be implemented using several 2×2 fiber couplers or beam splitters and/or formed using the fused biconical tapering method, as discussed in Chapter 6 of [40]. Due to energy conservation, the coefficients that relate the output signals to the input signals construct a unitary transfer matrix $\underline{B}$. The coefficient B_{ij} , the matrix element of $\underline{B}$ that connects input i to output j , is derived from the transmission and reflection coefficients of the used fiber couplers or beam splitters. References such as [41,42] studied the star coupler in the context of quantum mechanics, where the relation between the annihilation operator of the input modes and output modes is written similarly to Equation (7). In an $n \times n$ balanced star coupler, the input field is equally divided and transmitted to each output. Hence, $|B_{ij}|^2 = |B_n|^2 = 1/n$ for all i and j . By substituting Equation (7) in Equation (6), the evolved input state at the inputs of the WGR (or, equivalently, the state $|\psi\rangle^{(1)}$ in the bases of the output modes) becomes

$$\begin{aligned}
|\psi\rangle^{(2)} &= \frac{1}{\sqrt{2}} \iint d\Delta\omega_s d\Delta\omega_i f(\Delta\omega_s, \Delta\omega_i) \sum_{l,m=1}^3 B_{l1} B_{m1} \\
&\times [\hat{a}_{lH}^{\dagger(2)}(\omega_s) \hat{a}_{mV}^{\dagger(2)}(\omega_i) + \hat{a}_{lV}^{\dagger(2)}(\omega_s) \hat{a}_{mH}^{\dagger(2)}(\omega_i)] |0\rangle_1 |0\rangle_2 |0\rangle_3.
\end{aligned} \quad (8)$$

To better understand the different summation terms in Equation (8), we provide its expanded form in Equation (A1) of Appendix A.

- **Stage 3: Outputs of WGR**

The lossless 4×4 WGR input–output mode relations can be summarized in a frequency-dependent matrix form

$$\begin{bmatrix} \hat{a}_{1P}^{(2)}(\omega) \\ \hat{a}_{2P}^{(2)}(\omega) \\ \hat{a}_{3P}^{(2)}(\omega) \\ \hat{a}_{4P}^{(2)}(\omega) \end{bmatrix} = \underline{A}^\dagger \begin{bmatrix} \hat{a}_{1P}^{(3)}(\omega) \\ \hat{a}_{2P}^{(3)}(\omega) \\ \hat{a}_{3P}^{(3)}(\omega) \\ \hat{a}_{4P}^{(3)}(\omega) \end{bmatrix}, \quad (9)$$

where

$$\underline{A}^\dagger = \begin{bmatrix} A_{11}^*(\omega) & A_{21}^*(\omega) & A_{31}^*(\omega) & A_{41}^*(\omega) \\ A_{12}^*(\omega) & A_{22}^*(\omega) & A_{32}^*(\omega) & A_{42}^*(\omega) \\ A_{13}^*(\omega) & A_{23}^*(\omega) & A_{33}^*(\omega) & A_{43}^*(\omega) \\ A_{14}^*(\omega) & A_{24}^*(\omega) & A_{34}^*(\omega) & A_{44}^*(\omega) \end{bmatrix}. \quad (10)$$

For an ideal WGR, it is assumed

$$|A_{ij}^*(\omega)|^2 \approx \begin{cases} 1 & |\omega - \omega_{ij}| \leq \delta\omega/2 \\ 0 & \text{otherwise} \end{cases}, \quad (11)$$

where ω_{ji} is the central frequency of the passing bands for any pair of the input–output of the WGR with bandwidth $\delta\omega$ [43,44]. The distinguished central frequencies passing through the 4×4 WGR from its input j toward its output i are indicated by $\omega_{ij} = \omega_l$, where $l = i - j + 1 \bmod 4$ and $i, j \in \{1, 2, 3, 4\}$ [40]. Whenever $i - j + 1 = 4$ or $i - j + 1 = 0$, we assign $l = 4$. So, by choosing two signal frequencies (ω_{s_1} and ω_{s_2})

and two idler frequencies (ω_{i_1} and ω_{i_2}), which are set on the WGR-guided frequencies, as follows

$$\begin{aligned}\omega_{11} &= \omega_{22} = \omega_{33} = \omega_{44} = \omega_{s_1} = \frac{2\pi c}{\lambda_1}, \\ \omega_{21} &= \omega_{32} = \omega_{43} = \omega_{14} = \omega_{s_2} = \frac{2\pi c}{\lambda_2}, \\ \omega_{41} &= \omega_{12} = \omega_{23} = \omega_{34} = \omega_{i_1} = \frac{2\pi c}{\lambda_4}, \\ \omega_{31} &= \omega_{42} = \omega_{13} = \omega_{24} = \omega_{i_2} = \frac{2\pi c}{\lambda_3},\end{aligned}\quad (12)$$

one can approximate that the nonzero and dominant terms of Equation (9) occur at the WGR-passing frequencies as

$$\hat{a}_{jP}^{(2)}(\omega) \approx \begin{cases} A_{ij}^*(\omega) \hat{a}_{iP}^{(3)}(\omega) & |\omega - \omega_{ij}| \leq \delta\omega/2, \forall i \in \{1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}. \quad (13)$$

In stage 3, the WGR acts as a frequency filter with the bandwidth $\delta\omega$ of each output, and the passing signals satisfy the condition $\Delta\omega_s = -\Delta\omega_i$. Due to the phase-matching and energy-conservation conditions, the spectral joint amplitudes are $f(\Delta\omega_{s_1}, \Delta\omega_{i_2}) \approx 0$ and $f(\Delta\omega_{s_2}, \Delta\omega_{i_1}) \approx 0$. Thus, the nonzero terms $f(\Delta\omega_{s_1}, \Delta\omega_{i_1})$ and $f(\Delta\omega_{s_2}, \Delta\omega_{i_2})$, indicated by $f(\Delta\omega_{s_1})$ and $f(\Delta\omega_{s_2})$ from now on, can only be contributed in the following calculations. The spectral profiles of the WGR outputs are assumed to be a square with a width of $\delta\omega$. Therefore, we apply the following approximation in the integration terms that appear after inserting Equation (13) in Equation (8).

$$\begin{aligned}\int_{\Delta\omega_{s_1(2)} - \delta\omega/2}^{\Delta\omega_{s_1(2)} + \delta\omega/2} \int_{\Delta\omega_{i_1(2)} - \delta\omega/2}^{\Delta\omega_{i_1(2)} + \delta\omega/2} d\Delta\omega_s d\Delta\omega_i f(\Delta\omega_s, \Delta\omega_i) A_{j_1 l}(\omega_s) A_{j_2 m}(\omega_i) \hat{a}_{j_1 P}^{\dagger(3)}(\omega_s) \hat{a}_{j_2 P'}^{\dagger(3)}(\omega_i) \\ \approx \delta\omega f(\Delta\omega_{s_1(2)}) A_{j_1 l}(\omega_{s_1(2)}) A_{j_2 m}(\omega_{i_1(2)}) \hat{a}_{j_1 P, s_1(2)}^{\dagger(3)} \hat{a}_{j_2 P', i_1(2)}^{\dagger(3)},\end{aligned}\quad (14)$$

where

$$\begin{aligned}\hat{a}_{j_1 P, s_1(2)}^{\dagger(3)} &\approx \int_{\Delta\omega_{s_1(2)} - \delta\omega/2}^{\Delta\omega_{s_1(2)} + \delta\omega/2} d\Delta\omega_s \hat{a}_{j_1 P}^{\dagger(3)}(\omega_s) / \sqrt{\delta\omega}, \\ \hat{a}_{j_2 P', i_1(2)}^{\dagger(3)} &\approx \int_{\Delta\omega_{i_1(2)} - \delta\omega/2}^{\Delta\omega_{i_1(2)} + \delta\omega/2} d\Delta\omega_i \hat{a}_{j_2 P'}^{\dagger(3)}(\omega_i) / \sqrt{\delta\omega}.\end{aligned}\quad (15)$$

In Equation (14), $P, P' \in \{H, V\}$, $l, m \in \{1, 2, 3\}$, and $j_1, j_2 \in \{1, 2, 3, 4\}$. Moreover, according to Equations (12) and (13), for integral bounds around ω_{s_1} and ω_{i_1} , $j_1 = l$ and $j_2 = 3 + m \bmod 4$, and for integral bounds around ω_{s_2} and ω_{i_2} , $j_1 = l + 1 \bmod 4$ and $j_2 = m + 2 \bmod 4$. Note that whenever j_1 and j_2 become zero, we assign them 4. Inserting Equation (9) in Equation (8) and using Equations (12)–(15), the input state is rewritten based on the representation of the modes in stage 3 as

$$\begin{aligned}|\psi\rangle^{(3)} &= \frac{\delta\omega f(\Delta\omega_{s_1})}{\sqrt{2}} \sum_{l,m=1}^3 B_{l1} B_{m1} A_{ll}(\omega_{s_1}) A_{3+m \bmod 4, m}(\omega_{i_1}) \\ &\quad [\hat{a}_{lH, s_1}^{\dagger(3)} \hat{a}_{3+m \bmod 4, V, i_1}^{\dagger(3)} + \hat{a}_{lV, s_1}^{\dagger(3)} \hat{a}_{3+m \bmod 4, H, i_1}^{\dagger(3)}] \\ &\quad \times |0\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4 \\ &+ \frac{\delta\omega f(\Delta\omega_{s_2})}{\sqrt{2}} \sum_{l,m=1}^3 B_{l1} B_{m1} A_{l+1 \bmod 4, l, s_2} A_{m+2 \bmod 4, m, i_2} \\ &\quad [\hat{a}_{l+1 \bmod 4, H, s_2}^{\dagger(3)} \hat{a}_{m+2 \bmod 4, V, i_2}^{\dagger(3)} + \hat{a}_{l+1 \bmod 4, V, s_2}^{\dagger(3)} \hat{a}_{m+2 \bmod 4, H, i_2}^{\dagger(3)}] \\ &\quad \times |0\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4,\end{aligned}\quad (16)$$

Whenever the subindex in Equation (16) becomes a multiple of 4, we assign that label to 4 instead of 0. As a result, in Equation (16), each of the four users is indicated from the mode subindex set $\{1, 2, 3, 4\}$.

The first three lines on the right-hand side of Equation (16) are related to the photon entangled pair λ_1 and λ_4 (ω_{s_1} and ω_{i_1}), which is distributed between all the users. The second set of three lines in Equation (16) describes the entangled photon pair λ_2 and λ_3 (ω_{s_2} and ω_{i_2}) shared among all the users. In Appendix A, Equation (16) is expanded on the m and l indices, and the explicit form of the entangled state between each pair of users is specified in Equation (A3) (see also Figure 1). Therefore, from Equations (16) and (A3), it is clear that there are at least two entangled states between any two users. Furthermore, the pairs specified by subscripts (1,2) and (3,4) share three entangled states (see Equation (A3)). As a result, the fully connected quantum network is realized using only four wavelengths with a minimum number of utilized devices, i.e., a star coupler preceded by a waveguide grating router. This scheme paves the way for the miniaturization of a future quantum network.

2.3. Generalized Entanglement Distribution WDM Network Based on WGR for $2N$ Users

A generalized model of entanglement distribution for $2N$ users is illustrated in Figure 2. This generalized scheme inserts a broadband polarization-entangled photon source into an $(N + 1) \times (N + 1)$ star coupler from its first input port. Subsequently, a WGR distributes $N + 1$ out of $2N$ different wavelengths in the EPS spectrum to each subscriber, as depicted on the right-hand side of Figure 2. Consider the i th user with the wavelength set as

$$\Lambda_i = (\lambda_i, \lambda_{i-1 \bmod 2N}, \dots, \lambda_{i-l \bmod 2N}, \dots, \lambda_{i-N \bmod 2N}).$$

The modes related to the above wavelength set are entangled one by one, with the modes corresponding to the following wavelength set as

$$\Lambda_i^* = (\lambda_{2N-i+1 \bmod 2N}, \lambda_{2N-i+2 \bmod 2N}, \dots, \lambda_{2N-i+l \bmod 2N}, \dots, \lambda_{2N-i+N+1 \bmod 2N}),$$

which is called a conjugate wavelength set. The cardinal number of the sets Λ_i and Λ_i^* equals $N + 1$, while the total number of wavelengths is $2N$. Because $\Lambda_i \cup \Lambda_j^* \subseteq (\lambda_1, \dots, \lambda_{2N})$, the cardinal number of $\Lambda_i \cup \Lambda_j^*$ is equal or less than $2N$, i.e., $N + 1 \leq |\Lambda_i \cup \Lambda_j^*| \leq 2N$. Therefore, $\Lambda_i^* \cap \Lambda_j \neq \emptyset \forall i, j$ means at least one entangled pair exists between the i th and j th users. Note that whenever $i \bmod 2N$ becomes 0, we assign that label the value $2N$ (i.e., $\lambda_0 = \lambda_{2N}$).

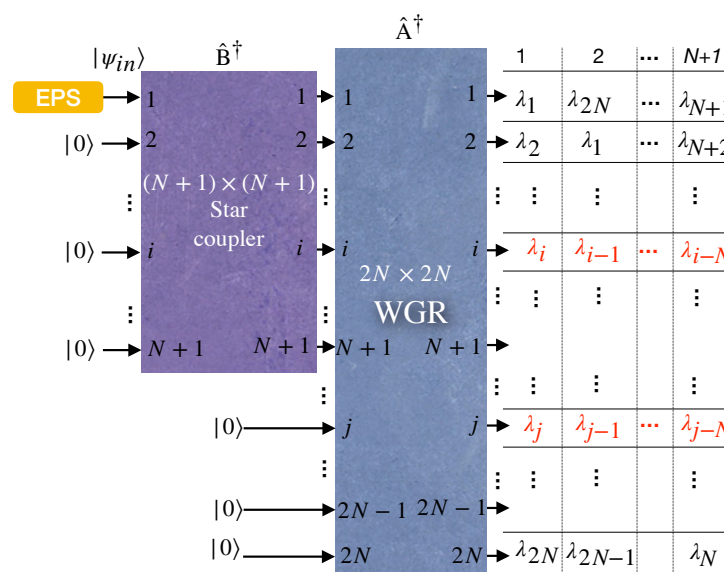


Figure 2. Schematic configuration of a quantum entanglement distribution network for $2N$ users.

The chief drawback of this scheme is related to the degradation of the entanglement distribution rate with the factor $1/(N+1)^2$ introduced by the $(N+1) \times (N+1)$ star coupler. The scheme presented in the next section improves this factor to $1/(N)^2$ (if we neglect the loss of the multiplexers). However, a salient trade-off still exists between the number of required wavelengths for running the fully connected network and the entanglement distribution rate of each pair in the network.

3. Entanglement-Based Quantum WDM Network Based on Wavelength Demultiplexing and Multiplexing

Here, we study a different configuration for distributing entanglement between end users in a quantum network. Figures 3 and 4 illustrate four- and eight-user entangled networks, respectively. Accordingly, the generalized setup shown in Figure 5 utilizes a $1 \times 2N$ wavelength demultiplexer and the N number of $N \times 1$ wavelength multiplexers. These wavelength multiplexers merge different wavelengths in a single fiber to send to the related long-distance user. Moreover, the N number of $N \times N$ balanced star couplers are deployed in each demultiplexer output. Substituting an $(N+1) \times (N+1)$ balanced beam splitter with $N \times N$ balanced star couplers in this new scheme leads to an entanglement rate increase, provided the loss arising from the multiplexer increment is neglected. The global state at the user's premises can be derived easily in a similar manner performed in the previous section. It is enough to write the input state $|\psi_{in}\rangle$ based on the output modes at the users' sites. For example, for the four-user network depicted in Figure 3, first, the input modes of the EPS evolve through the 1×4 demultiplexer according to Equation (9) (its quantum model is the same as the 4×4 WGR with the three unused input ports). Then, each output mode of the demultiplexer passes through a 2×2 beam splitter where the output modes are related to the input modes akin to Equation (7) but for a 2×2 system. Finally, two output modes related to two different beam splitters' outputs, as depicted in Figure 3, are combined by a 2×1 wavelength multiplexer. The relation between the input and output modes of the 2×2 wavelength multiplexer is the same as a 2×2 WGR. In Figure 3, for the sake of simplicity, all the 2×1 wavelength multiplexers are indicated by the same operator $\hat{A}^\dagger$, and the input and output ordering of their ports are chosen arbitrarily so that the desirable multiplexing happens. Indeed, we assume that all 2×1 wavelength multiplexers operating at their specified two wavelengths are constructed such that their $\hat{A}^\dagger$ operator has the same matrix elements, regardless of the two wavelengths chosen to operate for them.

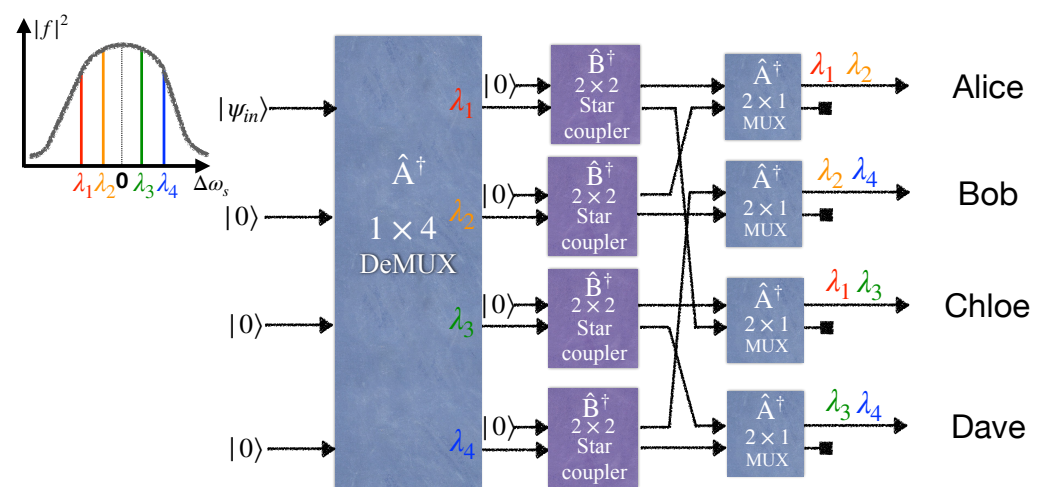


Figure 3. An entanglement distribution network between four end users via broadband entangled photon sources utilizing a wavelength demultiplexer (DeMUX), four 2×2 beam splitters, and four multiplexers (MUX).

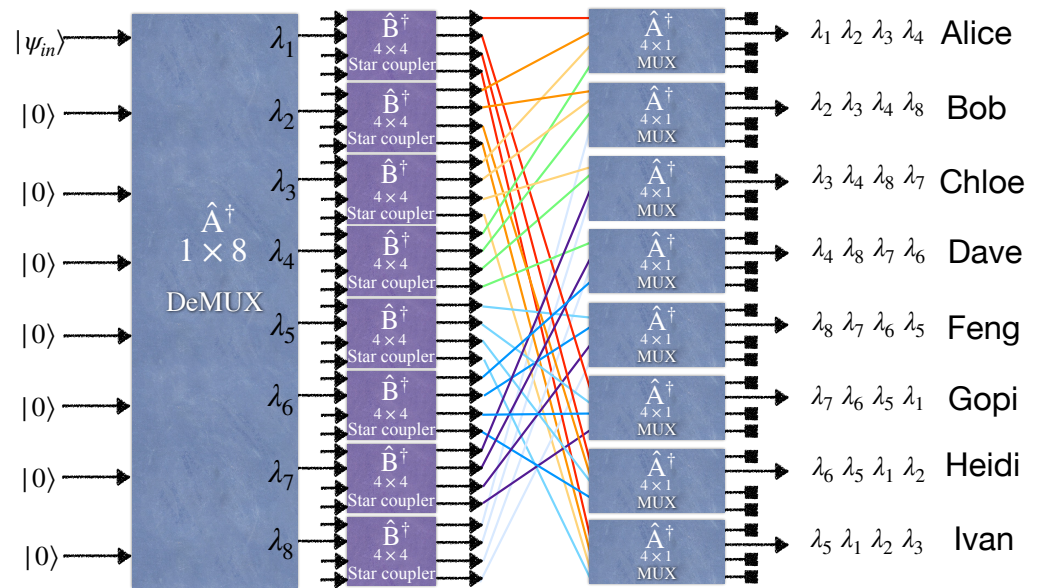


Figure 4. An entanglement distribution network between eight end users via broadband entangled photon sources utilizing a wavelength demultiplexer, eight 4×4 beam splitters, and eight multiplexers.

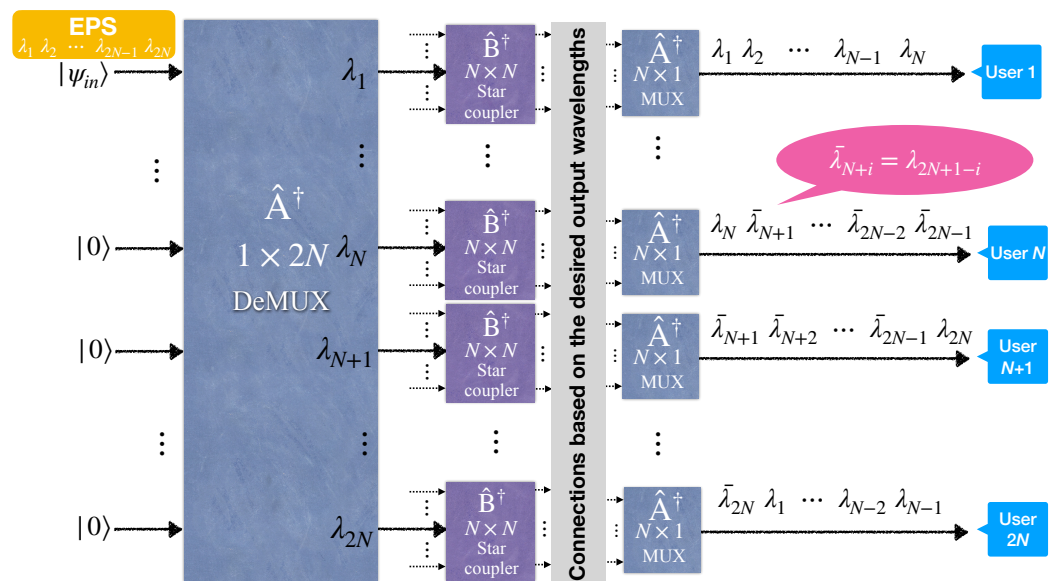


Figure 5. Schematic configuration of a quantum entanglement distribution network for $2N$ users based on wavelength DeMUX (demultiplexer) and MUX (multiplexer).

To prove why this scheme gives rise to a fully connected entanglement network, consider the i th user with the wavelength set as

$$\Lambda_i = (\lambda_i, \lambda_{i+1 \bmod 2N}, \dots, \lambda_{i+l \bmod 2N}, \dots, \lambda_{i+N-1 \bmod 2N}).$$

Here, we assume the modes related to the above wavelength set are entangled one by one, with the modes corresponding to the following wavelength set as

$$\Lambda_i^* = (\lambda_{i+N \bmod 2N}, \lambda_{i+1+N \bmod 2N}, \dots, \lambda_{i+l+N \bmod 2N}, \dots, \lambda_{i+N-1+N \bmod 2N}),$$

which is called a conjugate wavelength set. It is important to note that $\Lambda_i \cap \Lambda_i^* = \emptyset$ and $\Lambda_i \cup \Lambda_i^* = (\lambda_1, \dots, \lambda_{2N})$. It is straightforward to show that at least one common wavelength exists between each conjugate wavelength set (one user) and other wavelength

sets (other users). To achieve this, one can employ the method of proof by contradiction. Consider two different users with index i and j . If there is no common wavelength between sets Λ_i^* and Λ_j , one deduces $i = j$ because the total number of used wavelengths is $2N$. As a result, this configuration leads to entanglement distribution between every pair.

4. Comparison with Previous Realizations

This section will compare this paper's results with previous works that used passive optical WDM communication components. This comparison will focus on the number of wavelengths and devices used and the rate of entanglement distribution between users. This information is summarized in Table 1. Additionally, to provide a clearer comparison, we use similar block diagrams of the devices presented in this paper to illustrate the setups of the two previous experiments.

Table 1. Comparison of the entangled-based QWDM networks presented in this paper and the references [33,34]. The number of users is specified by N . Because ideal multiplexers, demultiplexers, and WGRs have the same mathematical model, their matrix elements' value is assumed to be equal and represented by $|A|$. The matrix elements' value of the $n \times n$ balanced star coupler is indicated by $|B_n| = \frac{1}{\sqrt{n}}$.

N-User Entangled-Based Network	Number of Required Wavelengths	Number of Employed Devices	Scale of Entanglement Distribution's Rate
Network presented in Figure 1: $N = 4(8)$	4(8)	One 3×3 (7×7) star coupler and one 4×4 (8×8) WGR	$2 B_{3(7)} ^4 A ^4$
Network presented in Figures 3 and 4: $N = 4(8)$	4(8)	Four (eight) 2×2 (4×4) star couplers, one 1×4 (1×8) demultiplexer, and four (eight) 2×1 (4×1) multiplexers	$ B_{2(4)} ^4 A ^8$
Ref. [33]: $N = 4(8)$	12(56)	One 1×12 (1×56) demultiplexer, and four (eight) 3×1 (7×1) multiplexers	$ A ^8$
Ref. [34]: $N = 4(8)$	4(16)	Four (eight) 2×2 star couplers, one 1×4 (1×16) demultiplexer, four (eight) 2×1 multiplexers, and zero (four) 3×1 multiplexers	$ B_2 ^4 A ^{8(12)}$

To compare our setup with the previous works [33,34], we depict the schematic configurations of these experimental setups based on the WDM demultiplexer/multiplexers and beam splitters, respectively, in Figures 6 and 7. Likewise, any entangled state comprises two single photons labeled by the wavelength pairs $(\lambda_i, \lambda_{N-i+1}) \forall i \in \{1, \dots, N/2\}$, which correspond to the signal and idler modes, respectively. It is assumed that the number of users, N , is even.

The setup presented in Figure 6 utilized twelve wavelengths to have a fully connected logical network layer among four users, while we used four wavelengths in our scheme presented in Figure 1. Nevertheless, it is essential to mention that because wavelength multiplexers substitute a star coupler in the setup related to Figure 6, the coincident photon count between any pair becomes proportional to $|A|^8$ in comparison to our model, which is proportional to $2|A|^4|B_3|^4$ (see Equation (16)). The coefficient 2 appearing in

the coincident photon count shows that at least two entangled states are shared between each pair in our model. Moreover, we assume $|A_{ij}| = |A|$ and $|B_{ij}| = |B_3| = 1/\sqrt{3}$ for all i and j . Suppose the loss corresponding to the multiplexers/demultiplexers is less than the loss introduced by the 3×3 star coupler in our model (roughly 4.8dB). In that case, the entanglement distribution rate of our model falls behind [33]. However, our scheme becomes dominant with respect to the number of required wavelengths when the number of users increases. For example, for eight-user networks, [33] needs 56 wavelengths, whereas our scheme utilizes only eight wavelengths. Note that Figure 1 can introduce a fully connected entangled network even by sharing two wavelengths, λ_1 and λ_4 , or λ_2 and λ_3 .

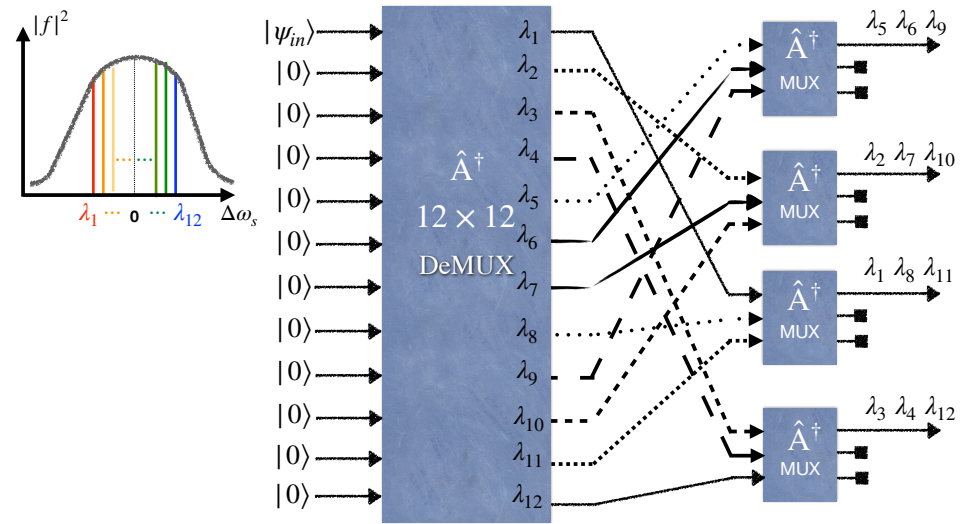


Figure 6. Schematic configuration of a quantum entanglement distribution network for 4 users according to [33].

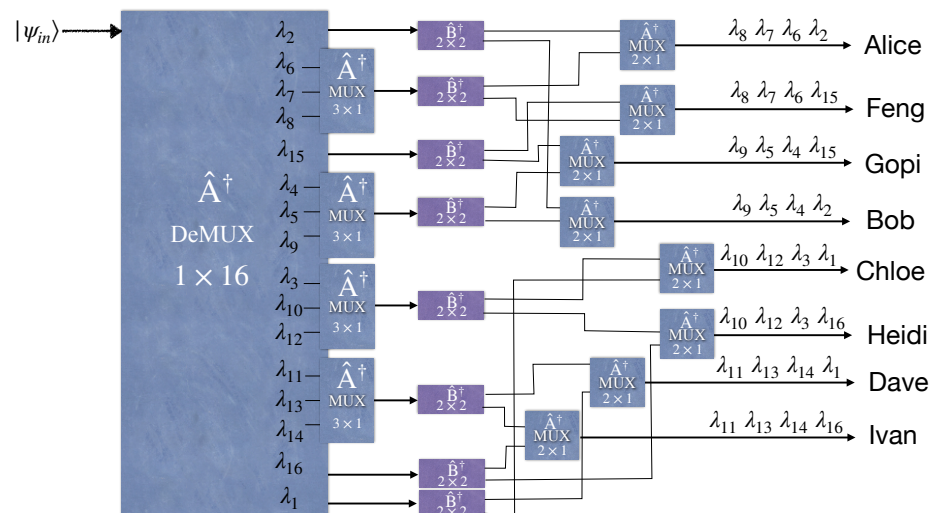


Figure 7. Schematic configuration of a quantum entanglement distribution network for eight users according to [34].

In 2020, [34] demonstrated that an eight-user entanglement-based quantum network requires only 16 wavelengths to become fully connected by utilizing beam splitters at the outputs of the wavelength demultiplexer, which separates the wavelength contents of the broadband entangled source. However, based on the generalized form of our scheme explained in Section 2.3, our proposed eight-user quantum network requires only eight wavelengths if a star coupler is placed before a WGR. Figure 3 illustrates the four-user

versions of the quantum network used by [34]. Hence, it is comparable with our four-user model now. In both models, the fully connected network runs only via four wavelengths.

Table 1 gives the scales of the entanglement distribution rate for different entangled-based network configurations. To obtain this scale, consider the entangled states containing two photons passing through each network device. Hence, the matrix elements of the network device appear two times as a coefficient of the entangled state (see Equations (16) and (A3)–(A7)). If the network has $m = m_1 + m_2 + m_3 + m_4$ WDM components, such as m_1 numbers of $n \times n$ balanced star couplers, m_2 wavelength multiplexers, m_3 wavelength demultiplexers, and m_4 WGRs, the coefficient of each shared entangled state is proportional to $|B_n|^{m_1} |A|^{m_2+m_3+m_4}$. Therefore, the probability of each entangled state produced is proportional to $|B_n|^{2m_1} |A|^{2(m_2+m_3+m_4)}$, which estimates the entanglement distribution rate. For simplicity, all the matrix elements in the wavelength multiplexers, demultiplexers, and WGRs with different numbers of ports are indicated by $|A|$.

5. Entanglement-Based BBM92 QKD Protocol Used in the WDM Network

The BBM92 QKD protocol [38] is the entangled version of the BB84 QKD protocol. In this protocol, an entangled bipartite state such as $|\psi_{in}\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) = \frac{1}{\sqrt{2}}(|DD\rangle - |AA\rangle)$ is distributed between two legitimate users by an untrusted entanglement provider, where $|m\rangle = \hat{a}_m^\dagger|0\rangle$ for the polarization modes $m \in \{H, V, D, A\}$ and $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$, $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. The users randomly choose rectilinear or diagonal bases for measuring the polarization of their received photon. As it is clear from the form of $|\psi_{in}\rangle$, if both users choose the rectilinear (diagonal) basis, their achieved information is anti-correlated (correlated). Like the BB84 QKD protocol, H and D polarization are related to the classical bit 0, while 1 is encoded by V or A polarization.

To understand how BBM92 is an entangled version of BB84, we can assume that Alice uses an entangled photon source as a transmitter in the BB84 protocol. She randomly measures the polarization of one photon of the bipartite system from a rectilinear or diagonal basis and sends the other to Bob. This procedure is equivalent to a random selection of the photon polarization via the polarization modulator. Because it can be assumed that Alice does not reveal the result of her measurement to herself until Bob performs the measurement, the BB84 protocol mimics the BBM92 protocol, where the entanglement provider is at the transmitter's site.

As depicted in Figure 8, a four-user entangled-based network consists of an entanglement provider as well as four users, Alice (A), Bob (B), Charlie (C), and David (D), equipped with measurement facilities. The entanglement provider distributes to four users a polarization-based bipartite entangled state produced by an entangled photon source (EPS), such as a PPSE, via a 2×2 star coupler and a WGR. Photon pairs with wavelengths (λ_1 and λ_4) as well as (λ_2 and λ_3) are entangled in their polarization degrees according to Equation (4). The table in Figure 8 displays the wavelength allocated to each user. For instance, with the help of Equations (16) and (A3), with the probability $\mathcal{P} \propto (|B|^2|A|^2)^2$, the entangled state shared between Alice and Charlie can be written as

$$|\psi\rangle_{AC} = \frac{1}{2}[(\hat{a}_{A,H,\lambda_4}^\dagger \hat{a}_{C,V,\lambda_1}^\dagger + \hat{a}_{A,V,\lambda_4}^\dagger \hat{a}_{C,H,\lambda_1}^\dagger)|00\rangle_{AC} + (\hat{a}_{A,H,\lambda_3}^\dagger \hat{a}_{C,V,\lambda_2}^\dagger + \hat{a}_{A,V,\lambda_3}^\dagger \hat{a}_{C,H,\lambda_2}^\dagger)|00\rangle_{AC}], \quad (17)$$

where the creation operator $\hat{a}_{i,p,\lambda_j}^\dagger$ is related to the photon in the site of $i \in \{A, C\}$ with the polarization $p \in \{H, V\}$ around the central wavelength λ_j where $j \in \{1, 2, 3, 4\}$. According to Equation (17), two entangled states are shared between Alice and Charlie in this network, which could also be understood by the fact that the common elements of the conjugate wavelength set of Alice's wavelength set $\{\lambda_1, \lambda_3, \lambda_4\}$ (i.e., $\{\lambda_4, \lambda_2, \lambda_1\}$) with Charlie's wavelength set $\{\lambda_1, \lambda_2, \lambda_3\}$ are $\{\lambda_1, \lambda_2\}$. Due to the fact that it is assumed that, in each mode, there is a single photon, photon coincidence counting happens probabilistically for only one pair at a time. Consequently, the secret key shared between pairs is unique.

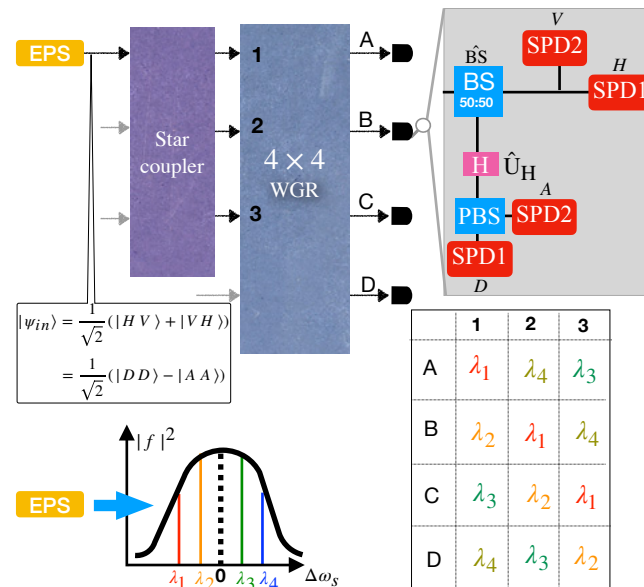


Figure 8. A four-user BBM92 QKD entangled-based network. The measurement components are BS (beam splitter), PBS (polarization beam splitter), SPD (single photon detector), and H (half-wave plate). EPS is an entangled photon source, and WGR is a waveguide grating router.

6. Conclusions

The authors of this paper studied the WDM network's entangled-based quantum state sharing between users by minimizing the required wavelengths for a fully connected entangled network. It was shown that each receiver could receive the encoded information in the quantum signal from every transmitter. Still, two receivers cannot receive the same data from the same transmitter at a specific time. In the entangled-based network, we proved that using an $N \times N$ star coupler behind a $2N \times 2N$ WGR, the minimum number of required wavelengths for distributing entanglement among each pair in the quantum WDM network becomes $2N$. The previous related entangled WDM networks were compared with our scheme. This comparison highlights the advantage of our work in requiring fewer wavelengths and devices, which is crucial in developing a cost-effective and compact quantum local access network. Finally, we concluded the paper by showing how the QKD protocol, BBM92, was employed in the WDM network.

Author Contributions: Conceptualization, M.B. and J.A.S.; methodology, M.B. and J.A.S.; formal analysis, M.B.; investigation, M.B.; writing—original draft preparation, M.B.; writing—review and editing, M.B. and J.A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Acknowledgments: The authors would like to thank M. Rezai of the Sharif University of Technology for many fruitful discussions and guidance in the course of this project.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

WDM	Wavelength Division Multiplexing
QWDM	Quantum Wavelength Division Multiplexing
WGR	Waveguide Grating Router
PPSF	Periodically Poled Silica Fiber

EPS	Entangled Photon Source
SPDC	Spontaneous Parametric Down Conversion
JSA	Joint Spectral Amplitude
MUX	Multiplexer
DeMUX	Demultiplexer

Appendix A. Quantum State at Stage 3 of the Four-User Entangled-Based QWDM

Here, we present the expanded equations for the four-user entangled-based QWDM network in stages 2 and 3 described in Section 2.2. By writing each term of summations over l and m in Equation (8) (the entangled state's evolution from the star coupler), it becomes

$$\begin{aligned}
 |\psi\rangle^{(2)} = \frac{1}{\sqrt{2}} \iint d\Delta\omega_s d\Delta\omega_i f(\Delta\omega_s) \{ & B_{11}B_{11}[\hat{a}_{1H}^{(2)}(\omega_s)\hat{a}_{1V}^{(2)}(\omega_i) + \hat{a}_{1V}^{(2)}(\omega_s)\hat{a}_{1H}^{(2)}(\omega_i)] \\
 & + B_{21}B_{21}[\hat{a}_{2H}^{(2)}(\omega_s)\hat{a}_{2V}^{(2)}(\omega_i) + \hat{a}_{2V}^{(2)}(\omega_s)\hat{a}_{2H}^{(2)}(\omega_i)] \\
 & + B_{11}B_{21}[\hat{a}_{1H}^{(2)}(\omega_s)\hat{a}_{2V}^{(2)}(\omega_i) + \hat{a}_{1V}^{(2)}(\omega_s)\hat{a}_{2H}^{(2)}(\omega_i)] \\
 & + B_{21}B_{11}[\hat{a}_{2H}^{(2)}(\omega_s)\hat{a}_{1V}^{(2)}(\omega_i) + \hat{a}_{2V}^{(2)}(\omega_s)\hat{a}_{1H}^{(2)}(\omega_i)] \\
 & + B_{11}B_{31}[\hat{a}_{1H}^{(2)}(\omega_s)\hat{a}_{3V}^{(2)}(\omega_i) + \hat{a}_{1V}^{(2)}(\omega_s)\hat{a}_{3H}^{(2)}(\omega_i)] \\
 & + B_{31}B_{11}[\hat{a}_{3H}^{(2)}(\omega_s)\hat{a}_{1V}^{(2)}(\omega_i) + \hat{a}_{3V}^{(2)}(\omega_s)\hat{a}_{1H}^{(2)}(\omega_i)] \\
 & + B_{21}B_{31}[\hat{a}_{2H}^{(2)}(\omega_s)\hat{a}_{3V}^{(2)}(\omega_i) + \hat{a}_{2V}^{(2)}(\omega_s)\hat{a}_{3H}^{(2)}(\omega_i)] \\
 & + B_{31}B_{21}[\hat{a}_{3H}^{(2)}(\omega_s)\hat{a}_{2V}^{(2)}(\omega_i) + \hat{a}_{3V}^{(2)}(\omega_s)\hat{a}_{2H}^{(2)}(\omega_i)] \\
 & + B_{31}B_{31}[\hat{a}_{3H}^{(2)}(\omega_s)\hat{a}_{3V}^{(2)}(\omega_i) + \hat{a}_{3V}^{(2)}(\omega_s)\hat{a}_{3H}^{(2)}(\omega_i)] \} \\
 & \times |0\rangle_1|0\rangle_2|0\rangle_3. \tag{A1}
 \end{aligned}$$

The nonzero terms of Equation (13), related to the mode evolution from the WGR at the signal and idler central frequencies, are

$$\begin{aligned}
 \hat{a}_{1P}^{(2)}(\omega_{s_1}) &\approx A_{11}^*(\omega_{s_1})\hat{a}_{1P}^{(3)}(\omega_{s_1}), \\
 \hat{a}_{1P}^{(2)}(\omega_{i_1}) &\approx A_{41}^*(\omega_{i_1})\hat{a}_{4P}^{(3)}(\omega_{i_1}), \\
 \hat{a}_{1P}^{(2)}(\omega_{s_2}) &\approx A_{21}^*(\omega_{s_2})\hat{a}_{2P}^{(3)}(\omega_{s_2}), \\
 \hat{a}_{1P}^{(2)}(\omega_{i_2}) &\approx A_{31}^*(\omega_{i_2})\hat{a}_{3P}^{(3)}(\omega_{i_2}), \\
 \hat{a}_{2P}^{(2)}(\omega_{s_1}) &\approx A_{22}^*(\omega_{s_1})\hat{a}_{2P}^{(3)}(\omega_{s_1}), \\
 \hat{a}_{2P}^{(2)}(\omega_{i_1}) &\approx A_{12}^*(\omega_{i_1})\hat{a}_{1P}^{(3)}(\omega_{i_1}), \\
 \hat{a}_{2P}^{(2)}(\omega_{s_2}) &\approx A_{32}^*(\omega_{s_2})\hat{a}_{3P}^{(3)}(\omega_{s_2}), \\
 \hat{a}_{2P}^{(2)}(\omega_{i_2}) &\approx A_{42}^*(\omega_{i_2})\hat{a}_{4P}^{(3)}(\omega_{i_2}), \\
 \hat{a}_{3P}^{(2)}(\omega_{s_1}) &\approx A_{33}^*(\omega_{s_1})\hat{a}_{3P}^{(3)}(\omega_{s_1}), \\
 \hat{a}_{3P}^{(2)}(\omega_{i_1}) &\approx A_{23}^*(\omega_{i_1})\hat{a}_{2P}^{(3)}(\omega_{i_1}), \\
 \hat{a}_{3P}^{(2)}(\omega_{s_2}) &\approx A_{43}^*(\omega_{s_2})\hat{a}_{4P}^{(3)}(\omega_{s_2}), \\
 \hat{a}_{3P}^{(2)}(\omega_{i_2}) &\approx A_{13}^*(\omega_{i_2})\hat{a}_{1P}^{(3)}(\omega_{i_2}). \tag{A2}
 \end{aligned}$$

We utilize Equations (A1) and (A2) and expand the summations over l and m in Equation (16). The following expression for the evolution of the entangled photon source passing through the star coupler and WGR is obtained.

$$\begin{aligned}
|\psi\rangle^{(3)} = & \frac{\delta\omega f(\Delta\omega_{s_1})}{\sqrt{2}} \times \\
& \{ B_{11}B_{11}A_{11}(\omega_{s_1})A_{41}(\omega_{i_1})[\hat{a}_{1H,s_1}^{+(3)}\hat{a}_{4V,i_1}^{+(3)} + \hat{a}_{1V,s_1}^{+(3)}\hat{a}_{4H,i_1}^{+(3)}] \\
& + B_{21}B_{21}A_{22}(\omega_{s_1})A_{12}(\omega_{i_1})[\hat{a}_{2H,s_1}^{+(3)}\hat{a}_{1V,i_1}^{+(3)} + \hat{a}_{2V,s_1}^{+(3)}\hat{a}_{1H,i_1}^{+(3)}] \\
& + B_{11}B_{21}A_{11}(\omega_{s_1})A_{12}(\omega_{i_1})[\hat{a}_{1H,s_1}^{+(3)}\hat{a}_{1V,i_1}^{+(3)} + \hat{a}_{1V,s_1}^{+(3)}\hat{a}_{1H,i_1}^{+(3)}] \\
& + B_{21}B_{11}A_{22}(\omega_{s_1})A_{41}(\omega_{i_1})[\hat{a}_{2H,s_1}^{+(3)}\hat{a}_{4V,i_1}^{+(3)} + \hat{a}_{2V,s_1}^{+(3)}\hat{a}_{4H,i_1}^{+(3)}] \\
& + B_{11}B_{31}A_{11}(\omega_{s_1})A_{23}(\omega_{i_1})[\hat{a}_{1H,s_1}^{+(3)}\hat{a}_{2V,i_1}^{+(3)} + \hat{a}_{1V,s_1}^{+(3)}\hat{a}_{2H,i_1}^{+(3)}] \\
& + B_{31}B_{11}A_{33}(\omega_{s_1})A_{41}(\omega_{i_1})[\hat{a}_{3H,s_1}^{+(3)}\hat{a}_{4V,i_1}^{+(3)} + \hat{a}_{3V,s_1}^{+(3)}\hat{a}_{4H,i_1}^{+(3)}] \\
& + B_{21}B_{31}A_{22}(\omega_{s_1})A_{23}(\omega_{i_1})[\hat{a}_{2H,s_1}^{+(3)}\hat{a}_{2V,i_1}^{+(3)} + \hat{a}_{2V,s_1}^{+(3)}\hat{a}_{2H,i_1}^{+(3)}] \\
& + B_{31}B_{21}A_{33}(\omega_{s_1})A_{12}(\omega_{i_1})[\hat{a}_{3H,s_1}^{+(3)}\hat{a}_{1V,i_1}^{+(3)} + \hat{a}_{3V,s_1}^{+(3)}\hat{a}_{1H,i_1}^{+(3)}] \\
& + B_{31}B_{31}A_{33}(\omega_{s_1})A_{23}(\omega_{i_1})[\hat{a}_{3H,s_1}^{+(3)}\hat{a}_{2V,i_1}^{+(3)} + \hat{a}_{3V,s_1}^{+(3)}\hat{a}_{2H,i_1}^{+(3)}] \} \\
& \times |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4 + \dots \\
& + \frac{\delta\omega f(\Delta\omega_{s_2})}{\sqrt{2}} \times \\
& \{ B_{11}B_{11}A_{21}(\omega_{s_2})A_{31}(\omega_{i_2})[\hat{a}_{2H,s_2}^{+(3)}\hat{a}_{3V,i_2}^{+(3)} + \hat{a}_{2V,s_2}^{+(3)}\hat{a}_{3H,i_2}^{+(3)}] \\
& + B_{21}B_{21}A_{32}(\omega_{s_2})A_{42}(\omega_{i_2})[\hat{a}_{3H,s_2}^{+(3)}\hat{a}_{4V,i_2}^{+(3)} + \hat{a}_{3V,s_2}^{+(3)}\hat{a}_{4H,i_2}^{+(3)}] \\
& + B_{11}B_{21}A_{21}(\omega_{s_2})A_{42}(\omega_{i_2})[\hat{a}_{2H,s_2}^{+(3)}\hat{a}_{4V,i_2}^{+(3)} + \hat{a}_{2V,s_2}^{+(3)}\hat{a}_{4H,i_2}^{+(3)}] \\
& + B_{21}B_{11}A_{32}(\omega_{s_2})A_{31}(\omega_{i_2})[\hat{a}_{3H,s_2}^{+(3)}\hat{a}_{3V,i_2}^{+(3)} + \hat{a}_{3V,s_2}^{+(3)}\hat{a}_{3H,i_2}^{+(3)}] \\
& + B_{11}B_{31}A_{21}(\omega_{s_2})A_{13}(\omega_{i_2})[\hat{a}_{2H,s_2}^{+(3)}\hat{a}_{1V,i_2}^{+(3)} + \hat{a}_{2V,s_2}^{+(3)}\hat{a}_{1H,i_2}^{+(3)}] \\
& + B_{31}B_{11}A_{43}(\omega_{s_2})A_{31}(\omega_{i_2})[\hat{a}_{4H,s_2}^{+(3)}\hat{a}_{3V,i_2}^{+(3)} + \hat{a}_{4V,s_2}^{+(3)}\hat{a}_{3H,i_2}^{+(3)}] \\
& + B_{21}B_{31}A_{32}(\omega_{s_2})A_{13}(\omega_{i_2})[\hat{a}_{3H,s_2}^{+(3)}\hat{a}_{1V,i_2}^{+(3)} + \hat{a}_{3V,s_2}^{+(3)}\hat{a}_{1H,i_2}^{+(3)}] \\
& + B_{31}B_{21}A_{43}(\omega_{s_2})A_{42}(\omega_{i_2})[\hat{a}_{4H,s_2}^{+(3)}\hat{a}_{4V,i_2}^{+(3)} + \hat{a}_{4V,s_2}^{+(3)}\hat{a}_{4H,i_2}^{+(3)}] \\
& + B_{31}B_{31}A_{43}(\omega_{s_2})A_{13}(\omega_{i_2})[\hat{a}_{4H,s_2}^{+(3)}\hat{a}_{1V,i_2}^{+(3)} + \hat{a}_{4V,s_2}^{+(3)}\hat{a}_{1H,i_2}^{+(3)}] \} \\
& \times |0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4.
\end{aligned} \tag{A3}$$

To understand better the meaning of different terms in Equation (A3), for instance, consider the first term of Equation (A3), which leads to the entangled state

$$\frac{1}{\sqrt{2}}(|H, \omega_{s_1}\rangle_1|V, \omega_{i_1}\rangle_4 + |V, \omega_{s_1}\rangle_1|H, \omega_{i_1}\rangle_4), \tag{A4}$$

between users 1 and 4, which is shared with the probability

$$|\delta\omega f(\Delta\omega_{s_1})B_{11}B_{11}A_{11}(\omega_{s_1})A_{41}(\omega_{i_1})|^2. \tag{A5}$$

Moreover, another entangled state related to the last line of Equation (A3) is shared between the users 1 and 4 as

$$\frac{1}{\sqrt{2}}(|H, \omega_{s_2}\rangle_4|V, \omega_{i_2}\rangle_1 + |V, \omega_{s_2}\rangle_4|H, \omega_{i_2}\rangle_1), \tag{A6}$$

with the probability equaling

$$|\delta\omega f(\Delta\omega_{s_2})B_{31}B_{31}A_{43}(\omega_{s_2})A_{13}(\omega_{i_2})|^2. \quad (A7)$$

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