

Supplementary Materials: Testing Equality of Multiple Population Means under Contaminated Normal Model Using the Density Power Divergence

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Supplementary Materials

A. Some Integrals

The following integrals are used in the DPD measure $d_\gamma(f_\theta, g)$ and simplify the J and K matrices at the model when $g = f_\theta$. As the integrals are over the entire real line, we will omit the subscripts from y_{ij} and call it y for simplicity. However, it should be noted that the mean function associated with y is μ_i .

$$\begin{aligned} \int_y f_\theta^{1+\gamma}(y)dy &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \int_y \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \frac{\sqrt{2\pi}\sigma}{\sqrt{1+\gamma}} \left[\frac{\sqrt{1+\gamma}}{\sqrt{2\pi}\sigma} \int_y \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\}\right] \quad (A.1) \\ &= (2\pi)^{-\frac{\gamma}{2}}\sigma^{-\gamma}(1+\gamma)^{-\frac{1}{2}}. \end{aligned}$$

$$\int_y (y-\mu_i)f_\theta^{1+\gamma}(y)dy = \int_y (y-\mu_i)^3f_\theta^{1+\gamma}(y)dy = 0. \quad (A.2)$$

$$\begin{aligned} \int_y (y-\mu_i)^2f_\theta^{1+\gamma}(y)dy &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \int_y (y-\mu_i)^2 \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \frac{\sqrt{2\pi}\sigma}{\sqrt{1+\gamma}} \left[\frac{\sqrt{1+\gamma}}{\sqrt{2\pi}\sigma} \int_y (y-\mu_i)^2 \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\}\right] \quad (A.3) \\ &= (2\pi)^{-\frac{\gamma}{2}}\sigma^{-\gamma}(1+\gamma)^{-\frac{1}{2}} \frac{\sigma^2}{1+\gamma} \\ &= (2\pi)^{-\frac{\gamma}{2}}\sigma^{-\gamma+2}(1+\gamma)^{-\frac{3}{2}}. \end{aligned}$$

$$\begin{aligned} \int_y (y-\mu_i)^4f_\theta^{1+\gamma}(y)dy &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \int_y (y-\mu_i)^4 \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1+\gamma} \frac{\sqrt{2\pi}\sigma}{\sqrt{1+\gamma}} \left[\frac{\sqrt{1+\gamma}}{\sqrt{2\pi}\sigma} \int_y (y-\mu_i)^4 \exp\left\{-\frac{1+\gamma}{2\sigma^2}(y-\mu_i)^2\right\}\right] \quad (A.4) \\ &= (2\pi)^{-\frac{\gamma}{2}}\sigma^{-\gamma}(1+\gamma)^{-\frac{1}{2}} \frac{3\sigma^4}{(1+\gamma)^2} \\ &= 3(2\pi)^{-\frac{\gamma}{2}}\sigma^{-\gamma+4}(1+\gamma)^{-\frac{5}{2}}. \end{aligned}$$

B. Estimating Equations

From Equation (4) of the main paper, we get

$$\begin{aligned}
 \frac{\partial}{\partial \mu_i} \hat{d}_\gamma(f_\theta, g) &= 0 \\
 \implies \frac{\partial}{\partial \mu_i} \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0 \\
 \implies \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \frac{\partial}{\partial \mu_i} (y_{ij} - \mu_i)^2 &= 0 \\
 \implies \sum_{j=1}^{n_i} (y_{ij} - \mu_i) \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0,
 \end{aligned}
 \tag{B.1}$$

and

$$\begin{aligned}
 \frac{\partial}{\partial \sigma^2} \hat{d}_\gamma(f_\theta, g) &= 0 \\
 \implies \frac{\partial}{\partial \sigma^2} \left\{ \sigma^{-\gamma} \left[1 - \frac{(1+\gamma)^{3/2}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \right] \right\} &= 0 \\
 \implies \left[\frac{\partial}{\partial \sigma^2} \sigma^{-\gamma} \right] \left[1 - \frac{(1+\gamma)^{3/2}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \right] \\
 - \frac{(1+\gamma)^{3/2} \sigma^{-\gamma}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \frac{\partial}{\partial \sigma^2} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0 \\
 \implies -\frac{\gamma}{2} \sigma^{-\gamma-2} \left[1 - \frac{(1+\gamma)^{3/2}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \right] \\
 - \frac{\gamma(1+\gamma)^{3/2} \sigma^{-\gamma-4}}{2N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0 \\
 \implies 1 - \frac{(1+\gamma)^{3/2}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \\
 + \frac{(1+\gamma)^{3/2}}{N\gamma\sigma^2} \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0.
 \end{aligned}
 \tag{B.2}$$

Thus, the estimating equations of θ are obtained from equation $\frac{\partial}{\partial \theta} \hat{d}_\gamma(f_\theta, g) = 0$ and they are simplified as

$$\begin{aligned}
 \sum_{j=1}^{n_i} (y_{ij} - \mu_i) \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0 \text{ for } i = 1, 2, \dots, k, \\
 1 - \frac{(1+\gamma)^{3/2}}{N\gamma} \sum_{i=1}^k \sum_{j=1}^{n_i} \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} \\
 + \frac{(1+\gamma)^{3/2}}{N\gamma\sigma^2} \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \mu_i)^2 \exp\left\{-\frac{\gamma}{2\sigma^2}(y_{ij} - \mu_i)^2\right\} &= 0.
 \end{aligned}
 \tag{B.3}$$

C. Score Functions

The probability density function is given by

$$f_\theta(y_{ij}) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2\sigma^2}(y_{ij} - \mu_i)^2\right\},
 \tag{C.1}$$

Let us define the score function as

$$\begin{aligned}
 u_{\theta}(y_{ij}) &= \frac{\partial}{\partial \theta} \log f_{\theta}(y_{ij}) \\
 &= \frac{\partial}{\partial \theta} \left[-\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} (y_{ij} - \mu_i)^2 \right].
 \end{aligned}
 \tag{C.2}$$

We write

$$u_{\theta}(y_{ij}) = (u_{\mu_1}(y_{ij}), u_{\mu_2}(y_{ij}), \dots, u_{\mu_k}(y_{ij}), u_{\sigma^2}(y_{ij}))^T.
 \tag{C.3}$$

For parameters μ_i and σ^2 , the score functions are given by

$$\begin{aligned}
 u_{\mu_i}(y_{ij}) &= \frac{\partial}{\partial \mu_i} \log f_{\theta}(y_{ij}) \\
 &= \frac{\partial}{\partial \mu_i} \left[-\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} (y_{ij} - \mu_i)^2 \right] \\
 &= \frac{1}{\sigma^2} (y_{ij} - \mu_i),
 \end{aligned}
 \tag{C.4}$$

$$u_{\mu_r}(y_{ij}) = \frac{\partial}{\partial \mu_r} \log f_{\theta}(y_{ij}) = 0, \text{ for } r \neq i,
 \tag{C.5}$$

$$\begin{aligned}
 u_{\sigma^2}(y_{ij}) &= \frac{\partial}{\partial \sigma^2} \log f_{\theta}(y_{ij}) \\
 &= \frac{\partial}{\partial \sigma^2} \left[-\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} (y_{ij} - \mu_i)^2 \right] \\
 &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} (y_{ij} - \mu_i)^2.
 \end{aligned}
 \tag{C.6}$$

D. J and K matrices at the Model

Note that, if the true distribution $g(y)$ is a member of the model family $f_{\theta}(y)$ for some $\theta \in \Theta$, then

$$J^{(ij)} = \int_y u_{\theta}(y) u_{\theta}^T(y) f_{\theta}^{1+\gamma}(y) dy.
 \tag{D.1}$$

In this case, the symmetric matrix $J^{(ij)}$ can be partitioned as

$$J^{(ij)} = \begin{bmatrix} J_{\mu_1}^{(ij)} & J_{\mu_1, \mu_2}^{(ij)} & J_{\mu_1, \mu_3}^{(ij)} & \dots & J_{\mu_1, \mu_k}^{(ij)} & J_{\mu_1, \sigma^2}^{(ij)} \\ \cdot & J_{\mu_2}^{(ij)} & J_{\mu_2, \mu_3}^{(ij)} & \dots & J_{\mu_2, \mu_k}^{(ij)} & J_{\mu_2, \sigma^2}^{(ij)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \dots & J_{\mu_k}^{(ij)} & J_{\mu_k, \sigma^2}^{(ij)} \\ \cdot & \cdot & \cdot & \dots & \cdot & J_{\sigma^2}^{(ij)} \end{bmatrix},
 \tag{D.2}$$

and in Appendix F, it is shown that

$$\begin{aligned}
 J_{\mu_i}^{(ij)} &= (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}}, \\
 J_{\mu_r}^{(ij)} &= 0, \text{ for } r \neq i, \\
 J_{\mu_r, \mu_s}^{(ij)} &= 0, \text{ for } r \neq s, \\
 J_{\sigma^2}^{(ij)} &= \frac{1}{4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-4} (1 + \gamma)^{-\frac{5}{2}} (2 + \gamma^2), \\
 J_{\mu_i, \sigma^2}^{(ij)} &= 0.
 \end{aligned}
 \tag{D.3}$$

The $J^{(ij)}$ matrix simplifies to

$$J^{(ij)} = (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}} \begin{bmatrix} D_i & 0_k \\ 0_k^T & \frac{(2+\gamma^2)}{4\sigma^2(1+\gamma)} \end{bmatrix}, \tag{D.4}$$

where D_i is a $k \times k$ dimensional matrix with (i, i) -th diagonal element 1 and 0 otherwise. Therefore,

$$J = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^k \sum_{j=1}^{n_i} J^{(ij)} = (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}} \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \frac{(2+\gamma^2)}{4\sigma^2(1+\gamma)} \end{bmatrix}, \tag{D.5}$$

where S is a $k \times k$ dimensional diagonal matrix with i -th diagonal element n_i/N .

Similarly, $\zeta^{(ij)}$ can be partitioned as $\zeta^{(ij)} = (\zeta_{\mu_1}^{(ij)}, \zeta_{\mu_2}^{(ij)}, \dots, \zeta_{\mu_k}^{(ij)}, \zeta_{\sigma^2}^{(ij)})^T$, and in Appendix E, it is shown that

$$\zeta_{\mu_i}^{(ij)} = 0, \text{ and } \zeta_{\sigma^2}^{(ij)} = -\frac{\gamma}{2} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}}. \tag{D.6}$$

Note that if we write the matrix $J^{(ij)}$ as a function of γ , i.e., $J^{(ij)} \equiv J^{(ij)}(\gamma)$, then we have

$$K^{(ij)} = J^{(ij)}(2\gamma) - \zeta^{(ij)} \zeta^{(ij)T}. \tag{D.7}$$

So K can be written as

$$K = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^k \sum_{j=1}^{n_i} J^{(ij)}(2\gamma) - \zeta^{(ij)} \zeta^{(ij)T}. \tag{D.8}$$

Using Equations (D.5) and (D.6), we get from Equation (D.8)

$$K = (2\pi)^{-\gamma} \sigma^{-2\gamma-2} (1 + 2\gamma)^{-\frac{3}{2}} \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \nu \end{bmatrix}, \tag{D.9}$$

where

$$\begin{aligned} \nu &= \frac{(1 + 2\gamma^2)}{2\sigma^2(1 + 2\gamma)} - \frac{\gamma^2}{4} (2\pi)^{-\gamma} \sigma^{-2\gamma-4} (1 + \gamma)^{-3} \frac{1}{(2\pi)^{-\gamma} \sigma^{-2\gamma-2} (1 + 2\gamma)^{-\frac{3}{2}}} \\ &= \frac{(1 + 2\gamma^2)}{2\sigma^2(1 + 2\gamma)} - \frac{\gamma^2(1 + 2\gamma)^{\frac{3}{2}}}{4\sigma^2(1 + \gamma)^3}. \end{aligned} \tag{D.10}$$

E. Vector $\zeta^{(ij)}$ at Model

From Equations (D.1) and (C.4), we get

$$\begin{aligned} \zeta_{\mu_i}^{(ij)} &= \int_y u_{\mu_i}(y) f_{\theta}^{1+\gamma}(y) dy \\ &= \int_y \frac{1}{\sigma^2} (y - \mu_i) f_{\theta}^{1+\gamma}(y) dy \\ &= 0, \text{ from (A.2)}. \end{aligned} \tag{E.1}$$

From Equations (D.1) and (C.6), we get

$$\begin{aligned}
 \zeta_{\sigma^2}^{(ij)} &= \int_y u_{\sigma^2}(y) f_{\theta}^{1+\gamma}(y) dy \\
 &= \int_y \frac{1}{\sigma^2} (y - \mu_i) f_{\theta}^{1+\gamma}(y) dy \text{ should not be here} \\
 &= \int_y \left[-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} (y - \mu_i)^2 \right] f_{\theta}^{1+\gamma}(y) dy \\
 &= -\frac{1}{2\sigma^2} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma} (1 + \gamma)^{-\frac{1}{2}} \text{ from (A.1)} \\
 &\quad + \frac{1}{2\sigma^4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma+2} (1 + \gamma)^{-\frac{3}{2}} \text{ from (A.3)} \\
 &= -\frac{\gamma}{2} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}}.
 \end{aligned}
 \tag{E.2}$$

F. Matrix $J^{(ij)}$ at Model

From Equations (D.1) and (C.4), we get

$$\begin{aligned}
 J_{\mu_i}^{(ij)} &= \int_y u_{\mu_i}^2(y) f_{\theta}^{1+\gamma}(y) dy \\
 &= \int_y \frac{1}{\sigma^4} (y - \mu_i)^2 f_{\theta}^{1+\gamma}(y) dy \\
 &= \frac{1}{\sigma^4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma+2} (1 + \gamma)^{-\frac{3}{2}} \text{ from (A.3)} \\
 &= (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}}.
 \end{aligned}
 \tag{F.1}$$

$$\begin{aligned}
 J_{\mu_r}^{(ij)} &= 0, \text{ for } r \neq i, \\
 J_{\mu_r, \mu_s}^{(ij)} &= 0, \text{ for } r \neq s,
 \end{aligned}
 \tag{F.2}$$

From Equations (D.1) and (C.6), we get

$$\begin{aligned}
 J_{\sigma^2}^{(ij)} &= \int_y u_{\sigma^2}^2(y) f_{\theta}^{1+\gamma}(y) dy \\
 &= \int_y \left[-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} (y - \mu_i)^2 \right]^2 f_{\theta}^{1+\gamma}(y) dy \\
 &= \frac{1}{4\sigma^4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma} (1 + \gamma)^{-\frac{1}{2}} \text{ from (A.1)} \\
 &\quad - \frac{1}{2\sigma^6} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma+2} (1 + \gamma)^{-\frac{3}{2}} \text{ from (A.3)} \\
 &\quad + \frac{1}{4\sigma^8} 3(2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma+4} (1 + \gamma)^{-\frac{5}{2}} \text{ from (A.4)} \\
 &= \frac{1}{4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-4} (1 + \gamma)^{-\frac{5}{2}} \left[(1 + \gamma)^2 - 2(1 + \gamma) + 3 \right] \\
 &= \frac{1}{4} (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-4} (1 + \gamma)^{-\frac{5}{2}} (2 + \gamma^2)
 \end{aligned}
 \tag{F.3}$$

From Equations (D.1) and (C.6), we get

$$\begin{aligned}
 J_{\mu_i, \sigma^2}^{(ij)} &= \int_y u_{\mu_i}(y) u_{\sigma^2}(y) f_{\theta}^{1+\gamma}(y) dy \\
 &= \int_y \frac{1}{\sigma^2} (y - \mu_i) \left[-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} (y - \mu_i)^2 \right] f_{\theta}^{1+\gamma}(y) dy \\
 &= 0 \text{ from (A.2)}.
 \end{aligned}
 \tag{F.4}$$

G. Test Statistics

From Equation (D.5), we get

$$J = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^k \sum_{j=1}^{n_i} J^{(ij)} = (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}} \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \frac{(2+\gamma^2)}{4\sigma^2(1+\gamma)} \end{bmatrix} = \lim_{N \rightarrow \infty} J_N, \quad (G.1)$$

where S is a $k \times k$ dimensional diagonal matrix with i -th diagonal element n_i/N . Using the inverse of a block matrix, we have

$$J_N^{-1} = d \begin{bmatrix} S^{-1} & 0_k \\ 0_k^T & \frac{4\sigma^2(1+\gamma)}{(2+\gamma^2)} \end{bmatrix}, \quad (G.2)$$

where $d = (2\pi)^{\frac{\gamma}{2}} \sigma^{\gamma+2} (1 + \gamma)^{\frac{3}{2}}$. From the definition of M in Section ??, we have

$$M = \begin{bmatrix} M_\mu \\ 0_{k-1}^T \end{bmatrix}. \quad (G.3)$$

Now

$$M^T J_N^{-1} = d \begin{bmatrix} M_\mu^T S^{-1} & 0_{k-1} \end{bmatrix}. \quad (G.4)$$

From Equation (D.9), we have

$$K = h \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \nu \end{bmatrix}, \quad (G.5)$$

where $h = (2\pi)^{-\gamma} \sigma^{-2\gamma-2} (1 + 2\gamma)^{-\frac{3}{2}}$. We write

$$K = \lim_{N \rightarrow \infty} K_N.$$

Then

$$\begin{aligned} M^T J_N^{-1} K_N M J_N^{-1} M &= d^2 h \begin{bmatrix} M_\mu^T S^{-1} & 0_{k-1} \end{bmatrix} \begin{bmatrix} S & 0_k \\ 0_k^T & \nu \end{bmatrix} \begin{bmatrix} M_\mu^T S^{-1} & 0_{k-1} \end{bmatrix}^T \\ &= d^2 h \begin{bmatrix} M_\mu^T & 0_{k-1} \end{bmatrix} \begin{bmatrix} M_\mu^T S^{-1} & 0_{k-1} \end{bmatrix}^T \\ &= d^2 h M_\mu^T S^{-1} M_\mu, \end{aligned} \quad (G.6)$$

where

$$\begin{aligned} d^2 h &= (2\pi)^\gamma \sigma^{2\gamma+4} (1 + \gamma)^3 \times (2\pi)^{-\gamma} \sigma^{-2\gamma-2} (1 + 2\gamma)^{-\frac{3}{2}} \\ &= \sigma^2 (1 + \gamma)^3 (1 + 2\gamma)^{-\frac{3}{2}}. \end{aligned} \quad (G.7)$$

Combining Equations (G.6) and (G.7), we get

$$M^T J_N^{-1} K_N M J_N^{-1} M = \sigma^2 (1 + \gamma)^3 (1 + 2\gamma)^{-\frac{3}{2}} M_\mu^T S^{-1} M_\mu. \quad (G.8)$$

H. Matrix Σ_μ

We have $\Sigma_\theta = J^{-1} K J^{-1}$, where

$$J = (2\pi)^{-\frac{\gamma}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}} \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \frac{(2+\gamma^2)}{4\sigma^2(1+\gamma)} \end{bmatrix} = d(\gamma) \lim_{N \rightarrow \infty} J_N, \quad (H.1)$$

$$K = (2\pi)^{-\gamma} \sigma^{-2\gamma-2} (1 + 2\gamma)^{-\frac{3}{2}} \lim_{N \rightarrow \infty} \begin{bmatrix} S & 0_k \\ 0_k^T & \nu \end{bmatrix} = d(2\gamma) \lim_{N \rightarrow \infty} K_N, \quad (H.2)$$

and $d(\gamma) = (2\pi)^{-\frac{k}{2}} \sigma^{-\gamma-2} (1 + \gamma)^{-\frac{3}{2}}$. Now,

$$J_N^{-1} = \begin{bmatrix} S^{-1} & 0_k \\ 0_k^T & \frac{1}{\eta} \end{bmatrix}, \tag{H.3}$$

where $\eta = \frac{(2+\gamma^2)}{4\sigma^2(1+\gamma)}$. Therefore,

$$\begin{aligned} J^{-1}KJ^{-1} &= \frac{d(2\gamma)}{d^2(\gamma)} \lim_{N \rightarrow \infty} J_N^{-1}K_NJ_N^{-1} \\ &= \frac{d(2\gamma)}{d^2(\gamma)} \lim_{N \rightarrow \infty} \begin{bmatrix} I_K & 0_k \\ 0_k^T & \frac{\nu}{\eta} \end{bmatrix} \begin{bmatrix} S^{-1} & 0_k \\ 0_k^T & \frac{1}{\eta} \end{bmatrix} \\ &= \frac{(1 + \gamma)^3 \sigma^2}{(1 + 2\gamma)^{\frac{3}{2}}} \lim_{N \rightarrow \infty} \begin{bmatrix} S^{-1} & 0_k \\ 0_k^T & \frac{\nu}{\eta^2} \end{bmatrix}. \end{aligned} \tag{H.4}$$

Thus, the covariance matrix of $\sqrt{N}\hat{\mu}$ is $\Sigma_\mu = \frac{(1+\gamma)^3 \sigma^2}{(1+2\gamma)^{\frac{3}{2}}} \lim_{N \rightarrow \infty} S^{-1}$. Similarly, the variance of $\sqrt{N}\hat{\sigma}$ is given by $\sigma_\gamma = \frac{(1+\gamma)^3 \sigma^2}{(1+2\gamma)^{\frac{3}{2}}} \frac{\nu}{\eta^2} = \frac{\sigma^4(1+\gamma)^2}{(2+\gamma^2)^2} \left\{ \frac{2(1+\gamma)^3(1+2\gamma^2)}{(1+2\gamma)^{\frac{5}{2}}} - \gamma^2 \right\}$. Finally, $\hat{\mu}$ and $\hat{\sigma}$ are asymptotically independent.