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Interpolating between Positive and Completely Positive Maps: A New Hierarchy of Entangled States

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Abstract: A new class of positive maps is introduced. It interpolates between positive and completely positive maps. It is shown that this class gives rise to a new characterization of entangled states. Additionally, it provides a refinement of the well-known classes of entangled states characterized in terms of the Schmidt number. The analysis is illustrated with examples of qubit maps.

Keywords: qubit maps; contractivity; Schwarz inequality; positive maps; Schmidt number

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1. Introduction

Both positive and completely positive maps play an essential role in quantum information theory [1]. Recall that a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is positive if $\Phi[X] \geq 0$ for $X \geq 0$ [2–4]. Moreover, if Φ is trace-preserving, then it maps quantum states (represented by density operators) into quantum states. In what follows, we consider only finite-dimensional Hilbert spaces $\mathcal{H}$, where $\dim \mathcal{H} = d$. Additionally, we denote a vector space of (bounded) operators acting on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Interestingly, quantum physics requires a more refined notion of positivity due to the fact that a tensor product $\Phi_1 \otimes \Phi_2$ of two positive maps is not necessarily a positive map. A map Φ is called *k-positive* if

$$\mathrm{id}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}) \to M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}), \tag{1}$$

is positive, where id_k denotes the identity map on $M_k(\mathbb{C})$, which is a vector space of $k \times k$ complex matrices. Finally, a map is completely positive if it is k-positive for all $k = 1, 2, \ldots$ Actually, in the finite-dimensional case, complete positivity is equivalent to d-positivity. Hence, if $\mathcal{P}_k$ denotes a (convex) set of k-positive, trace-preserving maps, then there is the following chain of inclusions,

$$\text{CPTP maps} = \mathcal{P}_d \subset \mathcal{P}_{d-1} \subset \ldots \subset \mathcal{P}_1 = \text{PTP maps}. \tag{2}$$

Completely positive maps play a key role in quantum information theory since they correspond to physical operations. In particular, completely positive, trace-preserving (CPTP) maps provide mathematical representations of quantum channels. Any CPTP map satisfies the data processing inequality [1,5,6]. Namely, for an arbitrary quantum channel $\mathcal{E}$ and any pair of states ρ, σ, one has [7]

$$D(\rho||\sigma) \geq D(\mathcal{E}[\rho]||\mathcal{E}[\sigma]), \tag{3}$$

where $D(\rho||\sigma)$ is the relative entropy. Interestingly, it turns out that condition (3) holds for any PTP map.

Maps that are positive but not completely positive find elegant applications in the theory of entanglement [8–11]. A state ρ_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if

$$(\mathrm{id} \otimes \Phi)[\rho_{AB}] \geq 0 \tag{4}$$

for all positive maps Φ. Hence, any violation of (4) witnesses entanglement of ρ_{AB}. The key property of any PTP map is its contractivity with respect to the trace norm [12],

$$\|\Phi[X]\|_{\mathrm{tr}} \leq \|X\|_{\mathrm{tr}}, \tag{5}$$

for any $X \in \mathcal{B}(\mathcal{H})$. This implies that distinguishability between any pair of density operators ρ and σ, defined by $\|\rho - \sigma\|_{\mathrm{tr}}$, cannot increase under the action of a PTP map. Similarly, if Φ is k-positive and trace-preserving, then $\mathrm{id}_k \otimes \Phi$ is contractive.

In this paper, we introduce a new family of maps such that $\mathrm{id} \otimes \Phi$ is contractive but only on the subspaces of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ of particular dimensions. We call such maps *k-partially contractive*, where now $k \in \{1, \ldots, d^2\}$. For $k = 1$ and $k = d^2$, one reproduces PTP maps and CPTP maps, respectively. Hence, this new family interpolates between these two important classes. The inspiration for k-partially contractive maps comes from [13], where the author considered the strength of non-Markovian evolution that lies between P and CP-divisible dynamical maps. We provide the characterisation of partially contractive maps and illustrate this concept with simple qubit maps. Interestingly, in the qubit case, we find a connection between partially contractive maps and the Schwarz maps. The class of partially contractive maps allows us to introduce a new hierarchy of entangled states in full analogy to the well-known characterization in terms of the Schmidt number [8–10,14]. In the qubit case, this new characterization interpolates between separable states (Schmidt number = 1) and entangled states (Schmidt number = 2). Hence, it provides a refinement of the Schmidt number classes. A simple illustration of two-qubit isotropic states is discussed. We hope that the new class of partially contractive maps introduced in this paper will also allow for a more refined analysis of entangled states in higher-dimensional quantum systems.

2. Partially Contractive Maps

Denote by $\mathcal{B}_{\mathrm{H}}(\mathcal{H})$ a real subspace of Hermitian operators in $\mathcal{B}(\mathcal{H})$. Let us recall the following characterisation of PTP maps [4,15].

Proposition 1. *Assume that $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a map that preserves both trace and Hermiticity. Then, Φ is positive if and only if*

$$\|\Phi[X]\|_{\mathrm{tr}} \leq \|X\|_{\mathrm{tr}}, \tag{6}$$

for all Hermitian operators $X \in \mathcal{B}_{\mathrm{H}}(\mathcal{H})$.

Let $\{\rho_1, \ldots, \rho_k\}$ be a set of linearly independent density operators in $\mathcal{B}_{\mathrm{H}}(\mathcal{H})$ and denote by $\mathcal{M}(\{\rho_1, \ldots, \rho_k\}) = \mathrm{span}_{\mathbb{R}}\{\rho_1, \ldots, \rho_k\}$ a real linear subspace of $\mathcal{B}_{\mathrm{H}}(\mathcal{H})$.

Definition 1. *A trace-preserving, Hermiticity-preserving map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called k-partially contractive if, for any set of linearly independent $\{\rho_1, \ldots, \rho_k\}$,*

$$\|(\mathrm{id} \otimes \Phi)[X]\|_{\mathrm{tr}} \leq \|X\|_{\mathrm{tr}} \tag{7}$$

for all $X \in \mathcal{B}_{\mathrm{H}}(\mathcal{H}) \otimes \mathcal{M}(\{\rho_1, \ldots, \rho_k\})$.

Corollary 1. *One has the following correspondence between partial contractivity criteria and positivity of quantum maps:*

1. *Φ is PTP iff it is 1-partially contractive;*
2. *Φ is CPTP iff it is d^2-partially contractive.*

Hence, k-partially contractive, trace-preserving maps are interpolated between PTP and CPTP maps.

Denote by $\mathcal{C}_k$ a set of k-partially contractive, trace-preserving maps. It is easy to show that there is an inclusion relation between different $\mathcal{C}_k$.

Proposition 2. *The set $\mathcal{C}_k$ is a convex subset of $\mathcal{P}_1$ (a set of PTP maps). Moreover,*

$$\text{CPTP maps} = \mathcal{C}_{d^2} \subset \mathcal{C}_{d^2-1} \subset \ldots \subset \mathcal{C}_2 \subset \mathcal{C}_1 = \mathcal{P}_1 = \text{PTP maps}. \tag{8}$$

If $\Phi_1 \in \mathcal{C}_k$ and $\Phi_2 \in \mathcal{C}_\ell$, then the composition $\Phi_1 \circ \Phi_2 \in \mathcal{C}_{\min\{k,\ell\}}$.

Given a set $\{\rho_1, \ldots, \rho_k\}$ and a map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, define a map restricted to $\mathcal{M}(\{\rho_1, \ldots, \rho_k\})$ by

$$\Phi_{\mathcal{M}}[X] = \Phi[X] \tag{9}$$

for any $X \in \mathcal{M}(\{\rho_1, \ldots, \rho_k\})$.

Theorem 1. *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a trace-preserving map. If for any linearly independent set $\{\rho_1, \ldots, \rho_k\}$ the restricted map $\Phi_{\mathcal{M}}$ can be extended to a CPTP map on $\mathcal{B}(\mathcal{H})$, then Φ is k-partially contractive.*

Proof. If $\widetilde{\Phi}_{\mathcal{M}}$ is a CPTP extension of $\Phi_{\mathcal{M}}$, then one has

$$\left\|(\mathrm{id} \otimes \widetilde{\Phi})[X]\right\|_{\mathrm{tr}} \leq \|X\|_{\mathrm{tr}} \tag{10}$$

for any $\mathcal{B}_{\mathrm{H}}(\mathcal{H}) \otimes \mathcal{B}_{\mathrm{H}}(\mathcal{H})$. Hence, if $X \in \mathcal{B}_{\mathrm{H}}(\mathcal{H}) \otimes \mathcal{M}(\{\rho_1, \ldots, \rho_k\})$, then

$$\|(\mathrm{id} \otimes \Phi)[X]\|_{\mathrm{tr}} = \left\|(\mathrm{id} \otimes \widetilde{\Phi})[X]\right\|_{\mathrm{tr}} \leq \|X\|_{\mathrm{tr}}, \tag{11}$$

which proves k-partial contractivity of Φ. □

The problem of finding extensions for positive and completely positive maps is well-studied in mathematical literature. Let us recall a seminal extension theorem of Arveson [16].

Theorem 2. *Assume that $\Phi : S \to \mathcal{B}(\mathcal{H})$ is a CP unital map, where S denotes an operator system in $\mathcal{B}(\mathcal{H})$ (i.e., $\mathbb{1} \in S$ and if $X \in S$, then $X^\dagger \in S$). Then, there exists a (not unique) CP unital extension $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of Φ to $\mathcal{B}(\mathcal{H})$.*

Actually, if S contains strictly positive operatora $X > 0$ and $\Phi : S \to \mathcal{B}(\mathcal{H})$ is CP, then there exists a CP extension $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of Φ [17]. Another interesting result was provided in [18].

Proposition 3. *Consider a CP map $\Phi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{M}$ is spanned by positive operators (e.g., density operators). Then, Φ can be extended to a CP map $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.*

However, the above result only guarantees the existence of a CP extension and says nothing about trace preservation.

3. Qubit Maps

For $d = 2$, we have the following seminal result due to Alberti and Uhlmann [19].

Theorem 3. *Let $\Phi : \mathcal{M}(\{\rho_1, \rho_2\}) \to \mathcal{B}(\mathcal{H})$ be a trace-preserving contraction. Then, Φ can be extended to a CPTP map $\widetilde{\Phi} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.*

This result was recently generalized in [20,21]. Using the Albert–Uhlmann theorem, one comes to the following conclusion.

Corollary 2. *If $d = 2$, then any PTP map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a two-partial contraction.*

Hence, in the qubit case, the hierarchy in (8) simplifies to

$$\text{CPTP maps} = \mathcal{C}_4 \subset \mathcal{C}_3 \subset \mathcal{C}_2 = \mathcal{C}_1 = \mathcal{P}_1 = \text{PTP maps}. \tag{12}$$

Therefore, there exists a class $\mathcal{C}_3$ that interpolates between PTP and CPTP maps. In this section, we analyze $\Phi \in \mathcal{C}_3$.

Consider a triple $\{\rho_1, \rho_2, \rho_3\}$ of linearly independent qubit density operators. From [22], we know that

$$\mathcal{M}(\{\rho_1, \rho_2, \rho_3\}) = \text{span}_{\mathbb{R}}\{UX_1U^\dagger, UX_2U^\dagger, UX_3U^\dagger\}, \tag{13}$$

where $X_1 = \sigma_1$, $X_2 = \sigma_2$, $X_3 = \text{diag}(p, 1-p)$ for some $p \in (0,1)$, and U is a unitary operator depending on the choice of $\{\rho_1, \rho_2, \rho_3\}$. From Definition 1, Φ is three-partially contractive if and only if

$$\left\| \sum_{k=1}^{3} A_k \otimes \Phi[\rho_k] \right\|_{\text{tr}} \leq \left\| \sum_{k=1}^{3} A_k \otimes \rho_k \right\|_{\text{tr}} \tag{14}$$

for all possible choices of $\{\rho_1, \rho_2, \rho_3\}$ and Hermitian qubit operators $\{A_1, A_2, A_3\}$. Using Equation (13), we see that for any set of $\{A_1, A_2, A_3\}$ there exists another set of Hermitian operators $\{B_1, B_2, B_3\}$ such that

$$\sum_{k=1}^{3} A_k \otimes \rho_k = \sum_{k=1}^{3} B_k \otimes UX_kU^\dagger. \tag{15}$$

Now, consider a special class of qubit maps with the following property: for any unitary operator U, there exists a unitary operator V such that

$$\Phi[UXU^\dagger] = V\Phi[X]V^\dagger \tag{16}$$

for any $X \in \mathcal{B}(\mathcal{H})$.

Lemma 1. *If a trace-preserving qubit map Φ satisfies Equation (16), as well as*

$$\left\| \sum_{k=1}^{3} B_k \otimes \Phi[X_k] \right\|_{\text{tr}} \leq \left\| \sum_{k=1}^{3} B_k \otimes X_k \right\|_{\text{tr}} \tag{17}$$

for any $p \in (0,1)$ and all Hermitian $\{B_1, B_2, B_3\}$, then $\Phi \in \mathcal{C}_3$.

Proof. It is enough to show that conditions (14) and (17) are equivalent if Φ satisfies Equation (16). Using Equation (15), one has

$$\left\| \sum_{k=1}^{3} A_k \otimes \rho_k \right\|_{\text{tr}} = \left\| \sum_{k=1}^{3} B_k \otimes X_k \right\|_{\text{tr}}. \tag{18}$$

Now, for a positive, trace-preserving map Φ satisfying Equation (16), it follows that

$$\left\| \sum_{k=1}^{3} A_k \otimes \Phi[\rho_k] \right\|_{\text{tr}} = \left\| \sum_{k=1}^{3} B_k \otimes \Phi[UX_kU^\dagger] \right\|_{\text{tr}} = \left\| \sum_{k=1}^{3} B_k \otimes V\Phi[X_k]V^\dagger \right\|_{\text{tr}} = \left\| \sum_{k=1}^{3} B_k \otimes \Phi[X_k] \right\|_{\text{tr}}. \tag{19}$$

□

Example 1. *Let us propose an example of a positive map that is not three-partially contractive. Consider the transposition map*

$$T[X] = X^T. \tag{20}$$

It can be easily seen that T satisfies condition (16). Now, take a set of three Hermitian operators,

$$B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{21}$$

The corresponding trace norms read

$$\left\| \sum_{k=1}^{3} B_k \otimes X_k \right\|_{\mathrm{tr}} = \left\| \begin{matrix} p & 0 & 0 & 0 \\ 0 & 1-p & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right\|_{tr} = p + \sqrt{(1-p)^2 + 16} \tag{22}$$

and

$$\left\| \sum_{k=1}^{3} B_k \otimes T[X_k] \right\|_{\mathrm{tr}} = \left\| \begin{matrix} p & 0 & 0 & 2 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{matrix} \right\|_{tr} = 1 - p + \sqrt{p^2 + 16}. \tag{23}$$

Hence, we show that T violates condition (17) for $p > \frac{1}{2}$ and thus, from Lemma 1, we conclude that $T \notin \mathcal{C}_3$.

Example 2. *Consider the qubit map*

$$\Lambda_a[X] = \frac{1}{2-a}(\mathbb{1}\mathrm{Tr}X - aX), \tag{24}$$

which is trace-preserving and positive if and only if $0 \le a \le 1$. Let us show when Λ_a is three-partially contractive but not completely positive. Observe that $0 \le a \le 1$ gives the primary constraint for a, as three-partially contractive maps are necessarily positive. The map restricted to the subspace $\mathcal{M}$ reads

$$\Lambda_a|_{\mathcal{M}}[X_1] = -\frac{a}{2-a}X_1, \qquad \Lambda_a|_{\mathcal{M}}[X_2] = -\frac{a}{2-a}X_2, \qquad \Lambda_a|_{\mathcal{M}}[X_3] = \frac{1}{2-a}(\mathbb{1} - aX_3). \tag{25}$$

Now, we extend $\Lambda_a|_{\mathcal{M}}$ to $\widetilde{\Lambda}_a : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ in the following way,

$$\widetilde{\Lambda}_a[X_k] = \Lambda_a|_{\mathcal{M}}[X_k], \quad k = 1,2,3, \qquad \widetilde{\Lambda}_a[X_4] = r\sigma_3, \tag{26}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$, which guarantees the trace preservation of $\widetilde{\Lambda}_a$. The complete positivity of the extension is equivalent to the positivity of its Choi matrix [23],

$$C(\widetilde{\Lambda}_a) = \frac{1}{2}\sum_{j,k=0}^{1} |j\rangle\langle k| \otimes \Lambda[|j\rangle\langle k|] = \frac{1}{2(2-a)} \begin{pmatrix} c(p) & 0 & 0 & -a \\ 0 & b(1-p) & 0 & 0 \\ 0 & 0 & b(p) & 0 \\ -a & 0 & 0 & c(1-p) \end{pmatrix}, \tag{27}$$

where

$$c(p) = 1 - pa + r(1-p)(2-a), \qquad b(p) = 1 - pa - rp(2-a). \tag{28}$$

From Sylvester's criterion [24], $C(\widetilde{\Lambda}_a) \ge 0$ if and only if all its minors have positive determinants. This translates to the condition that there exists a real number r such that

$$c(p) \ge 0, \qquad b(p) \ge 0, \qquad \det A(p) = \det\begin{pmatrix} c(p) & -a \\ -a & c(1-p) \end{pmatrix} \ge 0 \tag{29}$$

for all $0 \le p \le 1$. From the first two inequalities, one obtains

$$-\frac{1}{2-a} \le r \le \frac{1-a}{2-a}. \tag{30}$$

What remains is the condition for the determinant of $A(p)$. For the allowed range of p, the only local extrema of $\det A(p)$ *are maxima, as one has*

$$\frac{\partial^2 \det A(p)}{\partial p^2} = -2(ra - a - 2r)^2 < 0. \quad (31)$$

Therefore, the minimal value of $\det A(p)$ *is achieved at the end points, $p = 0$ and $p = 1$. The sufficient condition for the positivity of* $\det A(p)$ *is the positivity of* $\det A(0)$*, where*

$$\det A(0) = \det A(1) = (1-a)[1 + r(2-a)] - a^2. \quad (32)$$

The last inequality of Equation (29) gives the new upper bound on r,

$$r \geq \frac{a^2 + a - 1}{a^2 - 3a + 2}. \quad (33)$$

Thus, there exists a parameter r such that $\widetilde{\Lambda}_a[X_3] = r\sigma_3$ whenever

$$\frac{a^2 + a - 1}{a^2 - 3a + 2} \leq \frac{1-a}{2-a}, \quad (34)$$

which is satisfied for $a \in [0,1]$ if and only if

$$0 \leq a \leq \frac{2}{3}. \quad (35)$$

In this way, we arrive at the three-partial contractivity condition for Λ_a. Finally, Λ_a is completely positive if and only if $0 \leq a \leq 1/2$. Hence, it is three-partially contractive but not completely positive for

$$\frac{1}{2} < a \leq \frac{2}{3}. \quad (36)$$

Example 3. *Now, consider another trace-preserving qubit map,*

$$\Omega_\epsilon[X] = \frac{\epsilon}{2}\mathbb{1}\mathrm{Tr}X + (1-\epsilon)X^T. \quad (37)$$

This map is positive if and only if $0 \leq \epsilon \leq 2$ and completely positive if and only if $2/3 \leq \epsilon \leq 2$. Using the same method as in the previous example, one shows that Ω_ϵ is three-partially contractive but not completely positive for

$$\frac{1}{2} \leq \epsilon < \frac{2}{3}. \quad (38)$$

4. Partial Contractivity vs. Schwarz Qubit Maps

A positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called a *Schwarz map* if for any $X \in \mathcal{B}(\mathcal{H})$,

$$\|\Phi(\mathbb{1})\|_\infty \Phi[X^\dagger X] \geq \Phi[X^\dagger]\Phi[X], \quad (39)$$

where $\|A\|_\infty$ stands for the operator norm. Any Schwarz map satisfies $\|\Phi\|_\infty = \|\Phi(\mathbb{1})\|_\infty$, where

$$\|\Phi\|_\infty := \sup_{X \in \mathcal{B}(\mathcal{H})} \frac{\|\Phi(X)\|_\infty}{\|X\|_\infty}. \quad (40)$$

If Φ is unital ($\Phi[\mathbb{1}] = \mathbb{1}$), then Equation (39) reduces to [4,25,26]

$$\Phi[X^\dagger X] \geq \Phi[X^\dagger]\Phi[X]. \quad (41)$$

Note that condition (39) is sufficient for positivity and necessary for complete positivity. A composition $\Phi_1 \circ \Phi_2$ and a convex combination $q\Phi_1 + (1-q)\Phi_2$ of two unital Schwarz maps is also a unital Schwarz map. It was proven by Kadison that any positive unital map satisfies Equation (41) for Hermitian X (the celebrated Kadison inequality [27]). If Φ is not unital but $V = \Phi[\mathbb{1}] > 0$, then $\Psi(X) := V^{-1/2}\Phi[X]V^{-1/2}$ is inital and Ψ is a Schwarz map if and only if

$$\Phi[X^\dagger X] \geq \Phi[X^\dagger]\Phi[\mathbb{1}]^{-1}\Phi[X] \geq \frac{1}{\|\Phi\|_\infty}\Phi[X^\dagger]\Phi[X], \tag{42}$$

for arbitrary $X \in \mathcal{B}(\mathcal{H})$. Hence, Φ satisfies (39). Now, for PTP unital qubit maps, one finds the following hierarchy,

$$\text{CPTP unital maps} \subset S \subset \text{PTP unital maps}, \tag{43}$$

where S denotes the set of Schwarz maps. After comparing Equation (12) with Equation (43), it would be interesting to analyze the relation between two classes of maps: three-partially contractive and Schwarz maps.

A method of constructing the Schwarz maps was proposed in [28]. Take a positive, trace-preserving, unital map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is a contraction $\|\Psi[X]\|_2 \leq \|X\|_2$ in the Hilbert–Schmidt (Frobenius) norm, where $\|X\|_2 = \sqrt{\mathrm{Tr}(X^\dagger X)}$. Define the qubit map

$$\Phi[X] = \frac{q}{2}\mathbb{1}\mathrm{Tr}X + (1-q)\Psi[X]. \tag{44}$$

Now, Φ is a Schwarz map if

$$\frac{1}{2} \leq q \leq \frac{3}{2}. \tag{45}$$

The following stronger results were also proven.

Proposition 4 ([28])**.** *The map Λ_a defined in Equation (24) is the Schwarz map (that is not CP) if and only if $\frac{1}{2} < a \leq \frac{2}{3}$.*

Proposition 5 ([28])**.** *The map Ω_ϵ defined in Equation (37) is the Schwarz map (that is not CP) if and only if $\frac{1}{2} \leq \epsilon < \frac{2}{3}$.*

At this point, we make an important observation: the sufficient condition for three-partial contractivity and the necessary and sufficient conditions for Schwarz maps coincide for Λ_ϵ and Ω_ϵ. In other words, the following relations hold,

$$\text{CPTP unital maps} \subset \mathcal{C}_3 \subset S \subset \text{PTP unital maps}. \tag{46}$$

Therefore, one observes an intricate connection between the Schwarz maps and three-partially contractive maps. This problem deserves further analysis.

5. Partial Contractions vs. Quantum Entanglement

Any state vector $\psi \in \mathcal{H} \otimes \mathcal{H}$ gives rise to the Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{r} s_i |e_i\rangle \otimes |f_i\rangle, \tag{47}$$

where the Schmidt rank $r = \mathrm{SR}(\psi)$ satisfies $1 \leq r \leq d$. This concept can be easily generalized to density operators [14]: given ρ, one defines its Schmidt number

$$\mathrm{SN}(\rho) = \min_{\{p_k, \psi_k\}} \max_k \mathrm{SR}(\psi_k), \tag{48}$$

where the minimization is carried over all pure state decompositions $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. If $\rho = |\psi\rangle\langle\psi|$, then $\mathrm{SN}(\rho) = \mathrm{SR}(\psi)$. Furthermore, the generalized Schmidt number is used to measure entanglement in a multipartite scenario [29–33].

Proposition 6 ([10]). *A map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is k-positive if and only if*

$$(\mathrm{id} \otimes \Phi)[\rho] \geq 0 \tag{49}$$

for any $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\mathrm{SN}(\rho) \leq k$.

This result allows us to provide the following classification of entangled states in $\mathcal{H} \otimes \mathcal{H}$,

$$\text{separabale states} = \mathbb{E}_1 \subset \mathbb{E}_2 \subset \ldots \subset \mathbb{E}_{d-1} \subset \mathbb{E}_d = \text{all states}, \tag{50}$$

where $\mathbb{E}_k$ contains all states with $\mathrm{SN} \leq k$. The hierarchy in (50) is dual to (2).

Example 4. *Consider a map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by*

$$\Phi_p[X] = p\,\mathbb{1}_d \mathrm{Tr} X - X, \tag{51}$$

with $p > 0$. Choi showed that Φ_p is k-positive but not $(k+1)$-positive if and only if $k \leq p < k+1$ [34]. In particular, Φ is positive if $p \geq 1$. In the entanglement theory, the map Φ_1 is called the reduction map and plays an important role in classifying states of composite systems [8–10]. Now, consider the family of isotropic states in $\mathcal{H} \otimes \mathcal{H}$,

$$\rho_f = \frac{1-f}{d^2-1}(\mathbb{1}_d \otimes \mathbb{1}_d - P_d^+) + f P_d^+, \quad f \in [0,1], \tag{52}$$

where P_d^+ denotes the projector onto the maximally entangled state. One finds that if the fidelity $f > k/d$, then $\mathrm{SN}(\rho) \geq k+1$ [14].

A large class of k-positive maps based on spectral property of the Choi matrix was proposed in [35]. It provides a generalization of the Choi map from Example 4.

Note that one can easily define a hierarchy similar to (8) via

$$\text{separabale states} = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_{d^2-1} \subset \mathcal{E}_{d^2} = \text{all states}, \tag{53}$$

where $\mathcal{E}_k$ contains such states ρ for which

$$(\mathrm{id} \otimes \Phi)[\rho] \geq 0 \tag{54}$$

for all $\Phi \in \mathcal{C}_k$. In the qubit case, the above hierarchy reduces to

$$\text{separabale states} = \mathcal{E}_1 = \mathcal{E}_2 \subset \mathcal{E}_3 \subset \mathcal{E}_4 = \text{all states}. \tag{55}$$

Example 5. *Consider the action of Λ_a from Equation (24) on one part of the two-qubit isotropic states ρ_f defined in Equation (52). We find that*

$$(\mathrm{id} \otimes \Lambda_a)[\rho_f] = \frac{1}{6(2-a)} \begin{bmatrix} 3-a(1+2f) & 0 & 0 & -a(4f-1) \\ 0 & 3-2a(1-f) & 0 & 0 \\ 0 & 0 & 3-2a(1-f) & 0 \\ -a(4f-1) & 0 & 0 & 3-a(1+2f) \end{bmatrix}. \tag{56}$$

The above matrix is positive in the range $\frac{1}{2} < a \leq \frac{2}{3}$, where Λ_a is three-partially contractive but not completely positive, if and only if

$$0 \leq f \leq \frac{3}{4}. \tag{57}$$

The same result is obtained if one considers Ω_ϵ instead of Λ_a. Thus, the above inequality provides a necessary condition for $\rho_f \in \mathcal{E}_3$. Note that $\rho_f \notin \mathcal{E}_2$ (is entangled) for $1/2 < f \leq 1$, and if $f > 3/4$, then $\rho_f \notin \mathcal{E}_3$ (highly entangled).

6. Conclusions

We proposed a new classification of positive maps based on the contractivity with respect to the trace norm on subspaces of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. The k-partially contractive maps give rise to a hierarchy of d^2 classes of positive maps that interpolate between PTP and CPTP maps. The parameter k, which is the dimension of the subspace, can be interpreted as a strength of positivity. For qubit maps, we introduced an intermediate class of positive but not completely positive maps, corresponding to the choice $k = 3$. Moreover, we provided an analytical technique for assessing three-partial contractivity. Our concept was illustrated with examples of simple qubit maps. Interestingly, our analysis showed that there is a connection between partially contractive maps and the Schwarz maps. Finally, we applied these results to refine the hierarchy of entangled states based on the Schmidt number. In the qubit case, we obtained a single class that interpolates between separable and entangled states.

The topic of partially contractive maps requires further analysis. The first step would be to find less restrictive sufficient conditions for three-partially contractive qubit maps. Then, the full relation between these maps and the Schwarz maps could be established. Additionally, it is crucial to find a computational method of constructing k-partially contractive maps in any finite dimension d. To achieve this, one needs a relation analogical to Equation (13) for $d = 2$. Then, one would be able to find the connection between k-partial contractivity and k-positivity. Another open question concerns possible applications of our classification. One implementation, already touched upon in this paper, is a more refined analysis of entangled states.

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References

1. Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*; Cambridge University Press: Cambridge, UK, 2010.
2. Størmer, E. *Positive Linear Maps of Operator Algebras*; Springer: Berlin, Germany, 2013.
3. Størmer, E. Positive linear maps of operator algebras. *Acta Math.* **1963**, *110*, 233–278. [CrossRef]
4. Paulsen, V. *Completely Bounded Maps and Operator Algebras*; Cambridge University Press: Cambridge, UK, 2003.
5. Wilde, M.M. *Quantum Information Theory*; Cambridge University Press: Cambridge, UK, 2013.
6. Hayashi, M. *Quantum Information: An Introduction*; Springer: Berlin, Germany, 2006.
7. Lindblad, G. Completely positive maps and entropy inequalities. *Commun. Math. Phys.* **1975**, *40*, 147. [CrossRef]
8. Horodecki, R.; Horodecki, P.; Horodecki, M.; Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **2009**, *81*, 865. [CrossRef]
9. Gühne, O.; Tóth, G. Entanglement detection. *Phys. Rep.* **2009**, *474*, 1–75. [CrossRef]
10. Chruściński, D.; Sarbicki, G. Entanglement witnesses: Construction, analysis and classification. *J. Phys. A Math. Theor.* **2014**, *47*, 483001. [CrossRef]
11. Bengtsson, I.; Życzkowski, K. *Geometry of Quantum States: An Introduction to Quantum Entanglement*; Cambridge University Press: Cambridge, UK, 2007.
12. Kossakowski, A. On necessary and sufficient conditions for a generator of a quantum dynamical semi-group. *Bull. Acad. Polon. Sci. Sér. Math. Astr. Phys* **1972**, *20*, 1021–1025.
13. Chakraborty, S. Generalized formalism for information backflow in assessing Markovianity and its equivalence to divisibility. *Phys. Rev. A* **2018**, *97*, 032130. [CrossRef]
14. Terhal, B.M.; Horodecki, P. Schmidt number for density matrices. *Phys. Rev. A* **2000**, *61*, 040301. [CrossRef]

15. Ruskai, M.B.; Szarek, S.; Werner, E. An analysis of completely-positive trace-preserving maps on M_2. *Linear Algebra Appl.* **2002**, *347*, 159–187. [CrossRef]
16. Arveson, W. Subalgebra of C^*-algebras. *Acta Math.* **1969**, *123*, 141. [CrossRef]
17. Heinosaari, T.; Jivulescu, M.A.; Reeb, D.; Wolf, M.M. Extending quantum operations. *J. Math. Phys.* **2012**, *53*, 102208. [CrossRef]
18. Jenčová, A. Generalized channels: Channels for convex subsets of the state space. *J. Math. Phys.* **2012**, *53*, 012201. [CrossRef]
19. Alberti, P.M.; Uhlmann, A. A Problem Relating to Positive Linear Maps on Matrix Algebras. *Rep. Math. Phys.* **1980**, *18*, 163–176. [CrossRef]
20. Chakraborty, S.; Chruściński, D.; Sarbicki, G.; vom Ende, F. On the Alberti-Uhlmann Condition for Unital Channels. *Quantum* **2020**, *4*, 360. [CrossRef]
21. Dall'Arno, M.; Buscemi, F.; Scarani, V. Extension of the Alberti-Ulhmann criterion beyond qubit dichotomies. *Quantum* **2020**, *4*, 233. [CrossRef]
22. Chakraborty, S.; Chruściński, D. Information flow versus divisibility for qubit evolution. *Phys. Rev. A* **2019**, *99*, 042105. [CrossRef]
23. Choi, M.D. Completely Positive Linear Maps on Complex Matrices. *Linear Algebra Appl.* **1975**, *10*, 285–290. [CrossRef]
24. Gilbert, G. Positive Definite Matrices and Sylvester's Criterion. *Am. Math. Mon.* **1991**, *98*, 44–46. [CrossRef]
25. Bhatia, R. *Positive Definite Matrices*; Princeton University Press: Princeton, NJ, USA, 2006.
26. Kadison, R.V.; Ringrose, J. *Fundamentals of the Theory of Operator Algebras*; Academic Press: New York, NY, USA, 1956.
27. Kadison, R.V. A generalised Schwarz inequality and algebraic invariants for operator algebras. *Ann. Math.* **1952**, *56*, 494–503. [CrossRef]
28. Chruściński, D.; Mukhamedov, F.; Hajji, M.A. On Kadison-Schwarz Approximation to Positive Maps. *Open Sys. Inf. Dyn.* **2020**, *27*, 2050016. [CrossRef]
29. Eisert, J.; Briegel, H.J. Schmidt measure as a tool for quantifying multiparticle entanglement. *Phys. Rev. A* **2001**, *64*, 022306. [CrossRef]
30. Guo, Y.; Fan, H. A generalization of Schmidt number for multipartite states. *Int. J. Quantum Inf.* **2015**, *13*, 1550025. [CrossRef]
31. Chen, L.; Yang, Y.; Tang, W.-S. Schmidt number of bipartite and multipartite states under local projections. *Quantum Inf. Process.* **2017**, *16*, 75.
32. Wang, Y.; Liu, T.; Ma, R. Schmidt Number Entanglement Measure for Multipartite k-nonseparable States. *Int. J. Theor. Phys.* **2020**, *59*, 983–990. [CrossRef]
33. Szarek, S.J.; Werner, E.; Życzkowski, K. How often is a random quantum state k-entangled? *J. Phys. A Math. Theor.* **2011**, *44*, 045303. [CrossRef]
34. Choi, M.D. Positive linear maps on C^*-algebras. *Can. J. Math.* **1972**, *24*, 520–529. [CrossRef]
35. Chruściński, D.; Kossakowski, A. Spectral Conditions for Positive Maps. *Commun. Math. Phys.* **2009**, *290*, 1051. [CrossRef]