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Review

Properties of Nonnegative Hermitian Matrices and New Entropic Inequalities for Noncomposite Quantum Systems

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Abstract: We consider the probability distributions, spin (qudit)-state tomograms and density matrices of quantum states, and their information characteristics, such as Shannon and von Neumann entropies and q-entropies, from the viewpoints of both well-known purely mathematical features of nonnegative numbers and nonnegative matrices and their physical characteristics, such as entanglement and other quantum correlation phenomena. We review entropic inequalities such as the Araki–Lieb inequality and the subadditivity and strong subadditivity conditions known for bipartite and tripartite systems, and recently obtained for single qudit states. We present explicit matrix forms of the known and some new entropic inequalities associated with quantum states of composite and noncomposite systems. We discuss the tomographic probability distributions of qudit states and demonstrate the inequalities for tomographic entropies of the qudit states. In addition, we mention a possibility to use the discussed information properties of single qudit states in quantum technologies based on multilevel atoms and quantum circuits produced of Josephson junctions.

Keywords: von Neumann entropy; information and entropic inequalities; spin tomography; single-qudit subadditivity condition

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1. Introduction

Quantum states are characterized by entropies, which have the properties associated with well-known purely mathematical properties of nonnegative Hermitian matrices (see, for example, [1,2]). A review of some entropic and information theoretic inequalities is presented in [3]. Quantum correlations reflected in a phenomenon, such as the state entanglement in composite bipartite or multipartite systems are known to provide the resource for quantum technologies like quantum computing, quantum teleportation, *etc.* [4]. To characterize quantum correlations, one can employ the entropic and information characteristics given in terms of von Neumann entropies of the states of multipartite systems and their subsystems. The von Neumann entropy is determined by the quantum-state density matrix [5,6]. The notion of the density matrix was introduced in [7]; for mixed states of composite systems, the density matrix provides the density matrices of the subsystem states, which can be obtained using the partial tracing procedure. The qudit state can be described by the spin tomogram [8,9]. There exist q-entropies determined by the density matrices like Rényi [10] and Tsallis [11] entropies, which depend on the parameter q.

For diagonal density matrices, the von Neumann entropy and q-entropies provide the entropies associated with classical probability distributions, such as, for example, Shannon entropy [12]. The Shannon entropy determined by the probability distribution is a characteristic of the order in the system. The entropy takes a maximum value for a complete disorder in the system and is equal to zero for the complete order. In the limit $q \to 1$, the q-entropies under discussion become the Shannon entropy for classical probability distributions and are equal to the von Neumann entropy for quantum density matrices. The development of experimental techniques like quantum tomography [13] provided a possibility to measure the density matrices of quantum states and obtain, as the results of the experiments, numerical values of the matrix elements of the density matrices for qudits. Since the entropic and information characteristics of quantum states used in the experiments with superconducting circuits discussed, for example, in [14–16] are expressed in terms of the density matrix elements, it is desirable to have explicit formulas for the entropic inequalities containing the matrix elements. In this connection, we express some entropic inequalities known for multipartite systems in an explicit form to be applied for studying the states of the systems without subsystems. It is worth pointing out that all inequalities considered in this paper like the subadditivity and strong subadditivity conditions, as well as the relative entropy nonnegativity, are well known. In this paper, we focus on the fact that the same inequalities can be applied in experiments where the noncomposite systems are studied.

Entropic and information inequalities exist for both classical and quantum entropies, including the q-entropies [3,17–23]. These inequalities are related to correlations in the systems. For classical random variables, the entropic inequalities are related to classical correlations, and for quantum observables the inequalities are related to quantum correlations in the systems. On the other hand, it was pointed out in [24] that all entropies and informations are expressed only either in terms of the probability distributions for classical random variables, which present a set of nonnegative numbers satisfying the normalization condition, or in terms of nonnegative trace-class Hermitian matrices in the case of quantum states.

The known entropic inequalities, from the viewpoint of purely numerical relations, are the formulas containing either expressions with nonnegative numbers only or the expressions containing the matrix elements of nonnegative Hermitian matrices with unit trace. The inequalities written in the form of purely numerical relations for the nonnegative numbers and nonnegative matrices are valid per se and do not depend on the interpretation of these numbers as the probability distributions or the matrices as the density matrices of quantum states. Nevertheless, in quantum interpretation applications, the numerical properties can be translated as specific properties of quantum systems. In this context, the known inequalities have a new physical interpretation.

The aim of this paper is to review a recent approach employed in [25–34] to study the possibility of finding such new entropic inequalities like the subadditivity and strong subadditivity conditions for noncomposite quantum systems and obtaining some other entropic and information equalities and inequalities for conditional and relative entropies and q-entropies known for composite systems, which can be also introduced for the systems without subsystems like, e.g., a single qudit. We analyze which aspects of known quantum and classical entropic inequalities depend only on the properties of nonnegative numbers and matrices and which aspects depend on the interpretation of the numbers and matrices as probability distributions and density matrices of physical system states.

The other goal of this paper is to discuss which aspects of the entropic inequalities depend on the interpretation of the probability distributions and nonnegative matrices given in the form of a set of nonnegative numbers or a set of complex numbers organized in a table with columns and rows, as the joint probability distributions for composite classical systems or as the density matrix corresponding to the density operator acting in the Hilbert space of a tensor-product form $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ associated with the multipartite system states. We show that the known entropic inequalities valid for multipartite systems are also valid for the systems without subsystems; they characterize correlations of the degrees of freedom of, e.g., only one single qudit state. We illustrate this statement considering a two-qubit system (composite system) and qudit with spin $j = 3/2$ (noncomposite system) and demonstrate the same entropic subadditivity condition known for two-qubit states on an example of the state of the single qudit system.

This paper is organized as follows.

In Section 2, we review the approach where the bijective map of integers $1, 2, \ldots, N$ onto pairs of integers, triples of integers, *etc.* is used to interpret sets of nonnegative numbers p_s, $s = 1, 2, \ldots, N$ as the joint probability distributions describing random variables in bipartite, tripartite, *etc.* systems. In Section 3, we discuss the use of the map to interpret a $N \times N$ matrix (table of complex numbers) that is a Hermitian nonnegative matrix ρ with unit trace as the density matrix of the system without subsystems or the density matrix of the bipartite, tripartite, *etc.* quantum system state. In Section 4, we present the entropic subadditivity and strong subadditivity conditions in the form of numerical inequalities; this provides the subadditivity condition for the single qudit state. In Section 5, we demonstrate the equality known for bipartite system in the form of a matrix equality, which yields an analogous equality for the single-qudit pure state. In Section 6, we describe the strong subadditivity condition known for the tripartite systems in the form of a numerical matrix inequality, which provides the strong subadditivity condition for a single qudit. We obtain the subadditivity condition for weighted entropy [35] for a single qudit state in Section 7, discuss the entropic relation for spin tomograms in Section 8, and present relative entropy inequalities in Section 9. In Section 10, we give the chain relation for conditional entropy (known for multipartite systems) for a single qudit state. Finally, we present our conclusions and the prospectives in Section 11.

2. Set of Nonnegative Numbers as the Probability Distribution

Given N positive numbers, which we denote as $p_s \geq 0$, where $s = 1, 2, \ldots, N$ and $\sum_{s=1}^{N} p_s = 1$. Let the integer N be equal to the product of integers $(N = nm)$. Then we can use the map (or partition)

$$1 \leftrightarrow 11,\ 2 \leftrightarrow 12, \ldots, m \leftrightarrow 1m,\ m+1 \leftrightarrow 21,\ m+2 \leftrightarrow 22, \ldots,\ N-1 \leftrightarrow nm-1,\ N \leftrightarrow nm.$$

This means that we constructed the map of nonnegative numbers p_s onto nonnegative numbers P_{jk}, where $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. The introduced map $s \leftrightarrow jk$ means that we constructed a function of two variables $s(jk)$, which provides the value $s(jk)$ for the pair of integers j and k. The table of numbers P_{jk} can be interpreted as a joint probability distribution of two classical random variables. One has the normalization condition $\sum_{j=1}^{n} \sum_{k=1}^{m} P_{jk} = 1$.

The set p_s can be interpreted as the probability distribution of one random variable. The interpretation of the set P_{jk} as a joint probability distribution opens the possibility to introduce other sets of nonnegative numbers, which correspond to marginal probability distributions $\mathcal{P}_j^{(1)} = \sum_{k=1}^{m} P_{jk}$ and $\mathcal{P}_k^{(2)} = \sum_{j=1}^{n} P_{jk}$.

For example, if $N = 4$, then $p_1 = P_{11}$, $p_2 = P_{12}$, $p_3 = P_{21}$, $p_4 = P_{22}$ and $\mathcal{P}_1^{(1)} = p_1 + p_2$, $\mathcal{P}_2^{(1)} = p_3 + p_4$, $\mathcal{P}_1^{(2)} = p_1 + p_3$, $\mathcal{P}_2^{(2)} = p_2 + p_4$.

On the other hand, the conditional probability distributions known for bipartite systems with joint probability distributions P_{jk} can be presented as sets of nonnegative numbers expressed in terms of nonnegative numbers p_s.

For $N = 4$, one has the pairs of nonnegative numbers $P^A(1|1) = \dfrac{p_1}{p_1 + p_3}$, $P^A(2|1) = \dfrac{p_3}{p_1 + p_3}$ and $P^A(1|2) = \dfrac{p_2}{p_2 + p_4}$, $P^A(2|2) = \dfrac{p_4}{p_2 + p_4}$. The notation $P^A(j|k)$ corresponds to the conditional probability related to the joint probability distribution of the bipartite system state P_{jk} given by the Bayes formula for subsystem A. Thus, we introduce artificial subsystems A and B to associate sets of nonnegative numbers, which we denote as $P^A(j|k) = \dfrac{P_{jk}}{\sum_{j'=1}^{n} P_{j'k}}$ and $P^B(k|j) = \dfrac{P_{jk}}{\sum_{k'=1}^{n} P_{jk'}}$, with the set of numbers p_s. In view of the used map $s \leftrightarrow jk$, the numbers $P^A(j|k)$ and $P^B(k|j)$ are expressed in terms of nonnegative numbers p_s.

The other possibility to label numbers p_s takes place for $N = n_1 n_2 n_3$, where n_1, n_2, and n_3 are integers. Then we employ the bijective map $s \leftrightarrow jkl$, where $j = 1, 2, \ldots n_1$, $k = 1, 2, \ldots n_2$, and $l = 1, 2, \ldots n_3$.

For $N = 8$, one has the map

$$1 \leftrightarrow 111,\ 2 \leftrightarrow 112,\ 3 \leftrightarrow 121,\ 4 \leftrightarrow 122,\ 5 \leftrightarrow 211,\ 6 \leftrightarrow 212,\ 7 \leftrightarrow 221,\ 8 \leftrightarrow 222.$$

This means that we introduced a function of three variables $s(jkl)$, which provides the integer value $s(jkl)$ for the triple of integers j, k, and l. The map provides the possibility to interpret the set of N nonnegative numbers p_s as the joint probability distribution P_{jkl} of tripartite classical system with three random variables. Such an interpretation provides the possibility to construct a set of nonnegative numbers using numbers p_s, and the numbers constructed correspond to marginal probability distributions and conditional probability distributions known for tripartite classical systems.

Thus, one has "marginals" $\mathcal{P}_j^A = \sum_{k=1}^{n_2} \sum_{l=1}^{n_3} P_{jkl}$, $\mathcal{P}_k^B = \sum_{j=1}^{n_1} \sum_{l=1}^{n_3} P_{jkl}$, $\mathcal{P}_l^C = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} P_{jkl}$, $\mathcal{P}_{jk}^{AB} = \sum_{l=1}^{n_2} P_{jkl}$, and $\mathcal{P}_{kl}^{BC} = \sum_{j=1}^{n_1} P_{jkl}$. We use the notation corresponding to the interpretation of the set of numbers $p_s \leftrightarrow P_{jkl}$ as joint probability distributions corresponding to the description of the statistics of composite systems containing three subsystems A, B, and C. One can point out that there exist purely numerical relations of the numbers in the set of nonnegative numbers p_s. In view of different labeling of the numbers, the possibility to interpret the numbers as the probability distribution arises, but available numerical relations are valid per se independently of the interpretation.

3. Density Matrices of Quantum States

Given $N \times N$-matrix ρ with matrix elements $\rho_{ss'}$, where $s, s' = 1, 2, \ldots, N$, such that $\rho_{ss'}^* = \rho_{s's}$ and $\mathrm{Tr}\,\rho = 1$ with nonnegative eigenvalues. The matrix can be interpreted as the density matrix of a quantum state of the qudit with $j = (N-1)/2$. In this case, the matrix ρ can be considered as a matrix representing the density operator $\hat{\rho}$ acting in the Hilbert space H of the qudit (spin) states.

On the other hand, if $N = nm$, one can use the discussed map of integers $s \leftrightarrow jk$, $s' \leftrightarrow j'k'$, where $jj' = 1, 2, \ldots, n$ and $kk' = 1, 2, \ldots, m$, and identify the matrix elements $\rho_{ss'}$ with the matrix elements $\rho_{jk,j'k'}$ of the nonnegative matrix, which we denote as $\rho^{(AB)}$, representing the density operator $\hat{\rho}$ acting in the Hilbert space of bipartite system states $H = H_1 \otimes H_2$. This interpretation induces the possibility to introduce new matrices ρ^A with matrix elements $\rho_{jj'}^A = \sum_{k=1}^m \rho_{jk,j'k}$ and $\rho_{kk'}^B = \sum_{j=1}^n \rho_{jk,jk'}$.

Since matrix $\rho^{(AB)}$ is nonnegative Hermitian matrix with unit trace, matrices ρ^A and ρ^B are the $n \times n$-matrices and $m \times m$-matrices, respectively, such that $(\rho^A)_{j'j}^* = (\rho^A)_{jj'}$, $(\rho^B)_{k'k}^* = (\rho^B)_{kk'}$, $\sum_{j=1}^n \rho_{jj}^A = 1$, $\sum_{k=1}^m \rho_{kk}^B = 1$, and the eigenvalues of matrices ρ^A and ρ^B are nonnegative numbers. The numerical matrices obtained can be interpreted as density matrices of two qudit states, respectively, corresponding to partial tracing of the matrix $\rho^{(AB)}$. Numerical properties of the matrices $\rho_{ss'}$, $\rho_{jj'}^A$, and $\rho_{kk'}^B$ do not depend on the interpretation of these matrices as the density matrices of quantum system states. In view of this fact, the relations like equalities and inequalities valid for matrix elements of the matrices ρ, ρ^A, and ρ^B can be applied, if the matrix ρ is a nonnegative Hermitian matrix with nonnegative eigenvalues or if this matrix is interpreted as the density matrix of a single qudit state, or if the matrix is interpreted as the density matrix of a bipartite system state. This simple observation provides the possibility to extend the matrix relations known for density matrices of bipartite systems (systems of two qudits) to the case of a single qudit system.

Following the same approach, in which the map $s \leftrightarrow jkl$ is used, one can consider the numerical $N \times N$-matrix $\rho_{ss'}$ as a matrix with matrix elements $\rho_{jkl,j'k'l'}$. This can be done if the integer $N = n_1 n_2 n_3$, where n_1, n_2, and n_3 are integers. Thus, the same matrix ρ can be interpreted as the density matrix of a three-partite quantum system; in this case, we denote this matrix as $\rho^{(ABC)}$ and the matrix elements as $\rho_{ss'} \leftrightarrow \rho_{jkl,j'k'l'}$.

Similar to the case where we considered this matrix as the matrix corresponding to the bipartite system state, now we can consider this matrix as the density matrix of a tripartite quantum system, and there is a prescription how to obtain other nonnegative matrices by the partial tracing procedure.

Thus, one has the nonnegative matrices with unit trace denoted as $\rho^{(AB)}_{jk,j'k'} = \sum_{l=1}^{n_3} \rho_{jkl,j'k'l}$, $\rho^{(BC)}_{kl,k'l'} = \sum_{j=1}^{n_1} \rho_{jkl,jk'l'}$, and $\rho^{B}_{kk'} = \sum_{j=1}^{n_1}\sum_{l=1}^{n_3} \rho_{jkl,jk'l}$. Here, we introduce the notation corresponding to the notation we used in the case where the matrix ρ was the density matrix of the state of a composite system with three subsystems A, B, and C. Then the matrices $\rho^{(AB)}_{jk,j'k'}$, $\rho^{(BC)}_{kl,k'l'}$, and $\rho^{B}_{kk'}$ are the density matrices of the states of subsystems $\rho^{(AB)}$, $\rho^{(BC)}$, and ρ^{B}, respectively. However, if the matrix ρ is not associated with any density matrix of a tripartite quantum system, we simply have numerical matrices with unit traces and nonnegative eigenvalues, and their properties do not depend on the interpretation of the matrices as the density matrices of quantum states.

4. Entropic Subadditivity and Strong Subadditivity Conditions

For any probability distribution p_s, one has the Shannon entropy

$$H = -\sum_{s=1}^{N} p_s \ln p_s \tag{1}$$

and q-entropy

$$H_q = -\sum_{s=1}^{N} p_s^q \, \frac{p_s^{1-q}-1}{1-q}\,. \tag{2}$$

In the limit $q \to 1$, the entropy $H_q \to H$.

For any density matrix ρ, one has the von Neumann entropy

$$S = -\mathrm{Tr}\,\rho \ln \rho \tag{3}$$

and q-entropy

$$S_q = -\mathrm{Tr}\,\rho^q \, \frac{\rho^{1-q}-1}{1-q}\,. \tag{4}$$

Using the map (discussed in Section 2) of the probability distribution p_s on the table P_{jk}, we can write the inequality known as the subadditivity condition for bipartite system, which in terms of numbers p_s reads

$$-\sum_{j=1}^{n}\left(\sum_{k=1}^{m} p_{s(jk)}\right)\left(\ln \sum_{k'=1}^{m} p_{s(jk')}\right) - \sum_{k=1}^{m}\left(\sum_{j=1}^{n} p_{s(jk)}\right)\left(\ln \sum_{j'=1}^{n} p_{s(j'k)}\right) \geq -\sum_{s=1}^{N} p_{s(jk)} \ln p_{s(jk)}. \tag{5}$$

Here, we use the notation $p_{s(jk)}$ that shows what integer s corresponds to the pair jk according to the map we consider. Inequality (5) is the subadditivity condition for the probability distribution p_s (or for a set of nonnegative numbers p_s).

If $N = n_1 n_2 n_3$, we have the strong subadditivity condition for nonnegative numbers p_s or the probability distribution associated with these numbers; the inequality reads

$$\begin{aligned} & -\sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\sum_{m=1}^{n_3} p_{s(jkm)} \ln p_{s(jkm)} - \sum_{k=1}^{n_2}\left(\sum_{j=1}^{n_1}\sum_{m=1}^{n_3} p_{s(jkm)}\right) \ln\left(\sum_{j'=1}^{n_1}\sum_{m'=1}^{n_3} p_{s(j'km')}\right) \\ \leq\ & -\sum_{j=1}^{n_1}\sum_{k=1}^{n_2}\left(\sum_{m'=1}^{n_3} p_{s(jkm')}\right)\ln\left(\sum_{m=1}^{n_3} p_{s(jkm)}\right) - \sum_{k=1}^{n_2}\sum_{m=1}^{n_3}\left(\sum_{j'=1}^{n_1} p_{s(j'km)}\right)\ln\left(\sum_{j=1}^{n_1} p_{s(jkm)}\right). \end{aligned} \tag{6}$$

If $N \neq n_1 n_2 n_3$, we can consider a set of $\tilde{N}$ numbers $\tilde{p}_s = (p_1, p_2, \ldots, p_N, 0, \ldots, 0)$, with the number of added zeros k such that $\tilde{N} = N + k = n_1 n_2 n_3$. For this new set $\tilde{p}_s$, all the inequalities constructed are valid.

5. Projector Density Matrices and New *q*-Entropic Equalities for Pure Qudit States

Given the $N \times N$ matrix ρ in a block form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}, \tag{7}$$

where $N = nm$ and the blocks ρ_{jk} are $m \times m$ matrices. We construct matrices ρ_1 and ρ_2, using the numerical tool [24–26,28,30,31]; the $n \times n$ matrix ρ_1 reads

$$\rho_1 = \begin{pmatrix} \mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{12} & \cdots & \mathrm{Tr}\,\rho_{1n} \\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn} \end{pmatrix}, \tag{8}$$

and the $m \times m$ matrix ρ_2 is

$$\rho_2 = \sum_{k=1}^{n} \rho_{kk}. \tag{9}$$

The matrices ρ_1 and ρ_2 obtained from the matrix ρ (7) and given by Equations (8) and (9), respectively, corresponds to the partial tracing procedure. In fact, if the numerical matrix ρ coincides with the density matrix $\rho(1,2)$ of a two-qudit system with $n = 2j_1+1$ and $m = 2j_2+1$, one can check that the matrices ρ_1 and ρ_2 coincide with the matrices obtained by partial tracing, *i.e.*, $\rho_1 = \mathrm{Tr}_2\,\rho(1,2)$ and $\rho_2 = \mathrm{Tr}_1\,\rho(1,2)$. This observation provides the possibility to use all known entropic relations for the density matrices of two-qudit systems $\rho(1,2)$ and density matrices of its first subsystem ρ_1 and second subsystem ρ_2 and apply these numerical relations to an arbitrary matrix ρ, which has the properties $\rho = \rho^\dagger$, $\rho \geq 0$, and $\mathrm{Tr}\rho = 1$ characterizing the density matrices. This observation explains the principle and method of artificial subpartitioning. Once this is understood, all multipartite inequalities can immediately be applied. For example, the known inequality given by the nonnegativity property of mutual information $I = \mathrm{Tr}_1\,\rho(1,2)\ln\rho(1,2) - \mathrm{Tr}\,\rho_1\ln\rho_1 - \mathrm{Tr}\,\rho_2\ln\rho_2 \geq 0$ provides the inequality for matrices (7)–(9), *i.e.*, $\mathrm{Tr}\,\rho\ln\rho \geq \mathrm{Tr}\,\rho_1\ln\rho_1 + \mathrm{Tr}\,\rho_2\ln\rho_2$. Also the other entropic relations known for bipartite-system density matrices $\rho(1,2)$ can be extended to the matrices ρ (7). For example, specific equalities for entropies are available in the case of Hermitian nonnegative matrices ρ with unit trace and extra condition $\rho^2 = \rho$. The equalities follow from the Araki–Lieb inequality extended to a single qudit state [33]. The matrices ρ_1 and ρ_2 obtained by means of the rules (8) and (9) satisfy the following entropic equality:

$$\begin{aligned}
&\mathrm{Tr}\left[\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{12} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}\ln\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{12} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}\right]\\
&= \mathrm{Tr}\left[\left(\sum_{k=1}^{n}\rho_{kk}\right)\ln\left(\sum_{k=1}^{n}\rho_{kk}\right)\right]. \qquad (10)
\end{aligned}$$

This equality can be easily obtained using the same observation. If we identify the matrix ρ with the density matrix $\rho(1,2)$ of the pure state of a bipartite quantum system and ρ_1 and ρ_2 as the density matrices of the subsystem states of the bipartite quantum system (composite system), equality (10) means that the von Neumann entropies of the subsystem states are equal. The nonzero eigenvalues of matrices ρ_1 and ρ_2 coincide. These properties of pure states of bipartite systems are well known. Thus, we obtain this equality for the pure state of noncomposite system.

There exists the other q-entropic equality valid for such matrices ρ_1 and ρ_2, which can be easily checked, namely,

$$\begin{aligned}
&\mathrm{Tr}\left\{\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{12} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}^{q}\left[\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{12} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}^{1-q} - 1\right]\right\}\\
&= \mathrm{Tr}\left\{\left(\sum_{k=1}^{n}\rho_{kk}\right)^{q}\left[\left(\sum_{k=1}^{n}\rho_{kk}\right)^{1-q} - 1\right]\right\}. \qquad (11)
\end{aligned}$$

If we take into account the limit $q \to 1$ in the expression for q-entropy, equality (11) converts to (10).

Equality (11) is valid for Tsallis entropies of two density matrices ρ_1 and ρ_2 of the subsystem states in the case where the matrix ρ is the density matrix of the bipartite-system quantum state. It is clear that this equality is valid for the noncomposite-system state as well.

Such the equality can also be applied for the qutrit state,

$$\begin{aligned}
&\mathrm{Tr}\left\{\begin{pmatrix}\rho_{11}+\rho_{-1-1} & \rho_{10}\\ \rho_{01} & \rho_{00}\end{pmatrix}^{q}\left[\begin{pmatrix}\rho_{11}+\rho_{-1-1} & \rho_{10}\\ \rho_{01} & \rho_{00}\end{pmatrix}^{1-q} - 1\right]\right\}\\
&= \mathrm{Tr}\left\{\begin{pmatrix}\rho_{11}+\rho_{00} & \rho_{1-1}\\ \rho_{-11} & \rho_{-1-1}\end{pmatrix}^{q}\left[\begin{pmatrix}\rho_{11}+\rho_{00} & \rho_{1-1}\\ \rho_{-11} & \rho_{-1-1}\end{pmatrix}^{1-q} - 1\right]\right\}. \qquad (12)
\end{aligned}$$

The difference of the left and right hand sides of equality (12) can characterize the degree of coherence of the qutrit state, in addition to the purity parameter $\mu = \mathrm{Tr}\,\rho^2$. Analogously, for a mixed state of a single qudit, the difference $(\mathrm{Tr}\,\rho_1 \ln \rho_1 - \mathrm{Tr}\,\rho_2 \ln \rho_2)$ is a characteristic of the state purity.

The test of purity in Equation (10) is certainly not useful if one already knows the entire density matrix but, if one obtains from the experiment the matrices ρ_1 and ρ_2 only, equality (10) witnesses that the matrix ρ corresponds to the pure state. This fact is a motivation to discuss such entropic equality both for composite and noncomposite (qudit) systems.

If $N \neq nm$, one can introduce the matrix $\tilde{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$, using $\tilde{N} = N + k = nm$, analogously to the tool we employed in the case of the set of numbers; all the matrix relations obtained above are valid for the matrix $\tilde{\rho}$.

6. The Strong Subadditivity Condition for a Single Qudit State

If $N = n_1 n_2 n_3$, one can use the notation $\rho_{jkl,j'k'l'}$ for matrix elements of the matrix $\rho_{ss'}$, where $j', j = 1, 2, \ldots, n_1$, $k', k = 1, 2, \ldots, n_2$, and $l', l = 1, 2, \ldots, n_3$. This means that we apply the map of integers $1, 2, \ldots$ onto the triples of integers, *i.e.*, the integer $s(s')$ is considered as a function of three variables $s(jkl)$ $[s'(j'k'l')]$. Thus, one has the nonnegative matrix $\rho = \rho^\dagger$ with unit trace and matrix elements $\rho_{s,s'} \equiv \rho_{s(jkl),s'(j'k'l')}$. If this matrix is the density matrix of a tripartite quantum system (e.g., a system of three qudits), it satisfies the strong subadditivity condition for von Neumann entropies of the system $S(A,B,C)$ and three its subsystems $S(AB)$, $S(BC)$, and $S(B)$, where

$$S(A,B,C) = -\sum_{s=1}^{N} \big(\rho(A,B,C) \ln \rho(A,B,C)\big)_{ss} \equiv -\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{l=1}^{n_3} \big[(\rho \ln \rho)_{s(jkl),s(jkl)}\big], \tag{13}$$

with the density matrices of the subsystems

$$\rho(A,B) = \mathrm{Tr}_C\, \rho(A,B,C), \qquad \rho(B,C) = \mathrm{Tr}_A\, \rho(A,B,C), \qquad \rho(B) = \mathrm{Tr}_{AC}\, \rho(A,B,C). \tag{14}$$

The interpretation of an arbitrary nonnegative matrix $\rho_{s(jkl),\, s'(j'k'l')}$ as the density matrix of tripartite system provides the possibility to write the strong subadditivity condition for the matrix elements of this matrix even in the case where this $N \times N$-matrix is the density matrix $\rho_{ss'}$ of a single qudit state. An explicit form of the strong subadditivity condition for the $N \times N$-matrix $\rho_{ss'}$ reads

$$-\,\mathrm{Tr}\, \rho \ln \rho - \mathrm{Tr}\, \rho_2 \ln \rho_2 \leq -\mathrm{Tr}\, \rho_{12} \ln \rho_{12} - \mathrm{Tr}\, \rho_{23} \ln \rho_{23}, \tag{15}$$

where the matrices ρ_{12}, ρ_{23}, and ρ_2 have the matrix elements

$$\begin{aligned} (\rho_{12})_{jk,j'k'} &= \sum_{l=1}^{n_3} \rho_{s(jkl),s'(j'k'l)}, \\ (\rho_{23})_{kl,k'l'} &= \sum_{j=1}^{n_1} \rho_{s(jkl),s'(jk'l')}, \\ (\rho_2)_{k,k'} &= \sum_{j=1}^{n_1} (\rho_{12})_{jk,jk'} = \sum_{l=1}^{n_3} (\rho_{23})_{kl,k'l}. \end{aligned} \tag{16}$$

For example, if the matrix ρ is the density matrix of the qudit state with $j = 3$, we obtain the strong subadditivity condition

$$-\,\mathrm{Tr}\, \rho \ln \rho - \sum_{k=1}^{2} \big(\rho_2 \ln \rho_2\big)_{kk} \leq -\sum_{s=1}^{4} \big(\rho_{12} \ln \rho_{12}\big)_{ss} - \sum_{s=1}^{4} \big(\rho_{23} \ln \rho_{23}\big)_{ss}, \tag{17}$$

using the 8×8-matrix $\tilde{\rho}$ of the form $\tilde{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ and employing the equality $\mathrm{Tr}\,\rho \ln \rho = \mathrm{Tr}\,\tilde{\rho} \ln \tilde{\rho}$. Here, the 8×8-matrix $\tilde{\rho}$ reads

$$\tilde{\rho} = \left(\begin{array}{cc|cc|cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} & 0 \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} & \rho_{27} & 0 \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} & \rho_{37} & 0 \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} & \rho_{47} & 0 \\ \hline \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} & \rho_{57} & 0 \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} & \rho_{67} & 0 \\ \hline \rho_{71} & \rho_{72} & \rho_{73} & \rho_{74} & \rho_{75} & \rho_{76} & \rho_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{18}$$

and 4×4-matrices ρ_{12} and ρ_{23} and 2×2-matrix ρ_2 are

$$\rho_{12} = \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} & \rho_{15}+\rho_{26} & \rho_{17} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} & \rho_{35}+\rho_{46} & \rho_{37} \\ \rho_{51}+\rho_{62} & \rho_{53}+\rho_{64} & \rho_{55}+\rho_{66} & \rho_{57} \\ \rho_{71} & \rho_{73} & \rho_{75} & \rho_{77} \end{pmatrix}, \tag{19}$$

$$\rho_{23} = \begin{pmatrix} \rho_{11}+\rho_{55} & \rho_{12}+\rho_{56} & \rho_{13}+\rho_{58} & \rho_{14} \\ \rho_{21}+\rho_{65} & \rho_{22}+\rho_{66} & \rho_{23}+\rho_{67} & \rho_{24} \\ \rho_{31}+\rho_{75} & \rho_{32}+\rho_{76} & \rho_{33}+\rho_{77} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{20}$$

$$\rho_2 = \begin{pmatrix} \rho_{11}+\rho_{22}+\rho_{55}+\rho_{66} & \rho_{13}+\rho_{24}+\rho_{57} \\ \rho_{31}+\rho_{42}+\rho_{75} & \rho_{33}+\rho_{44}+\rho_{77} \end{pmatrix}. \tag{21}$$

Thus, we presented the strong subadditivity condition for the qudit density matrix $\rho_{mm'} \leftrightarrow \rho_{ss'}$ in the case of $j = 3$, where the notation for spin projections $m, m' = -3, -2, -1, 0, 1, 2, 3$ is mapped as $mm' \leftrightarrow ss' = 1, 2, 3, 4, 5, 6, 7$.

Inequality (17) can be checked in the experiments with superconducting circuits where the states of the seven-level systems can be constructed.

7. The Subadditivity Condition for Quantum Weighted Entropy of a Single Qudit State

Recently [35], the notion of weighted entropy was introduced for quantum states. For the density matrix ρ, the entropy is defined as

$$S_\varphi(\rho) = -\mathrm{Tr}\,(\varphi\rho \ln \rho), \tag{22}$$

where φ is called the weight, being the Hermitian positively definite matrix.

The subadditivity condition for entropy of the bipartite-system state with the density matrix $\rho(A, B)$ has been proven in [35].

When the subadditivity condition is formulated for a bipartite system, the weight matrix φ_{AB} is chosen in the product form corresponding to the density matrix $\rho(AB)$. There are two conditions for the weight matrix, namely, $\varphi_{AB} = \varphi_A \otimes \varphi_B$ and $\mathrm{Tr}\,\varphi_{AB}\rho(A,B) \geq \mathrm{Tr}_A\left(\varphi_A\rho(A)\right)\mathrm{Tr}_B\left(\varphi_B\rho(B)\right)$. Then one has the inequality

$$S_{\varphi_{AB}}\,\rho(A,B) \leq S_{\psi_A}\,\rho(A) + S_{\psi_B}\,\rho(B), \tag{23}$$

where

$$\psi_A\,\rho_A = \mathrm{Tr}_B\left(\varphi_{AB}\,\rho_{AB}\right), \qquad \psi_B\,\rho_B = \mathrm{Tr}_A\left(\varphi_{AB}\,\rho_{AB}\right). \tag{24}$$

The inequality for $\varphi_{AB} = 1$ yields the standard subadditivity condition for bipartite system.

Now we extend this inequality to the case of a single qudit state using the map of integers onto pairs of the integers, $N = nm$. In this approach, the weighted subadditivity condition is formulated as a matrix inequality for given $N\times N$-matrices, where φ, ρ, and $N = nm$ are as follows. We take a specific matrix $\varphi = \varphi_1 \times \varphi_2$, where φ_1 and φ_2 are the $n\times n$-matrix and $m\times m$-matrix, respectively. Then an analog of the weighted subadditivity condition (23) for the single-qudit density matrix (7), presented in the block form with blocks ρ_{jk}, reads

$$-\,\mathrm{Tr}\left(\varphi_1\otimes\varphi_2\,\rho\ln\rho\right) \leq -\mathrm{Tr}\left(\tilde{\varphi}_1\rho_1\right)\ln\rho_1 - \mathrm{Tr}\left(\tilde{\varphi}_2\rho_2\right)\ln\rho_2. \tag{25}$$

Here, $\tilde{\varphi}_1\rho_1 = \begin{pmatrix} \mathrm{Tr}\,\tilde{\rho}_{11} & \mathrm{Tr}\,\tilde{\rho}_{11} & \cdots & \mathrm{Tr}\,\tilde{\rho}_{1n} \\ \mathrm{Tr}\,\tilde{\rho}_{21} & \mathrm{Tr}\,\tilde{\rho}_{22} & \cdots & \mathrm{Tr}\,\tilde{\rho}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Tr}\,\tilde{\rho}_{n1} & \mathrm{Tr}\,\tilde{\rho}_{n2} & \cdots & \mathrm{Tr}\,\tilde{\rho}_{nn} \end{pmatrix}$, $\tilde{\varphi}_2\rho_2 = \sum_{k=1}^{n}\tilde{\rho}_{kk}$, matrices ρ_1 and ρ_2 are given by Equations (8) and (9), respectively, and the matrix $\varphi_1\otimes\varphi_2\,\rho$ is presented in a block form analogous to Equation (7) with $m\times m$ blocks $\tilde{\rho}_{jk}$.

We point out that the $N\times N$-matrix φ, in general, is not obligatory expressed as $\varphi = \varphi_1\otimes\varphi_2$. To formulate inequality (25) for the single-qudit state, we employ weights φ having the product form only.

For unit matrix φ, the weighted inequality (25) becomes an inequality, which is the subadditivity condition for the matrix ρ, e.g., for the single qudit state.

The weighted subadditivity condition for the single qudit state (25) can be checked experimentally.

8. Spin Tomography Inequality

The density operators $\hat{\rho}$ of qudit states can be described by tomographic-probability-distribution functions (spin tomograms) [8,9,36].

For a single qudit with $j = (N-1)/2$, the tomogram $w(m|u)$ reads

$$w(m|u) = \mathrm{Tr}\,\hat{U}(m,u)\hat{\rho}, \tag{26}$$

where dequantizer operator $\hat{U}(m,u) = (u \mid m\rangle\langle m \mid u^\dagger)$ with $m = -j, -j+1, \ldots, j-1, j$ and u the unitary $N\times N$-matrix. The matrix can be considered also as the matrix of irreducible representation of the $SU(2)$ group. In this case, the tomogram is the function $w(m \mid \vec{n})$, where $\vec{n}$ is unit vector ($\vec{n}^2 = 1$) defined by two angles, *i.e.*, $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

The physical meaning of the tomogram is that it is equal to the probability distribution to get in the state $\hat{\rho}$ the spin projection m on the direction $\vec{n}$. The tomogram $w(m|\vec{n})$ can be interpreted as the conditional probability distribution [33] to obtain the projection m on a given direction $\vec{n}$.

If the qudit is associated with N-level atom, the qudit tomogram $w(m|u)$ is the probability to get the population of the levels in a reference frame in the atomic-state Hilbert space rotated by the global unitary transform matrix u. In this case, we map the indices $m = -j, -j+1, \ldots, j-1, j$ to integers $s = 1, 2, \ldots, N$ which label the atomic levels; the integers s are functions $s(jk)$.

Now we are in the position to obtain a new entropic inequality for the quantum qudit state applying the approach discussed in Section 4 to spin tomograms.

Thus, for tomogram of the qudit state with $j = (N-1)/2$, where $N = nm$, a new subadditivity condition for Tsallis entropy reads

$$\begin{aligned} & -\sum_{j=1}^{n}\sum_{k=1}^{m} w^{q}\big(s(jk)|u\big)\,\frac{w^{1-q}\big(s(jk)|u\big)-1}{1-q} \\ \leq\; & -\sum_{j=1}^{n}\left(w_1^{q}\big(s(j)|u\big)\,\frac{w_1^{1-q}\big(s(j)|u\big)-1}{1-q}\right)-\sum_{k=1}^{m}\left(w_2^{q}\big(s(k)|u\big)\,\frac{w_2^{1-q}\big(s(k)|u\big)-1}{1-q}\right), \end{aligned} \tag{27}$$

where

$$w_1\big(s(j)|u\big) = \sum_{k=1}^{m} w\big(s(jk)|u\big), \qquad w_2\big(s(k)|u\big) = \sum_{j=1}^{n} w\big(s(jk)|u\big). \tag{28}$$

In the limit $q \to 1$, one has the subadditivity condition for Shannon entropies of the form (5) associated with the tomogram. Inequality (27) can be checked experimentally.

As an example of this inequality, we consider the state of spin with $j = 3/2$. The tomogram of this state is the probability distribution with values

$$\begin{aligned} & w\big(+3/2|u\big) \equiv w\big(s(11)|u\big), \quad w\big(+1/2|u\big) \equiv w\big(s(12)|u\big), \\ & w\big(-1/2|u\big) \equiv w\big(s(21)|u\big), \quad w\big(-3/2|u\big) \equiv w\big(s(22)|u\big), \end{aligned} \tag{29}$$

where the function $s(11) = 1$, $s(12) = 2$, $s(21) = 3$, and $s(22) = 4$.

The subadditivity condition for this qudit-state tomogram written in terms of tomographic Shannon entropic inequality reads

$$\begin{aligned} & -\big[w\big(+3/2\,|\,\vec{n}\big)\ln w\big(+3/2\,|\,\vec{n}\big) + w\big(+1/2\,|\,\vec{n}\big)\ln w\big(+1/2\,|\,\vec{n}\big) \\ & + w\big(-1/2\,|\,\vec{n}\big)\ln w\big(-1/2\,|\,\vec{n}\big) + w\big(-3/2\,|\,\vec{n}\big)\ln w\big(-3/2\,|\,\vec{n}\big)\big] \\ \leq\; & -\big\{\big[w\big(+3/2\,|\,\vec{n}\big) + w\big(+1/2\,|\,\vec{n}\big)\big]\ln\big[w\big(+3/2\,|\,\vec{n}\big) + w\big(+1/2\,|\,\vec{n}\big)\big] \\ & + \big[w\big(-1/2\,|\,\vec{n}\big) + w\big(-3/2\,|\,\vec{n}\big)\big]\ln\big[w\big(-1/2\,|\,\vec{n}\big) + w\big(-3/2\,|\,\vec{n}\big)\big]\big\} \\ & -\big\{\big[w\big(+3/2\,|\,\vec{n}\big) + w\big(-1/2\,|\,\vec{n}\big)\big]\ln\big[w\big(+3/2\,|\,\vec{n}\big) + w\big(-1/2\,|\,\vec{n}\big)\big] \\ & \big[w\big(+1/2\,|\,\vec{n}\big) + w\big(-3/2\,|\,\vec{n}\big)\big]\ln\big[w\big(+1/2\,|\,\vec{n}\big) + w\big(-3/2\,|\,\vec{n}\big)\big]\big\}; \end{aligned} \tag{30}$$

this new inequality can also be checked in the experiments with superconducting circuits nuclear magnetic resonance discussed, e.g., in [37,38].

It is worth noting that other inequalities of the form (30) where one uses the permutation of spin projections m are valid.

If we interpret the density operator $\hat{\rho}$ being correspondent to the state of two qudits with j_1 and j_2, the tomogram $w(m_1, m_2|u)$ is given by the formula

$$w(m_1, m_2|u) = \mathrm{Tr}\,\big(\hat{\rho} u \mid m_1 m_2\rangle\langle m_1 m_2 \mid u^{\dagger}\big);$$

it is the conditional probability distribution to get the spin projections m_1 and m_2 for given global transform u. If $u = u_1 \otimes u_2$, where u_1 and u_2 are local unitary transforms depending on the angles determined by two directions $\vec{n}_1$ and $\vec{n}_2$, the tomogram $w(m_1, m_2|\vec{n}_1, \vec{n}_2)$ is the conditional probability for the given directions to get the corresponding projections m_1 and m_2, respectively. For such an interpretation of the matrix ρ, one has inequality (27), where the map $m_1, m_2 \leftrightarrow jk \leftrightarrow s(jk)$ is used.

Now we present an analog of inequality (30) for the density matrix ρ of the two-qubit state; in this case, the density matrix has matrix elements $\rho_{m_1 m_2, m_1' m_2'}$, where $m_1 m_2 (m_1' m_2')$ take values $\pm 1/2$, and the tomogram $w(m_1, m_2|\vec{n}_1, \vec{n}_2)$ satisfies the known entropic inequality:

$$\begin{aligned}
& -\big[w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)\ln w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big) \\
& +w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\ln w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big) \\
& +w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)\ln w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big) \\
& +w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\ln w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
\leq\ & -\big\{\big[w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& \times\ln\big[w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& +\big[w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& \times\ln\big[w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big]\big\} \\
& -\big\{\big[w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& \times\ln\big[w\big(+1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ +1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& +\big[w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big] \\
& \times\ln\big[w\big(+1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)+w\big(-1/2\ -1/2\,|\,\vec{n}_1\vec{n}_2\big)\big]\big\}.
\end{aligned} \tag{31}$$

This inequality is the subadditivity condition for the joint tomographic probability distribution of the two-qubit state. The entanglement property of two-qubit states exists; e.g., in this case, the pure state reads $2^{-1/2}\big(\ |\ +1/2\ +1/2\rangle+\ |\ -1/2\ -1/2\rangle\big)$. An analogous entanglement property takes place for the qudit state with $j = 3/2$, which is the pure state $2^{-1/2}\big(\ |\ +3/2\rangle+\ |\ -3/2\rangle\big)$. The density matrices of these states are identical numerical matrices; they cannot be presented in the form of convex sum of tensor products of $2{\times}2$-matrices $\rho_1^{(k)} \otimes \rho_2^{(k)}$, where numerical matrices $\rho_1^{(k)}$ and $\rho_2^{(k)}$ are nonnegative Hermitian matrices with unit trace.

The entanglement of two-qubit states reflects the presence of strong quantum correlations of the subsystem degrees of freedom. The analogous entanglement of the single qudit state with $j = 3/2$ reflects the strong quantum correlations of the degrees of freedom (spin projections m) of the same noncomposite system. The entanglement of composite system is the resource for quantum technologies. The strong correlations (analogous to entanglement) of the single-qudit states can also provide an additional resource for quantum technologies. The presence of strong correlations in the qudit state with $j = 3/2$ corresponds to the subadditivity condition (30).

We address the question how the numerical $4{\times}4$-matrix $\rho_{ss'}$ contains information on correlations in two different quantum systems. One system is a bipartite system of two qubits, and the other one is a single qudit with the $j = 3/2$ state. The answer to this question is related to the interpretation of the numerical matrix elements.

For two qubits, the matrix elements are interpreted as the matrix elements of the density operator in the basis $\mid m, m'\rangle$, $m, m' = \pm 1/2$.

For a qudit, the same matrix elements are interpreted as the matrix elements of the density operator in the basis $\mid m\rangle$, $m = +3/2, +1/2, -1/2, -3/2$.

Quantum correlations in two-qubit systems are the correlations between the events where one measures the spin projection $m_1 = \pm 1/2$ of the first qubit or $m_2 = \pm 1/2$ of the second qubit; thus, the correlations are interpreted as correlations between the degrees of freedom of two different subsystems.

For a qudit with $j = 3/2$, analogous measurements mean that one measures correlations between the following events. Analogs of the events, where $m_1 = \pm 1/2$, are the events, where the spin projections $m = +3/2, +1/2$ and $m = -3/2, -1/2$ are observed, while analogs of the events, where $m_2 = \pm 1/2$, are the events, where the spin projections $m = +3/2, -1/2$ and $m = -3/2, +1/2$, are observed.

Thus, for a single qudit with $j = 3/2$, the quantum correlations between such events are described by the same numerical matrix $\rho_{ss'}$, which describes the quantum correlations between the subsystems (two qubits) in the composite quantum system.

9. Relative Entropy Inequality for a Single Qudit

The relative entropy for states with the density matrices ρ and σ reads (see, e.g., [37])

$$S(\rho\|\sigma) = \mathrm{Tr}(\rho\ln\rho - \rho\ln\sigma) \geq 0. \tag{32}$$

The relative entropy for the bipartite system has the property

$$S\big(\rho(AB)\|\sigma(AB)\big) \geq S\big(\rho(A)\|\sigma\rho(A)\big). \tag{33}$$

We use the map of integers $s \rightarrow s(jk)$ to write the property of the relative entropy in numerical form, which can be used to formulate an analog of the property (33) for a single qudit state. In fact, Equation (33) can be rewritten in the form of inequality

$$\begin{aligned}
&\mathrm{Tr}\left\{\begin{pmatrix}\rho_{11} & \rho_{12} & \cdots & \rho_{1n}\\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn}\end{pmatrix}\left[\ln\begin{pmatrix}\rho_{11} & \rho_{12} & \cdots & \rho_{1n}\\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn}\end{pmatrix} - \ln\begin{pmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}\end{pmatrix}\right]\right\}\\
\geq\ &\mathrm{Tr}\left\{\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{11} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}\left[\ln\begin{pmatrix}\mathrm{Tr}\,\rho_{11} & \mathrm{Tr}\,\rho_{11} & \cdots & \mathrm{Tr}\,\rho_{1n}\\ \mathrm{Tr}\,\rho_{21} & \mathrm{Tr}\,\rho_{22} & \cdots & \mathrm{Tr}\,\rho_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\rho_{n1} & \mathrm{Tr}\,\rho_{n2} & \cdots & \mathrm{Tr}\,\rho_{nn}\end{pmatrix}\right.\right.\\
&\left.\left.- \ln\begin{pmatrix}\mathrm{Tr}\,\sigma_{11} & \mathrm{Tr}\,\sigma_{11} & \cdots & \mathrm{Tr}\,\sigma_{1n}\\ \mathrm{Tr}\,\sigma_{21} & \mathrm{Tr}\,\sigma_{22} & \cdots & \mathrm{Tr}\,\sigma_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{Tr}\,\sigma_{n1} & \mathrm{Tr}\,\sigma_{n2} & \cdots & \mathrm{Tr}\,\sigma_{nn}\end{pmatrix}\right]\right\}.
\end{aligned} \tag{34}$$

This inequality is valid for an arbitrary $N \times N$-matrix ρ and matrix σ of the block form (7); the matrices have the matrix elements $\mathrm{Tr}\,(\rho_{jk})$ and $\mathrm{Tr}\,(\sigma_{jk})$. If matrices ρ and σ are density matrices of the single qudit state with $j = (N-1)/2$, inequality (34) characterizes quantum correlations in the system under study. For example, if $j = 3/2$, the relative entropy inequality for the qudit state with matrices $\rho_{mm'}$ and $\sigma_{mm'}$ $(mm' = 3/2, 1/2, -1/2, -3/2)$ reads

$$\begin{aligned}
&-\mathrm{Tr}\begin{pmatrix} \rho_{3/2\,3/2} & \rho_{3/2\,1/2} & \rho_{3/2\,-1/2} & \rho_{3/2\,-3/2} \\ \rho_{1/2\,3/2} & \rho_{1/2\,1/2} & \rho_{1/2\,-1/2} & \rho_{1/2\,-3/2} \\ \rho_{-1/2\,3/2} & \rho_{-1/2\,1/2} & \rho_{-1/2\,-1/2} & \rho_{-1/2\,-3/2} \\ \rho_{-3/2\,3/2} & \rho_{-3/2\,1/2} & \rho_{-3/2\,-1/2} & \rho_{-3/2\,-3/2} \end{pmatrix} \left[\ln \begin{pmatrix} \rho_{3/2\,3/2} & \rho_{3/2\,1/2} & \rho_{3/2\,-1/2} & \rho_{3/2\,-3/2} \\ \rho_{1/2\,3/2} & \rho_{1/2\,1/2} & \rho_{1/2\,-1/2} & \rho_{1/2\,-3/2} \\ \rho_{-1/2\,3/2} & \rho_{-1/2\,1/2} & \rho_{-1/2\,-1/2} & \rho_{-1/2\,-3/2} \\ \rho_{-3/2\,3/2} & \rho_{-3/2\,1/2} & \rho_{-3/2\,-1/2} & \rho_{-3/2\,-3/2} \end{pmatrix} \right. \\
&\left. -\ln \begin{pmatrix} \sigma_{3/2\,3/2} & \sigma_{3/2\,1/2} & \sigma_{3/2\,-1/2} & \sigma_{3/2\,-3/2} \\ \sigma_{1/2\,3/2} & \sigma_{1/2\,1/2} & \sigma_{1/2\,-1/2} & \sigma_{1/2\,-3/2} \\ \sigma_{-1/2\,3/2} & \sigma_{-1/2\,1/2} & \sigma_{-1/2\,-1/2} & \sigma_{-1/2\,-3/2} \\ \sigma_{-3/2\,3/2} & \sigma_{-3/2\,1/2} & \sigma_{-3/2\,-1/2} & \sigma_{-3/2\,-3/2} \end{pmatrix} \right] \\
\geq &-\mathrm{Tr}\begin{pmatrix} \rho_{3/2\,3/2} + \rho_{1/2\,1/2} & \rho_{3/2\,-1/2} + \rho_{1/2\,-3/2} \\ \rho_{-1/2\,3/2} + \rho_{-3/2\,1/2} & \rho_{-1/2\,-1/2} + \rho_{-3/2\,-3/2} \end{pmatrix} \\
&\times \left[\ln \begin{pmatrix} \rho_{3/2\,3/2} + \rho_{1/2\,1/2} & \rho_{3/2\,-1/2} + \rho_{1/2\,-3/2} \\ \rho_{-1/2\,3/2} + \rho_{-3/2\,1/2} & \rho_{-1/2\,-1/2} + \rho_{-3/2\,-3/2} \end{pmatrix} \right. \\
&\quad \left. -\ln \begin{pmatrix} \sigma_{3/2\,3/2} + \sigma_{1/2\,1/2} & \sigma_{3/2\,-1/2} + \sigma_{1/2\,-3/2} \\ \sigma_{-1/2\,3/2} + \sigma_{-3/2\,1/2} & \sigma_{-1/2\,-1/2} + \sigma_{-3/2\,-3/2} \end{pmatrix} \right]. \qquad (35)
\end{aligned}$$

The inequalities presented are explicit inequalities for density matrices of single qudit states; they can be checked experimentally.

10. Chain Rule for Conditional Entropy of Single Qudit States

As we discussed in Section 2, the notion of conditional entropy is applied to the system with subsystems and it is related to the joint probability distributions. In addition, one can introduce it for the system without subsystems. We use this tool to consider the quantum system, which is a single qudit with $j = (N-1)/2$, where $N = nm$. Then, one can associate conditional probability distributions with the qudit-state tomogram $w(m|u)$, where u is unitary $N \times N$-matrix and $m = -j, -j+1, \ldots, j-1, j$.

We demonstrate such a procedure on the example of $j = 3/2$. In this case, the tomographic probability distribution is given by four nonnegative numbers p_s, $s = 1, 2, 3, 4$, where $p_1 = w(+3/2|u)$, $p_2 = w(+1/2|u)$, $p_3 = w(-1/2|u)$, $p_4 = w(-3/2|u)$, and $\sum_{s=1}^{4} p_s = 1$. Using the map discussed in Section 2, we obtain the entropic chain rule in the form of equality

$$H(AB) = H(A|B) + H(B), \qquad (36)$$

where the Shannon tomographic entropy of the system state reads

$$\begin{aligned}
H(AB) \;=\; & -w(+3/2|u)\ln w(+3/2|u) - w(+1/2|u)\ln w(+1/2|u) \\
& -w(-1/2|u)\ln w(-1/2|u) - w(-3/2|u)\ln w(-3/2|u), \qquad (37)
\end{aligned}$$

and the marginal Shannon tomographic entropy $H(B)$ for the single-qudit state with $j = 3/2$ is

$$\begin{aligned}
H(B) \;=\; & -\big[w(+3/2|u) + w(+1/2|u)\big]\ln\big[w(+3/2|u) + w(+1/2|u)\big] \\
& -\big[w(-1/2|u) + w(-3/2|u)\big]\ln\big[w(-1/2|u) + w(-3/2|u)\big]. \qquad (38)
\end{aligned}$$

The conditional tomographic entropy $H(A|B)$ for the single-qudit state with $j = 3/2$ can be expressed in terms of the numbers $p_s \leftrightarrow w(u|m)$; in an explicit form, it is given as follows:

$$\begin{aligned} H(A|B) \;=\; & \big[w(+3/2|u)+w(+1/2|u)\big] \ln \big[w(+3/2|u)+w(+1/2|u)\big] \\ & +\big[w(-1/2|u)+w(-3/2|u)\big] \ln \big[w(-1/2|u)+w(-3/2|u)\big] \\ & -w(+3/2|u) \ln w(+3/2|u) - w(+1/2|u) \ln w(+1/2|u) \\ & -w(-1/2|u) \ln w(-1/2|u) - w(-3/2|u) \ln w(-3/2|u). \end{aligned} \tag{39}$$

Thus, we introduced the notion of conditional entropy for systems without subsystems. The chain rule (36) can be extended, if one interprets the probability $w(m|u)$ for the single-qudit state as the joint probability distribution for an artificial multipartite system.

11. Conclusions

To conclude, we list the main results of our work.

First of all, it is worth pointing out that in this work we did not derive or discover new inequalities but did use the well-known for bipartite and multipartite systems entropic and information relations to attract attention of the researchers to the fact that, by employing the subpartition tools, these inequalities can be also applied to the system which does not contain subsystems, e.g., for a single qudit. For noncomposite systems, the relations discussed here, like the subadditivity condition, were not discussed in the literature.

The motivation of writing such entropic relations in explicit matrix forms is connected with recent discussions of experimental studies of qudit density matrices given, for example, in [37–39], where the information matrix formulas can be checked using the data characterizing the superconducting quantum circuits realized by Josephson junctions [14–16].

We reviewed the approach to the set of nonnegative numbers and Hermitian nonnegative matrices with unit trace, in view of the interpretation of the numbers and the matrices as the probability distributions and the density matrices, respectively, elaborated in [17–26]. We showed that the known entropic inequalities, which are applied to composite systems, both classical and quantum, can be also applied to the systems without subsystems. We obtained a new entropic equality, in view of the interpretation of the density matrix of this state as the density matrix of an artificial bipartite system. The inequalities and equalities obtained can be checked experimentally. Equalities (10) and (11) can be used to characterize the purity of mixed states.

In fact, the approach presented provides the possibility to extend all known entropic and information relations for classical and quantum composite systems to the case of systems without subsystems, and the relations reflect the presence of correlations either classical or quantum ones of the system degrees of freedom. Tomographic inequality (30) is a very simple inequality known for the nonnegative numbers $w(m \mid \vec{n})$ but, from the viewpoint of the tomographic analysis of a single qudit state, it is a new relation that should be satisfied by the experimental data. The quantum correlations of the single qudit states can be used in quantum technologies, analogously to the usage of entanglement as quantum resource.

We conclude that there are several aspects in the presented approach [32]. One has a set of nonnegative numbers and numerical relations (equalities and inequalities) for these numbers. Analogously, there exist sets of complex numbers organized as tables or nonnegative Hermitian matrices with different labeling of their matrix elements and the numerical relations for the matrix elements. The nonnegative numbers and matrices can be interpreted as the probability distributions or density matrices of physical systems. The same set of numbers and the same matrices can be interpreted as the probability distributions or density matrices of quantum states of different composite or noncomposite systems. This provides the possibility to extend the numerical matrix relations known for particular physical systems, e.g., composite ones, to the other systems, including noncomposite systems, since the relations do not depend on the interpretation of the numerical equalities and inequalities. This tool was used in our work.

On the other hand, the interpretation of the mathematical equalities and inequalities as properties of classical and quantum correlations provides a tool to suggest the experimental check of new equalities and inequalities for the physical systems and employ the found quantum correlations in single-qudit states as a possible resource for quantum technologies. The discussed quantum correlations, which can be employed in quantum technologies, are available in such systems as multilevel atoms and quantum circuits modeled by Josephson junctions [37]. One can use the approach reviewed in this work in applications for constructing the Deutsch algorithm [38,39]. We consider these aspects of the presented approach in a future publication.

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Conflicts of Interest

The authors declare no conflict of interest.

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