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Article

Fractional Heat Conduction in an Infinite Medium with a Spherical Inclusion

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Abstract: The problem of fractional heat conduction in a composite medium consisting of a spherical inclusion $(0 < r < R)$ and a matrix $(R < r < \infty)$ being in perfect thermal contact at $r = R$ is considered. The heat conduction in each region is described by the time-fractional heat conduction equation with the Caputo derivative of fractional order $0 < \alpha \le 2$ and $0 < \beta \le 2$, respectively. The Laplace transform with respect to time is used. The approximate solution valid for small values of time is obtained in terms of the Mittag-Leffler, Wright, and Mainardi functions.

Keywords: fractional calculus; non-Fourier heat conduction; fractional diffusion-wave equation; perfect thermal contact; Laplace transform; Mittag-Leffler function; Wright function; Mainardi function

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1. Introduction

The standard heat conduction (diffusion) equation for temperature T

$$\frac{\partial T}{\partial t} = a\Delta T \tag{1}$$

is obtained from the balance equation for energy

$$\rho C \frac{\partial T}{\partial t} = -\operatorname{div} \mathbf{q}, \tag{2}$$

where ρ is the mass density, C is the specific heat capacity, **q** **is** the heat flux vector, and the classical Fourier law which states the linear dependence between the heat flux vector **q** and the temperature gradient

$$\mathbf{q} = -k \operatorname{grad} T \tag{3}$$

with k being the thermal conductivity. In the heat conduction Equation (1) $a = k/(\rho C)$ is the heat diffusivity coefficient.

To describe heat conduction in media with complex internal structure, the standard parabolic Equation (1) is no longer accurate enough. In nonclassical theories, the Fourier law Equation (3) and the parabolic heat conduction Equation (1) are replaced by more general equations (see [1–6]). The time-nonlocal dependence between the heat flux vector **q** and the temperature gradient [7,8]

$$\mathbf{q}(t) = -k \int_0^t K(t-\tau) \operatorname{grad} T(\tau) d\tau \tag{4}$$

results in the heat conduction with memory [7,8]

$$\frac{\partial T}{\partial t} = a \int_0^t K(t-\tau) \Delta T(\tau) d\tau. \tag{5}$$

Several particular cases of choice of the memory kernel $K(t-\tau)$ were analyzed in [9–12]. The time-nonlocal dependence between the heat flux vector **q** and the temperature gradient with the long-tail power kernel [9–12]

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \operatorname{grad} T(\tau) d\tau, \qquad 0 < \alpha \le 1, \tag{6}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \operatorname{grad} T(\tau) d\tau, \qquad 1 < \alpha \le 2, \tag{7}$$

where $\Gamma(\alpha)$ is the gamma function, can be interpreted in terms of fractional calculus:

$$\mathbf{q}(t) = -k D_{RL}^{1-\alpha} \operatorname{grad} T, \qquad 0 < \alpha \le 1, \tag{8}$$

$$\mathbf{q}(t) = -k I^{\alpha-1} \operatorname{grad} T, \qquad 1 < \alpha \le 2, \tag{9}$$

where $I^{\alpha} f(t)$ and $D_{RL}^{\alpha} f(t)$ are the Riemann–Liouville fractional integral and derivative of the order α, respectively [13–16]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha > 0, \tag{10}$$

$$D_{RL}^{\alpha} f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau \right], \qquad m-1 < \alpha < m. \tag{11}$$

The balance Equation (2) and the constitutive Equations (8) and (9) yield the time-fractional equation

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\Delta T, \qquad 0 < \alpha \le 2, \tag{12}$$

with the Caputo fractional derivative

$$D_C^{\alpha} f(t) \equiv \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, \qquad m-1 < \alpha < m. \tag{13}$$

The details of obtaining the time-fractional heat conduction Equation (12) from the balance Equation (2) and the constitutive Equations (8) and (9) can be found in [17].

Equations with fractional derivatives, in particular the time-fractional heat conduction equation (diffusion-wave equation), describe many important physical phenomena in different media (see [9,18–32], among many others). Fractional calculus plays a significant part in studies of entropy [33–38]. It should be noted that entropy is also used in analysis of anomalous diffusion processes and fractional diffusion equation [39–45].

Different kinds of boundary conditions for Equation (12) in a bounded domain were analyzed in [46,47]. It should be emphasized that due to the generalized constitutive equations for the heat flux (8) and (9) the boundary conditions for the time-fractional heat conduction equation have their traits in comparison with those for the standard heat conduction equation. The Dirichlet boundary condition specifies the temperature over the surface of a body

$$T\big|_S = g(\mathbf{x}_S, t). \tag{14}$$

For time-fractional heat conduction Equation (12) two types of Neumann boundary condition can be considered: the mathematical condition with the prescribed boundary value of the normal derivative of temperature

$$\left.\frac{\partial T}{\partial n}\right|_s = g(\mathbf{x}_S, t) \tag{15}$$

and the physical condition with the prescribed boundary value of the heat flux

$$D_{RL}^{1-\alpha} \left.\frac{\partial T}{\partial n}\right|_s = g(\mathbf{x}_S, t), \qquad 0 < \alpha \le 1, \tag{16}$$

$$I^{\alpha-1} \left.\frac{\partial T}{\partial n}\right|_s = g(\mathbf{x}_S, t), \qquad 1 < \alpha \le 2. \tag{17}$$

Here $\mathbf{n}$ is the outer unit normal the boundary surface. Similarly, the mathematical Robin boundary condition is a specification of a linear combination of the values of temperature and the values of its normal derivative at the boundary of the domain

$$\left.\left(c_1 T + c_2 \frac{\partial T}{\partial n}\right)\right|_s = g(\mathbf{x}_S, t) \tag{18}$$

with some nonzero constants c_1 and c_2, while the physical Robin boundary condition specifies a linear combination of the values of temperature and the values of the heat flux at the boundary of the domain.

For example, the Newton condition of convective heat exchange between a body and the environment with the temperature T_E

$$\mathbf{q}\cdot\mathbf{n}\big|_S = h\big(T\big|_S - T_E\big), \tag{19}$$

where h is the convective heat transfer coefficient, leads to

$$\left(hT + kD_{RL}^{1-\alpha}\frac{\partial T}{\partial n}\right)\Big|_s = hT_E(\mathbf{x}_s,t), \qquad 0<\alpha\le 1, \tag{20}$$

$$\left(hT + kI^{\alpha-1}\frac{\partial T}{\partial n}\right)\Big|_s = hT_E(\mathbf{x}_s,t), \qquad 1<\alpha\le 2. \tag{21}$$

If the surfaces of two solids are in perfect thermal contact, the temperatures on the contact surface and the heat fluxes through the contact surface are the same for both solids, and the boundary conditions of the fourth kind are obtained:

$$T_1\big|_S = T_2\big|_S, \tag{22}$$

$$k_1 D_{RL}^{1-\alpha}\frac{\partial T_1}{\partial n}\Big|_s = k_2 D_{RL}^{1-\beta}\frac{\partial T_2}{\partial n}\Big|_s, \qquad 0<\alpha\le 2, \qquad 0<\beta\le 2, \tag{23}$$

where subscripts 1 and 2 refer to the first and second solid, respectively, and **n** is the common unit normal at the contact surface. In fractional calculus, where integrals and derivatives of arbitrary (not only integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors [15,25] do not use a separate notation for the fractional integral $I^{\alpha}f(t)$. The fractional integral of the order $\alpha>0$ is denoted as $D_{RL}^{-\alpha}f(t)$. In the equation of perfect thermal contact (23) $D_{RL}^{1-\alpha}f(t)$, $0<\alpha\le 2$, and $D_{RL}^{1-\beta}f(t)$, $0<\beta\le 2$, are understood in this sense.

Starting from the pioneering papers [48–52], considerable interest has been shown in solutions to time-fractional heat conduction equation. In the literature, there are only a few papers in which the fractional heat conduction equation (fractional diffusion-wave equation) is studied in composite medium [47,53,54]. In the present paper, the problem of fractional heat conduction in a composite medium consisting of a spherical inclusion ($0<r<R$) and a matrix ($R<r<\infty$) being in perfect thermal contact at $r=R$ is considered. The heat conduction in each region is described by the time-fractional heat conduction equation with the Caputo derivative of fractional order $0<\alpha\le 2$ and $0<\beta\le 2$, respectively.

2. Statement of the Problem

Consider the time-fractional heat conduction equations in a spherical inclusion

$$\frac{\partial^{\alpha}T_1}{\partial t^{\alpha}} = a_1\left(\frac{\partial^2 T_1}{\partial r^2}+\frac{2}{r}\frac{\partial T_1}{\partial r}\right), \qquad 0<r<R, \tag{24}$$

and in a matrix

$$\frac{\partial^{\beta} T_2}{\partial t^{\beta}} = a_2 \left(\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r} \frac{\partial T_2}{\partial r} \right), \qquad R < r < \infty, \tag{25}$$

under the initial conditions

$$t = 0: \quad T_1 = f_1(r), \qquad 0 < r < R, \quad 0 < \alpha \le 2, \tag{26}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = F_1(r), \qquad 0 < r < R, \quad 1 < \alpha \le 2, \tag{27}$$

$$t = 0: \quad T_2 = f_2(r), \qquad R < r < \infty, \quad 0 < \beta \le 2, \tag{28}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = F_2(r), \qquad R < r < \infty, \quad 1 < \beta \le 2, \tag{29}$$

and the boundary condition of perfect thermal contact

$$r = R: \qquad T_1(r,t) = T_2(r,t), \tag{30}$$

$$r = R: \qquad k_1 D_{RL}^{1-\alpha} \frac{\partial T_1(r,t)}{\partial r} = k_2 D_{RL}^{1-\beta} \frac{\partial T_2(r,t)}{\partial r}. \tag{31}$$

The boundedness condition at the origin and the zero condition at infinity are also assumed:

$$\lim_{r \to 0} T_1(r,t) \neq \infty, \qquad \lim_{r \to \infty} T_2(r,t) = 0. \tag{32}$$

The limitations on α and β in Equations (26–29) express the fact that if $1 < \alpha \le 2$ or $1 < \beta \le 2$, then the additional condition on the first time derivative should be also imposed.

In what follows we restrict ourselves to the particular case when a sphere $0 \le r < R$ is at initial uniform temperature T_0 and the matrix $R < r < \infty$ is at initial zero temperature

$$t = 0: \quad T_1 = T_0, \qquad 0 < r < R, \quad 0 < \alpha \le 2, \tag{33}$$

$$t = 0: \quad \frac{\partial T_1}{\partial t} = 0, \qquad 0 < r < R, \quad 1 < \alpha \le 2, \tag{34}$$

$$t = 0: \quad T_2 = 0, \qquad R < r < \infty, \quad 0 < \beta \le 2, \tag{35}$$

$$t = 0: \quad \frac{\partial T_2}{\partial t} = 0, \qquad R < r < \infty, \quad 1 < \beta \le 2. \tag{36}$$

The Laplace transform with respect to time t applied to Equations (24) and (25) leads to two ordinary differential equations

$$s^{\alpha} T_1^* - s^{\alpha-1} T_0 = a_1 \left(\frac{\partial^2 T_1^*}{\partial r^2} + \frac{2}{r} \frac{\partial T_1^*}{\partial r} \right), \qquad 0 < r < R, \tag{37}$$

$$s^{\beta} T_2^* = a_2 \left(\frac{\partial^2 T_2^*}{\partial r^2} + \frac{2}{r} \frac{\partial T_2^*}{\partial r} \right), \qquad R < r < \infty, \tag{38}$$

having the solutions

$$T_1^*(r,s)=\frac{A_1}{r}\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}r\right)+\frac{B_1}{r}\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}r\right)+\frac{T_0}{s},\qquad 0<r<R, \tag{39}$$

$$T_2^*(r,s)=\frac{A_2}{r}\exp\left(\sqrt{\frac{s^\beta}{a_2}}r\right)+\frac{B_2}{r}\exp\left(-\sqrt{\frac{s^\beta}{a_2}}r\right),\qquad R<r<\infty. \tag{40}$$

It follows from conditions at the origin and at infinity Equation (32) that

$$A_1=0,\qquad A_2=0. \tag{41}$$

The integration constants B_1 and B_2 are obtained from the perfect thermal contact boundary conditions Equations (30) and (31)

$$B_1=\frac{k_2T_0R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)s^{-1}}{\left[k_1s^{\beta-\alpha}-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)-Rk_1s^{\beta-\alpha}\sqrt{\frac{s^\alpha}{a_1}}\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)}, \tag{42}$$

$$B_2=\frac{T_0R}{s}\exp\left(\sqrt{\frac{s^\beta}{a_2}}R\right)+\frac{k_2T_0R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)s^{-1}\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)\exp\left(\sqrt{\frac{s^\beta}{a_2}}R\right)}{\left[k_1s^{\beta-\alpha}-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)-Rk_1s^{\beta-\alpha}\sqrt{\frac{s^\alpha}{a_1}}\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)}. \tag{43}$$

Hence, the solution is written as

$$T_1^*=\frac{T_0}{s}+\frac{k_2T_0R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}r\right)s^{-1}r^{-1}}{\left[k_1s^{\beta-\alpha}-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)-Rk_1s^{\beta-\alpha}\sqrt{\frac{s^\alpha}{a_1}}\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)}, \tag{44}$$

$$T_2^*=\frac{T_0R}{rs}\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]+\frac{k_2T_0R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)s^{-1}r^{-1}\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]}{\left[k_1s^{\beta-\alpha}-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]\sinh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)-Rk_1s^{\beta-\alpha}\sqrt{\frac{s^\alpha}{a_1}}\cosh\left(\sqrt{\frac{s^\alpha}{a_1}}R\right)}. \tag{45}$$

Now we will investigate the approximate solution of the considered problem for small values of time. In the case of classical heat conduction this method was described in [55,56]. Based on Tauberian theorems for the Laplace transform (see, for example [57]), for small values of time t (the large values of the transform variable s) we can neglect the exponential term in comparison with 1,

$$1\pm\exp\left[-2\sqrt{\frac{s^\alpha}{a_1}}R\right]\approx 1, \tag{46}$$

thus obtaining

$$T_1^* \approx \frac{T_0}{s} + \frac{k_2 T_0 R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\left\{\exp\left[-\sqrt{\frac{s^\alpha}{a_1}}(R-r)\right]-\exp\left[-\sqrt{\frac{s^\alpha}{a_1}}(R+r)\right]\right\}}{rs\left[k_1 s^{\beta-\alpha}\left(1-R\sqrt{\frac{s^\alpha}{a_1}}\right)-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]}, \tag{47}$$

$$T_2^* \approx \frac{T_0 R}{rs}\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right] + \frac{k_2 T_0 R\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\exp\left[-\sqrt{\frac{s^\beta}{a_2}}(r-R)\right]}{rs\left[k_1 s^{\beta-\alpha}\left(1-R\sqrt{\frac{s^\alpha}{a_1}}\right)-k_2\left(1+R\sqrt{\frac{s^\beta}{a_2}}\right)\right]}. \tag{48}$$

In the following particular cases $\alpha = 2/3,\ \beta = 4/3;\ \alpha = 1,\ \beta = 2;\ \alpha = 2,\ \beta = 1$ the denominator in Equations (47) and (48) can be treated as a cubic equation and the decomposition into the sum of partial fractions can be obtained similar to that used in [58].

Now we will consider another particular case when $\alpha = \beta$.

To invert the Laplace transform the following formula will be used [14–16]

$$L^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + c}\right\} = t^{\beta-1}E_{\alpha,\beta}\left(-ct^\alpha\right), \tag{49}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha > 0, \quad \beta > 0, \quad z \in C. \tag{50}$$

Additionally [51,52,59–61]

$$L^{-1}\left\{\exp\left(-\lambda s^\gamma\right)\right\} = \frac{\gamma\lambda}{t^{\gamma+1}}M\left(\gamma;\lambda t^{-\gamma}\right), \qquad 0<\gamma<1, \quad \lambda>0, \tag{51}$$

$$L^{-1}\left\{s^{\gamma-1}\exp\left(-\lambda s^\gamma\right)\right\} = \frac{1}{t^\gamma}M\left(\gamma;\lambda t^{-\gamma}\right), \qquad 0<\gamma<1, \quad \lambda>0, \tag{52}$$

$$L^{-1}\left\{s^{-\beta}\exp\left(-\lambda s^\gamma\right)\right\} = t^{\beta-1}W\left(-\gamma,\beta;-\lambda t^{-\gamma}\right), \qquad 0<\gamma<1, \quad \lambda>0. \tag{53}$$

Here $W(\gamma,\beta;z)$ is the Wright function [1,51,52,62]

$$W(\gamma,\beta;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\gamma k+\beta)}, \qquad \gamma > -1, \quad z \in C, \tag{54}$$

whereas $M(\gamma;z)$ is the Mainardi function [15,51,52]

$$M(\gamma;z) = W(-\gamma,1-\gamma;-z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!\Gamma(-\gamma k+1-\gamma)}, \qquad 0<\gamma<1, \quad z \in C. \tag{55}$$

From Equations (47) and (48) we get:

$$T_1(r,t) \approx T_0 - \frac{RT_0k_2}{(k_2-k_1)r}\left[W\left(-\frac{\alpha}{2},1\,;-\frac{R-r}{\sqrt{a_1}t^{\alpha/2}}\right)-W\left(-\frac{\alpha}{2},1;-\frac{R+r}{\sqrt{a_1}t^{\alpha/2}}\right)\right]$$

$$+\frac{CRT_0}{r}\int_0^t\frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}}\left[M\left(\frac{\alpha}{2};\frac{R-r}{\sqrt{a_1}\tau^{\alpha/2}}\right)-M\left(\frac{\alpha}{2};\frac{R+r}{\sqrt{a_1}\tau^{\alpha/2}}\right)\right]E_{\alpha/2,\,\alpha/2}\left[-b\,(t-\tau)^{\alpha/2}\right]d\tau, \tag{56}$$

$$T_2(r,t) \approx -\frac{RT_0k_1}{(k_2-k_1)r}W\left(-\frac{\alpha}{2},1\,;-\frac{r-R}{\sqrt{a_2}t^{\alpha/2}}\right)$$

$$+\frac{CRT_0}{r}\int_0^t\frac{(t-\tau)^{\alpha/2-1}}{\tau^{\alpha/2}}M\left(\frac{\alpha}{2};\frac{r-R}{\sqrt{a_2}\tau^{\alpha/2}}\right)E_{\alpha/2,\,\alpha/2}\left[-b\,(t-\tau)^{\alpha/2}\right]d\tau, \tag{57}$$

where

$$b=\frac{(k_2-k_1)\sqrt{a_1a_2}}{R\left(k_1\sqrt{a_2}+k_2\sqrt{a_1}\right)},\qquad C=\frac{k_1k_2\left(\sqrt{a_1}+\sqrt{a_2}\right)}{(k_2-k_1)\left(k_1\sqrt{a_2}+k_2\sqrt{a_1}\right)}. \tag{58}$$

It should be emphasized that the solution is expressed in terms of the Mainardi function $M(\alpha/2;z)$ and the Wright function $W(-\alpha/2,\beta;z)$. The limitation $0<\gamma<1$ in Equations (51–53) means that $0<\alpha<2$ in Equations (56) and (57).

4. Conclusions

We have obtained the approximate solution to the time-fractional heat conduction equations in a composite body consisting of a matrix and spherical inclusion with different thermophysical properties. The conditions of perfect thermal contact have been assumed: the temperatures at the boundary surface are equal and the heat fluxes through the contact surface are the same. The Laplace integral transform allows us to obtain the ordinary differential equations for temperatures. Inversion of the Laplace transform has been carried out analytically for small values of time.

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Conflicts of Interest

The author declares no conflict of interest.

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