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Article

Corrections to Bekenstein-Hawking Entropy — Quantum or not-so Quantum?

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Abstract: Hawking radiation and Bekenstein-Hawking entropy are the two robust predictions of a yet unknown quantum theory of gravity. Any theory which fails to reproduce these predictions is certainly incorrect. While several approaches lead to Bekenstein-Hawking entropy, they all lead to different sub-leading corrections. In this article, we ask a question that is relevant for any approach: Using simple techniques, can we know whether an approach contains quantum or semi-classical degrees of freedom? Using naive dimensional analysis, we show that the semi-classical black-hole entropy has the same dimensional dependence as the gravity action. Among others, this provides a plausible explanation for the connection between Einstein's equations and thermodynamic equation of state, and that the quantum corrections should have a different scaling behavior.

Keywords: black-holes; entanglement; Bekenstein-Hawking

Entropy is a derived quantity and does not show up in any fundamental equation of motion. However, in any physical theory, entropy takes unique position amongst other quantities. This is due to the fact that entropy relates the macroscopic and microscopic degrees of freedom (DOF) through Boltzmann relation [1]:

$$S = k_B \ln \Omega \tag{1}$$

where k_B is Boltzmann constant and Ω is total number of micro-states. Hence, it is not surprising that there has been intense research activity in obtaining the microscopic description of Bekenstein–Hawking entropy [2]:

$$\mathcal{S}_{\rm BH} = \frac{k_B}{4}\frac{\mathcal{A}_{\rm H}}{l^2_{\rm Pl}} = \frac{k_B}{4} M^2_{\rm Pl}\,\mathcal{A}_{\rm H} \tag{2}$$

where, $\mathcal{A}_{\rm H}$ is the area of black-hole horizon, $M_{\rm Pl}^2 \equiv 1/(8\pi G)$, $l_{\rm Pl}^2$ are Planck mass and Planck length, respectively.

While several approaches lead to $\mathcal{S}_{\rm BH}$, none of these approaches can be considered to be complete. For instance, in string computations, BPS states are well-defined only for (near) extremal black-holes [3,4]. In conformal field theory approach [5,6], where the horizon is treated as boundary, the vector fields (which generate the symmetries) do not have a well-defined limit at the horizon [7]. Besides, all these approaches lead to different sub-leading corrections to $\mathcal{S}_{\rm BH}$. For instance, conformal field theory [6] and quantum geometry approaches [8] lead to logarithmic corrections while the string [4] and entanglement computations [9,10] lead to power-law corrections.

In hindsight, one can say this is probably expected; different approaches count different microscopic states that are valid in domains of their applicability. In the absence of a consistent quantum theory of gravity, it is not possible to know the microscopic DOF, and hence, the subject of black-hole thermodynamics resembles a *jig-saw puzzle*. To put several pieces together in this puzzle, the best strategy is to slowly build a coherent picture and hope to understand/solve some of these problems. The purpose of this article is an attempt in this direction and we ask: Is there a way one can classify these different subleading corrections to the Bekenstein-Hawking entropy? In other words, *Using simple techniques, can we know whether an approach contains semi-classical or quantum DOF?* The answer to this question is relevant for any approach to black-hole entropy.

We show that the naive dimensional analysis [11] provides crucial information about the semi-classical and(or) quantum nature of the sub-leading terms. Among others, this provides an explanation for the connection between Einstein's equations and thermodynamic equation of state. Using this, we argue that quantum entanglement is crucial to gain insights on black-hole thermodynamics.

But, why is it important to understand sub-leading corrections to $\mathcal{S}_{\rm BH}$? In order to exemplify this, let us compare black-hole entropy with ideal gas entropy. The classical entropy of mono-atomic ideal gas is given by

$$\frac{S_{\rm ideal}}{k_{_B}\, N} = \ln\left(V\, T^{3/2}\right) \tag{3}$$

where V, T, N correspond to volume, temperature and number of particles, respectively.

Assuming that all atoms move independently, we can obtain the number of quantum states and, hence, the Sackur-Tetrode entropy [13]

$$\frac{S_{\rm ST}}{k_{_B}\, N} = \ln\left(V\, T^{3/2}\right) + \frac{1}{2}\ln\frac{M^3}{N^5} + \frac{3}{2}\ln\left(\frac{4\pi k_{_B}}{3\hbar^2}\right)\,, \tag{4}$$

where M is the mass of the gas.

Among others, there are two main reasons for the relevance of $S_{\rm ST}$ to black-hole entropy: Firstly, $S_{\rm ST}$ depends on the mass of the DOF of an ideal gas—*the individual atom*; the classical expression has no explicit mass dependence. In other words, varied DOF can lead to identical classical expression while sub-leading terms which contain information about the DOF will be different. Similarly, several approaches to black-hole entropy lead to identical $\mathcal{S}_{\rm BH}$; however, they lead to different sub-leading corrections.

Secondly, quantum correction to classical entropy [third term in the RHS of Equation (4)] does not depend on the macroscopic quantities. It is needless to say that this could not have been foreseen by

physical arguments. In the same manner, it would be impossible to predict quantum corrections to $\mathcal{S}_{\rm BH}$. If one uses symmetry arguments based on classical action, then what we may obtain, as discussed below, will be proportional to the form obtained from dimensional analysis [14].

Having addressed the importance of sub-leading corrections, we show that naive dimensional analysis provides information about the quantum/semi-classical nature of the sub-leading terms. Let us consider the Einstein-Hilbert action in 4-dimensions:

$$S_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\int d^4x\,\sqrt{-g}\,R\,. \tag{5}$$

Dimensional analysis of the above action leads to

$$S_{\rm EH} \propto M_{\rm Pl}^2 \times [L]^2 \tag{6}$$

Dimensional analysis of 4-dimensional Bekenstein-Hawking entropy Equation (2) leads to

$$\mathcal{S}_{\rm BH} \propto M_{\rm Pl}^2 \times [L]^2 \tag{7}$$

Attentive reader might realize that the Einstein-Hilbert action Equation (6) and Equation $\mathcal{S}_{\rm BH}$ (7) have the same dimensional dependence, indicating that the semi-classical black-hole entropy seem to follow the (classical) gravity action and ask, does this relation hold for a general gravity action?

Let us now consider a D-dimensional gravity action:

$$\begin{aligned} S_{\rm Gen} &= \frac{1}{16\pi G_{_D}}\int d^Dx\sqrt{-g}\left[R+\alpha\,F(R^2)+\beta\,G(R^3)+\cdots\right] \\ &= \frac{M_{\rm P}^{(D-2)}}{16\pi}\int d^Dx\sqrt{-g}\left[R+\alpha\,F(R^2)+\beta\,G(R^3)+\cdots\right] \end{aligned} \tag{8}$$

where $G_D, M_{\rm P}$ are D-dimensional Newton's constant and Planck mass, respectively, $F(R^2)$ include combination of R^2, $R_{AB}R^{AB}$, $R_{ABCD}R^{ABCD}$ terms, $G(R^3)$ includes cubic terms, and α,β are dimension-full constants. The above action includes Lovelock gravity whose equations of motion are quasi-linear [15,16]. Dimensional analysis of this action leads to:

$$S_{\rm Gen} \propto M_{\rm P}^{(D-2)} \times [L]^{D-2}\left[1+\frac{\alpha}{[L]^2}+\frac{\beta}{[L]^4}+\cdots\right] \tag{9}$$

The Noether charge entropy corresponding to this action is given by [17]:

$$S_{\rm NC} = M_{\rm P}^{(D-2)}\frac{\mathcal{A}_{\rm D}}{4}\left[1+\alpha\mathcal{A}_{\rm D}^{2/(D-2)}+\beta\mathcal{A}_{\rm D}^{2/(D-2)}+\cdots\right], \tag{10}$$

where $\mathcal{A}_{\rm D}$ is the horizon area of black-holes in D-dimensional space-time. It is important to note that in deriving Bekenstein-Hawking and Noether charge entropy, it is assumed that the back-reaction of the Hawking particles are negligible.

Dimensional analysis of the entropy leads to:

$$S_{\rm NC} \propto M_{\rm P}^{(D-2)} \times [L]^{D-2}\left[1+\frac{\alpha}{[L]^2}+\frac{\beta}{[L]^4}+\cdots\right] \tag{11}$$

This observation indicates that the (semi-classical) black-hole—like $\mathcal{S}_{\rm BH}$ and No-ether charge—entropy in any gravity theory follow the form of the classical gravity action. So, what are the physical

consequences of this observation? Firstly, this feature is specific to gravity and, to author's knowledge, can not be seen in other fundamental interactions. Comparing Equation (4) with the electromagnetic action:

$$S_{EM} = -\frac{1}{4}\int d^4\, x\, F_{\mu\nu}F^{\mu\nu}\,, \tag{12}$$

it is clear that the entropy of ideal gas and the electromagnetic action do not have same dimensional dependence.

The above observation can be viewed as the primary reason as to why Einstein's equations can be viewed as thermodynamic equation of state [18,19]. The crucial input, which leads to thermodynamic equation from Einstein equations, is the form of the entropy Equation (2), Equation (10). For instance, if we consider the power-law corrections to the Bekenstein-Hawking entropy in 4-dimensional space-times arising from entanglement [9], and use the approach of Jacobson or Padmanabhan, Einstein equations *can not* be rewritten as first law of thermodynamics. In the recent proposal by Verlinde [20,21], interpreting gravity as entropic force, there is an implicit assumption about the entropy-area relation [22].

Secondly, in ideal gas, the quantum corrections to the (semi)classical entropy do not have any volume dependence. For black-hole entropy, this suggests that quantum gravitational corrections to $\mathcal{S}_{\rm BH}$ will include terms which may not follow the form of the classical gravity action. At least two of the approaches to black-hole entropy do seem to agree with this observation: (i) In quantum geometry approach, it was shown that Hilbert space of the horizon of spherically symmetric space-time is $2d\, SU(2)_k$ Wess-Zumino model leading to generic logarithmic corrections [8]. (ii) Entanglement entropy of the metric perturbations, about the black-hole background in 4-dimensional general relativity, lead to power-law corrections [9,10].

Lastly, this provides a simple way to classify approaches which predict corrections to Bekenstein-Hawking entropy. For instance, approaches discussed in [4,23] lead to the form which is similar to Equation (10) while that obtained in [9] do not follow Noether charge entropy. In other words, this suggests that if any approach to black-hole entropy predicts the same dimensional form as the classical action of gravity, then this approach *only* provides semi-classical, and not quantum, structure of gravity. This might seem a strong assertion, however, it would be an even stronger claim if one says that the quantum corrections to $\mathcal{S}_{\rm BH}$ follow the same dimensionality of the classical action. For instance, the power-law corrections to the Bekenstein-Hawking entropy obtained by Demers *et al.* [23] have the same form as the Noether charge entropy Equation (10).

These conclusions raise a related question: *Why the entropy of a black-hole, and not (neutron) star, has the same dimensional form as the classical gravity action?* Classically, stars and black-holes are described by spherically symmetric solutions of gravity and matter action Equation (5), Equation (8). However, it is the existence of the event-horizon which distinguishes black-holes and stars. Hence, quantum gravity should have a mechanism to account the existence of the horizon which would imply that the semi-classical entropy of black-holes, and not stars, has the same dimensional form as gravity action.

This raises another question: *Is there one universal feature which is common to the microscopic theory which distinguishes black-hole and star?* Entanglement, the quantum correlation that exist between subsystems of a quantum system, is a feature of quantum system. The presence of the event-horizon

gives rise to natural emergence of entanglement entropy [24,25] and, hence, distinguishing the entropy associated to black-holes and stars.

Interestingly, entanglement provides natural explanation for the area-dependence of black-hole entropy. For a bipartite system in a pure state, tracing over given subsystem and its complementary system yield identical entanglement entropies [26]. As shown in [9], the interaction terms across the boundary contribute significantly to the entanglement entropy, hence, entanglement entropy is a function of the boundary [$S_{\rm ent} \propto F(A)$] which in the case of black-holes is the event-horizon. It is now known that, for fields in (i) *vacuum*: $F(A) = A$ [24,25], (ii) *excited states*: $F(A) = c_0 A + c_1/A^{\mu}$ (c_0, c_1, μ are positive real numbers) [9,10].

The central thesis in this article has been to put together some pieces of the *jig-saw puzzle* which one encounters in obtaining a microscopic description of black-hole entropy. Starting from the observation that the entropy of black-hole has the same dimensional dependence as that of classical gravity action, we have shown that it is plausible to differentiate between different approaches. Interestingly, this provides a plausible understanding for the connection between Einstein's equations and thermodynamic equation of state. We also have provided arguments as to how entanglement provides a natural framework to understand black-hole thermodynamics.

There are several conceptual issues which are unresolved: What makes gravity special that the dimensional analysis of the gravity action almost directly implies the maximum entropy the gravitational object can have? How to show from fundamental principle that the above observation indeed is the key to rewrite Einstein equation as thermodynamics of space-time? These are currently under investigation.

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